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ON ALMOST CONTINGENT MANIFOLDS OF SECOND CLASS WITH APPLICATIONS IN RELATIVITY

BY
K. L. DUGGAL

1. **Introduction.** D. E. Blair [1] has introduced the notion of K-manifolds as an analogue of the even dimensional Kähler manifolds and of the odd dimensional quasi-Sasakian manifolds. These manifolds have been studied with respect to a *positive definite metric*. In this paper, we study a more general case of K-manifolds carrying *an arbitrary* non-degenerate metric, in particular, a metric of Lorentz signature. This theory is then applied within the frame-work of general relativity. Using the Ruse–Synge classification [8, 9] of non-null electromagnetic fields with source, we develop a geometric proof for the existence of either two space like or one space like and one time like Killing vector fields on the space-time manifold.

2. K-**contingent manifolds.** Consider a differentiable manifold V_{2n+q}, of class C^∞, which carries a tensor field J of type (1,1) whose minimum recurrent relation is:

$$J^3+\phi^2 J=0, \qquad \text{rank } J=2n, \tag{2.1}$$

where ϕ is a non-zero C^∞ function on V_{2n+q}. In the above case, we say that V_{2n+q} is an almost contingent manifold[(1)] of second class. Corresponding to two complementary projection operators π and $\tilde{\pi}$, on the tangent space at each point of V_{2n+q}, defined by

$$\phi^2\pi=J^2+\phi^2 I, \qquad \phi^2\tilde{\pi}=-J^2, \tag{2.2}$$

where I denotes the identity operator, there exist two complementary distributions L and $\tilde{L}$ respectively such that $\dim L=q$ and $\dim \tilde{L}=2n$. Following relations can easily be verified.

$$J\pi=\pi J=0, \qquad J\tilde{\pi}=\tilde{\pi}J=J, \qquad J^2\pi=0, \qquad J^2\tilde{\pi}=-\phi^2\tilde{\pi}. \tag{2.3}$$

Let us assume that L is parallelizable [5] which allows us to take an ordered set of vector fields ξ_a $(a, b=1,\ldots,q)$ spanning L at each point. Thus, their exists an ordered set of 1- forms η^a such that $\pi(X)=\sum_a \eta^a(X)\xi_a$ and $\eta^a(\xi_b)=\delta^a_b$ for

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(1) A special case, where ϕ is a non-zero constant, has been discussed in [2, 3].

an arbitrary vector field X. Using these results, we get

(2.4)
$$J^2X+\phi^2X-\phi^2\sum_a \eta^a(X)\xi_a=0$$
$$J\xi_a=0, \qquad \eta^a JX=0$$

In this way, we say that V_{2n+q} is endowed with an almost contingent structure (J, ξ_a, η^a, ϕ) of the second class[(2)] [3].

DEFINITION 1. $(J, \xi_a, \eta^a, \phi, g, \sigma_a)$ is called an almost contingent metric structure[(3)] on V_{2n+q}, if V_{2n+q} carries a (J, ξ_a, η^a, ϕ)-structure and a non-degenerate metric g such that

(2.5) $g(\xi_a, \xi_a)=\sigma_a$, each σ_a is a non-zero function,

(2.6) $$g(X, \xi_a)=\sigma_a\eta^a(X),$$

(2.7) $$g(JX, Y)+g(X, JY)=0.$$

Replacing Y by JY in (2.7) and using (2.4) and (2.6), we get

(2.8) $$g(JX, JY)=\phi^2 g(X, Y)-\phi^2\sum_a \sigma_a\eta^a(X)\eta^a(Y).$$

Let us define a 2-form F on V_{2n+q} by

(2.9) $$F(X, Y)=g(X, JY).$$

The skew-symmetry of F is immediate from (2.7). We call F the fundamental 2-form of the structure. In the sequel, we assume that F is closed, i.e. $dF=0$, where d is the operator of exterior differentiation. If we define a (1,1) tensor field f on V_{2n+q} such that $f=\phi^{-1}J$, then it is easy to check that the manifold has an underlying f-structure $f^3+f=0$ [6]. It is well-known that such an f-structure is *normal* if $[f, f](X, Y)+\sum_a d\eta^a(X, Y)\xi_a=0$, where $[f, f]$ is the Nijenhuis torsion of f [6]. Thus the normality of the almost contingent structure may be defined in the following way.

