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A DECOMPOSITION THEOREM FOR CERTAIN BIPOLYNOMIAL HOPF ALGEBRAS

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ABSTRACT. In this note we generalise a result of D. Husemoller to certain bipolynomial Hopf algebras and are able to give Hopf algebra decompositions for these. As an easy consequence of our approach we give a simplified derivation of recent results of P. Hoffman on polynomial generators for these algebras; we also give explicit systems of "Borel generators" for a related family of quotient Hopf algebras considered by Hoffman.

Introduction. In this note we generalise a splitting due to D. Husemoller [3] for a certain bipolynomial Hopf algebra to a family of sub-Hopf algebras; in doing so, we are able to give an explicit procedure for choosing polynomial generators for these, and "Borel generators" for associated quotient Hopf algebras. We are thus able to illuminate and simplify results of P. Hoffman [2].

The main motivation for this work is the identification of the Hopf algebras P and S (see below) with $H_*(BU;\mathbb{Z})$ and $H^*(BU;\mathbb{Z})$, the homology and cohomology of the space BU—see [3]. In fact, $(S/\langle c_1\rangle)^*$ can be identified with $H_*(BSU,\mathbb{Z})$, the homology of BSU, which is considered in [1] and [4]. However, the other sub-Hopf algebras of P considered below are of less immediate interest to topologists, since their $(\text{mod}\, p)$ reductions are not invariant under the action of the Steenrod algebra on P. We will describe the appropriate topologically interesting generalisation in "Husemoller splittings and actions of the Steenrod algebra" (in preparation).

Let $P=\mathbb{Z}[b_j \mid j\geq 1]$ be the graded Hopf algebra with $|b_j|=2j$, and coproduct $\Delta(b_j)=\sum_{0\leq r\leq j} b_r\otimes b_{j-r}$; let $S=\mathbb{Z}[c_j \mid j\geq 1]$ be the dual of P where, relative to the monomial basis of P, c_j is dual to b_1^j. It is well known that there exists an isomorphism of Hopf algebras $P\cong S$ with $b_j\leftrightarrow c_j$, and hence P is a *self dual* Hopf algebra [3]. We denote by $s_j\in P_{2j}$ the element dual to c_j; recall that this element is primitive under the diagonal Δ (i.e. $\Delta(s_j)=s_j\otimes 1+1\otimes s_j$) and there is a recursive formula (due to I. Newton!)

$$s_j=b_1s_{j-1}-b_2s_{j-2}+\cdots+(-1)^{j-2}b_{j-1}s_1+(-1)^{j-1}jb_j. \tag{1}$$

Let $G_{(p)}$ denote the localisation of G at a prime p for an abelian group G.

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Recall from [3] that there are elements $a_{n,k} \in P_{(p)}$ with $|a_{n,k}| = 2np^k$, $(n, p) = 1$, and $k \geq 0$, defined recursively by

(2) $$s_{np^k} = p^k a_{n,k} + p^{k-1} a_{n,k-1}^p + \cdots + p a_{n,1}^{p^{k-1}} + a_{n,0}^{p^k}.$$

Let $B_{(p)}[2n] = \mathbb{Z}_{(p)}[a_{n,k} \mid k \geq 0] \subset P_{(p)}$ for $(n, p) = 1$; this is a sub-Hopf algebra, with coproduct defined recursively using (2). Then we have an isomorphism of Hopf algebras

$$P_{(p)} \cong \prod_{(n,p)=1} B_{(p)}[2n].$$

Now define $y_{n,k}$ with $|y_{n,k}| = 2np^k$ and for $(n, p) = 1$, by

(3) $$\begin{cases} y_{n,1} = s_{np} & \text{if} \quad k = 1, \\ y_{n,k} = \dfrac{1}{p^{k-1}} [s_{np^k} - p^{k-2} y_{n,k-1}^p - p^{k-3} y_{n,k-2}^{p^2} + \cdots - y_{n,1}^{p^{k-1}}] & \text{if} \quad k \geq 2. \end{cases}$$

These exist *a priori* in $B_{(p)}[2n] \otimes \mathbb{Q}$ and generate over $\mathbb{Z}_{(p)}$ a sub-Hopf algebra $B_{(p)}^{(1)}[2n]$ which is easily seen to be polynomial on the $y_{n,j}$.

THEOREM A. *Each $y_{n,k}$ is in $B_{(p)}[2n]$ and $B_{(p)}^{(1)}[2n]$ is a sub-Hopf algebra of $B_{(p)}[2n]$, polynomial on the $y_{n,k}$ for $k > 0$.*

Define $\Phi^{(1)}: B_{(p)}[2n] \to B_{(p)}[2n]$ by $\Phi^{(1)}(a_{n,k}) = y_{n,k+1}$ for $k \geq 0$; note that $\Phi^{(1)}$ multiplies degrees by p. Then $B_{(p)}^{(1)}[2n] = \operatorname{im} \Phi^{(1)}$.

More generally define $\Phi^{(r)}: B_{(p)}^{(r-1)}[2n] \to B_{(p)}[2n]$ by restricting $\Phi^{(1)}$, and set $B_{(p)}^{(r)}[2n] = \operatorname{im} \Phi^{(r)}$. For convenience, set $\Phi^{(0)} = \mathrm{Id}$.

THEOREM B. *There is an isomorphism of Hopf algebras (which multiplies degrees by p^r)*

$$\Psi^{(r)}: B_{(p)}[2n] \cong B_{(p)}^{(r)}[2n]$$

given by $\Psi^{(r)} = \Phi^{(r)} \cdot \Phi^{(r-1)} \cdots \Phi^{(1)}$. Hence, $B_{(p)}^{(r)}[2n]$ is self dual.

THEOREM C. *As sub-Hopf algebras of $P_{(p)}$*

$$(S_{(p)}/\langle c_1, \ldots, c_k \rangle)^* = \left[\prod_{(n,p)=1} B_{(p)}^{(r_n)}[2n] \right],$$

where r_n is the minimum r such that $k < np^r$ for $(n, p) = 1$.

COROLLARY. $(S_{(p)}/\langle c_1, \ldots, c_k \rangle)^* \cong S_{(p)}/\langle c_1, \ldots, c_k \rangle$*—hence, these are self dual Hopf algebras.*

Proof of Theorem A. We will prove by induction on $k \geq 1$ the statement "For each $n \geq 1$ we have:

(4) $$\begin{cases} y_{n,k} \in P_{(p)} \\ y_{n,k} \equiv a_{n,k-1}^p \quad (\text{mod } p) \\ y_{n,k} \equiv p a_{n,k} \quad (\text{mod decomposables})." \end{cases}$$

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For $k=1$ this is immediate from the definitions, so suppose it true for $k<m$; then we have

$$\begin{aligned} p^{m-1}y_{n,m} &= s_{np^m} - [p^{m-2}y_{n,m-1}^{p} + p^{m-3}y_{n,m-2}^{p^2} + \cdots + y_{n,1}^{p^{m-1}}] \\ &\equiv [p^{m-2}a_{n,m-2}^{p^2} + p^{m-3}a_{n,m-3}^{p^3} + \cdots + a_{n,0}^{p^m}] \\ &\quad - [p^{m-2}a_{n,m-2}^{p^2} + \cdots + a_{n,0}^{p^m}] \pmod{p^{m-1}} \\ &= 0. \end{aligned}$$

By direct inspection, the coefficients of monomials in the $a_{n,j}$'s are exactly p^m for $a_{n,m}$ and $p^{m-1} (\bmod p^m)$ for $a_{n,m-1}^p$.