DEFINITION 2. (J, ξ_a, η^a, ϕ)-structure is normal if

(2.10) $$[J, J](X, Y)+\phi^2\sum_a d\eta^a(X, Y)\xi_a=0$$

We know that an f-structure is integrable iff $[f, f]=0$ [6]. This allows us to say that an almost contingent structure is integrable iff $[J, J]=0$.

DEFINITION 3. A normal almost contingent metric manifold V_{2n+q}, whose fundamental 2-form is closed, will be called a K-contingent manifold[(4)] and its structure a K-contingent structure.

(2) In the sequel, we shall drop the words "second class".

(3) A special case, where each $\sigma_a=1$, has been discussed in [2, 3].

(4) For related literature on K-manifolds, we refer to [1].

LEMMA 1. *If V_{2n+q} has a normal $(J, \xi_a, \eta^a, \phi, g, \sigma_a)$-structure, then*

$$(2.11) \qquad (i)\ \mathcal{L}_{\xi_a} J = 0, \qquad (ii)\ \mathcal{L}_{\xi_b} \xi_a = 0,$$

where $\mathcal{L}$ denotes the operator of Lie derivation.

Proof. The proof follows the pattern of the proof of [4, Lemma 3].

LEMMA 2. *If V_{2n+q} has a $(J, \xi_a, \eta^a, \phi, g, \sigma_a)$-structure, then*

$$(2.12) \qquad \mathcal{L}_{\xi_a} F = 0,$$

where F is the closed fundamental 2-form of the structure.

Proof. Using the formula $\mathcal{L}_{\xi_a} = doi(\xi_a) + (i\xi_a)od$ where $i(\xi_a)$ is the inner product by ξ_a and d is the operator of exterior derivative, we get $\mathcal{L}_{\xi_a} F = do(i\xi_a F) + (i\xi_a)\, dF = 0$. Indeed, $dF = 0$ and $(i\xi_a F)X = F(\xi_a, X) = \sigma_a \eta_0^a J(X) = 0$.

3. **Pseudo-Riemannian connection on** V_{2n+q}. We consider a product manifold $M_{2m} = V_{2n+q} X R^q$, where R^q is a q-dimensional affine space, $m = n+q$ and V_{2n+q} has $(J, \xi_a, \eta^a, \phi, g, \sigma_a)$-structure. We denote a vector field on M_{2m} by $\tilde{X} = (X, \Psi^a \partial/\partial x^a)$ where X is tangent to V_{2n+q}, (x^a) are coordinates of R^q and Ψ^a are arbitrary C^∞ functions on M_{2m}. Let us define a (1,1) tensor field $\tilde{J}$ and a metric $\tilde{g}$ on M_{2m} by

$$(3.1) \qquad \tilde{J}\left(X, \Psi^a \frac{\partial}{\partial x^a}\right) = \left(JX - \Psi^a \xi_a, \phi^2 \eta^b(X) \frac{\partial}{\partial x^b}\right),$$

$$(3.2) \qquad \tilde{g}\left(\left(X, \Psi^a \frac{\partial}{\partial x^a}\right), \left(Y, \theta^b \frac{\partial}{\partial x^b}\right)\right) = \phi^2 g(X, Y) + \sum_a \theta^a \Psi^a \sigma_a,$$