To see that $B_{(p)}^{(1)}[2n]$ is a coalgebra, note that the coaction on $y_{n,k}$ is recursively defined by

$$(5) \qquad s_{np^k} = p^{k-1}y_{n,k} + p^{k-2}y_{n,k-1}^{p} + \cdots + y_{n,1}^{p^{k-1}}$$

and so except for a change in indexing $\Delta(y_{n,k})$ is given by the same expression in the $y_{n,j}$ as $\Delta(a_{n,k})$ is in the $a_{n,j}$. Note that this implies that $\Phi^{(1)}$ is a coalgebra homomorphism! (QED)

The proof of Theorem B is now immediate.

Observe that if we reduce $(\bmod p)$ and work in $B_{(p)}[2n] \otimes \mathbb{Z}/p$ then $\Psi^{(r)}$ becomes the p^r-th power map and the cokernel $(B_{(p)}[2n] \otimes \mathbb{Z}/p)//\Psi^{(r)}$ is the algebra

$$\mathbb{Z}/p[a_{n,k} \mid k \geq 0]/\langle a_{n,k}^{p^r} \mid k \geq 0 \rangle.$$

Proof of Theorem C. It will suffice to show that each of the factors in the stated decomposition are in $(S_{(p)}/\langle c_1, \ldots, c_k \rangle)^*$ and the generators of form $\Psi^{(r_n)}(a_{n,k})$ are indivisible in the indecomposable quotient.

Recall that we have $s_t \in (S_{(p)}/\langle c_1, \ldots, c_k \rangle)^*$ iff c_t is non-zero in $S_{(p)}/\langle c_1, \ldots, c_k \rangle$, i.e. iff $t>k$. Thus $s_{np^r} \in (S_{(p)}/\langle c_1, \ldots, c_k \rangle)^*$ with $(n,p)=1$ iff $np^r>k$. Now note that for $t \geq 0$, the equation

$$s_{np^{t+r_n}} = p^t Z_{n,t} + p^{t-1} Z_{n,t-1}^{p} + \cdots + Z_{n,0}^{p^t}$$

is satisfied by $Z_{n,j} = \Psi^{(r_n)}(a_{n,j})$. This follows on applying $\Phi^{(1)}$ to (2) repeatedly. Finally observe that since $(S/\langle c_1, \ldots, c_k \rangle)^*$ is a direct summand of P, we can now show that $\Psi^{(r_n)}(a_{n,t})$ is in $(S/\langle c_1, \ldots, c_k \rangle)^*$ by induction on $t \geq 0$.

Upon tensoring with $\mathbb{Z}/p$ we obtain a polynomial subalgebra

$$\prod_{(n,p)=1} \mathbb{Z}/p[a_{n,j}^{p^{r_n}} \mid j \geq 0]$$

of the algebra $(S/\langle c_1, \ldots, c_k \rangle)^* \otimes \mathbb{Z}/p$ and a Poincaré series argument shows that these algebras are equal. The proof is completed by observing that $(S_{(p)}/\langle c_1, \ldots, c_k \rangle)^*$ is a summand in $P_{(p)}$ and therefore the last remark is also true for the corresponding subalgebras of $P_{(p)}$. (QED)

The corollary now follows from the fact that each $B_{(p)}[2n]$ is self dual by [3].

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Theorem C gives criteria for choosing polynomial generators of these subalgebras of $P_{(p)}$ in terms of the $a_{n,j}$. Suppose we have a sequence of elements x_m in $(S_{(p)}/\langle c_1, \ldots, c_k\rangle)^*_{2m}$ for $m > k$. Then this is a set of polynomial generators iff for $(n, p) = 1$

$$x_{np^r} \equiv up^r a_{n,r} \quad \text{(mod decomposables)} \tag{6}$$

where $u \in \mathbb{Z}_{(p)}$ with $(u, p) = 1$. For the case $k = 1$ this recovers results of Adams [1] and Kochman [4], and more generally of Hoffman [2]. Observe that we can also rederive the results of Hoffman on $(\mathbb{Z}/p[c_1, \ldots, c_k])^*$; indeed the projections of the $a_{n,j}$ form a system of Borel generators for this Hopf algebra (viewed as a quotient of $P_{(p)}$) whereas the projections of the b_j into this quotient of P satisfy complicated polynomial relations—see [2].

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