where θ^a are also arbitrary C^∞ functions on M_{2m}. It is easy to check that $\tilde{J}^2 = -\phi^2 I$ and $\tilde{g}(\tilde{J}\tilde{X}, \tilde{J}\tilde{Y}) = \phi^2 \tilde{g}(\tilde{X}, \tilde{Y})$. Consequently, $(\tilde{J}, \tilde{g})$ defines on M_{2m} a structure whose properties will be similar to an almost complex metric structure. Let $\tilde{\nabla}$ be the symmetric (torsion free) connection of $\tilde{g}$ such that $\tilde{\nabla}\tilde{g} = 0$. We further assume that $\tilde{\nabla}\tilde{J} = 0$. Let ∇ be a pseudo-Riemannian connection of g on V_{2n+q}. A straightforward computation of

$$2\tilde{g}\left(\tilde{\nabla}_{(X,0)}(Y, 0), \left(Z, \frac{\partial}{\partial x^a}\right)\right) \quad \text{and} \quad 2\tilde{g}\left(\tilde{\nabla}_{(X,0)}\left(0, \frac{\partial}{\partial x^a}\right), \left(Z, \frac{\partial}{\partial x^b}\right)\right)$$

provides the following explicit relations between $\tilde{\nabla}$ and ∇.

$$(3.3) \qquad \tilde{\nabla}_{(X,0)}(Y, 0) = \nabla_X Y + (X \ln \phi^2) Y + (Y \ln \phi^2) X - g(X, Y) \operatorname{grad} \phi^2,$$

$$(3.4) \qquad \tilde{\nabla}_{(X,0)}\left(0, \frac{\partial}{\partial x^a}\right) = (X\sigma_a) \frac{\partial}{\partial x^a}, \quad \text{where} \quad g(\operatorname{grad} \phi^2, X) \overset{\text{def}}{=} X\phi^2.$$

Consequently,

$$(\tilde{\nabla}_{(X,0)}\tilde{J})\left(0, \frac{\partial}{\partial x^a}\right) = \tilde{\nabla}_{(X,0)}(-\xi_a, 0) - \tilde{J}\left(0, (X\sigma_a) \frac{\partial}{\partial x^a}\right)$$

implies that

(3.5) $$\nabla_X \xi_a = g(\xi_a, X)\text{grad } \phi^2 + (X\sigma_a)\xi_a - (X \ln \phi^2)\xi_a - (\xi_a \ln \phi^2)X.$$

4. **Applications.** Let us consider the space-time manifold V_4 of general relativity whose metric $g_{ij}(1 \le i, j \le 4)$ is of signature $(+---)$. It is well-known [8, 9] that at each point of V_4 one can introduce a null tetrad $\{l, n, m, \bar{m}\}$ such that l and n are real, m and $\bar{m}$ are conjugate complex vectors and $l_i n^i = -m_i \bar{m}^i = 1$ (all other products zero). Thus, g_{ij} can be expressed as:

(4.1) $$g_{ij} = l_i n_j + n_i l_j - m_i \bar{m}_j - \bar{m}_i m_j.$$

If F_{ij} is the electromagnetic field tensor of V_4, then the Maxwell equations, with a source term W, are expressed as:

(4.2) $$\text{(i) } \nabla_j F^{ij} = W^i, \qquad \text{(ii) } F_{[ij,k]} = 0,$$

where the vector W satisfies the conservation law $\nabla_i W^i = 0$ and ∇ is the symbol of pseudo-Riemannian connection on V_4. Well-known Maxwell scalars [11] are given by:

(4.3) $$\phi_o = 2F_{ij}l^i m^j, \quad \phi_1 = F_{ij}(l^i n^j + \bar{m}^i m^j), \quad \phi_2 = 2F_{ij}\bar{m}^i n^i.$$

In the sequel, we assume that F_{ij} is *non-null*. Therefore, $\phi_0 = \phi_2 = 0$ and $\phi_1 \ne 0$ [8, 9]. Moreover, due to the presence of a non-zero source term W, ϕ_1 is either real or pure imaginary [11, theorem 2]. Let us define a (1,1) tensor field $J_i^j = g^{jk} F_{ik}$ on V_4. Under above-mentioned conditions, J_i^j can be expressed as:

(4.4) $$J_i^j = \phi_1(n_i l^j - l_i n^j), \quad \text{if } \phi_1 \text{ is real,}$$

or

(4.5) $$J_i^j = \phi_1(\bar{m}_i m^j - m_i \bar{m}^j), \quad \text{if } \phi_1 \text{ is imaginary.}$$

It is important to note that, for both cases, J is real. The minimum recurrent relation[(5)] of powers of J, for both cases, is:

(4.6) $$J^3 + \phi^2 J = 0, \qquad \phi^2 = -\phi_1^2,$$

for any vector field X and rank $J = 2$. Comparing (4.6) with (2.1), we conclude that the space-time V_4 is an example of an almost contingent manifold.

Case 1. (ϕ_1 real). Using (2.1)~(2.4), we say that V_4 has $(J, \xi_\alpha, \eta^\alpha, \phi_1)$-structure for $\alpha = 1, 2$. Comparing ξ_1, ξ_2 with $m, \bar{m}$ (locally), we state the following proposition (proof is straightforward).

[(5)] In the sequel, index free notation will be used.

PROPOSITION 1. *The metric* g_{ij} *of the space-time* V_4 *is compatible with the metric of its associated* $(J, \xi_\alpha, \eta^\alpha, \phi_1, g, \sigma_\alpha)$*-structure, satisfying* (2.4)~(2.7), *iff* ξ_1 *and* ξ_2 *are space like such that* $\sigma_1=\sigma_2=\sigma<0$ *and*

$$m=\frac{\xi_1-i\xi_2}{\sqrt{2\sigma}},\qquad \bar{m}=\frac{\xi_1+i\xi_2}{\sqrt{2\sigma}},\qquad i=\sqrt{-1}. \tag{4.7}$$

Now the electromagnetic tensor field F, satisfying (4.2(ii)) can be associated as a closed fundamental form of V_4 and in order to clarify the integrability conditions, the associated almost contingent metric structure on V_4 must be normal. Thus, V_4 qualifies to be a K-contingent manifold.

LEMMA 3. *If the space-time* V_4 *is endowed with a* $(J, \xi_\alpha, \eta^\alpha, \phi_1, g, \sigma)$*-structure, then* ϕ_1 *is constant along the* ξ_α*-curves. Consequently,* grad $\phi_1^2 \perp \xi_\alpha$.

Proof. Substituting (4.7) in the value of ϕ_1 and then using (2.9) and (2.4), we get

$$\phi_1=F(l,n)-\frac{i}{\sigma}F(\xi_1,\xi_2)=F(l,n)-\frac{i}{\sigma}g(\xi_1,F\xi_2)=F(l,n).$$

Since $\{l, n\}$ are not in the plane of $\{\xi_1, \xi_2\}$, we conclude that $\xi_\alpha(\phi_1)=0$ and, therefore, grad $\phi_1^2\perp\xi_\alpha$.

THEOREM 4.1. *Let* F *be a non-null electromagnetic field with source and* g *a metric of signature* $(+---)$ *of the space-time* V_4. *Let* V_4 *be endowed with a* K*-contingent structure* $(J, \xi_\alpha, \eta^\alpha, g, \phi_1, \sigma)$ *satisfying* (2.4)~(2.10), (4.4) *and* (4.7) *for* $\alpha=1,2$. *If* V_4 *is embedded in* $M_6=V_4\times R^2$ *so that the respective connections* $\tilde{\nabla}$ *and* ∇ *of* M_6 *and* V_4 *are related by* (3.3), (3.4), *then*

(a) σ *is constant along the* ξ_α*-curves if* $\sigma\neq\frac{1}{2}$ *at any point of* V_4.

(b) ξ_1 *and* ξ_2 *are space like Killing vector fields.*

Proof. We first prove that, under the conditions of the theorem, (a) holds. Setting $X=\xi_\alpha$ in (3.5), replacing a by β, σ_α by σ and then using lemma 3, we get $\nabla_{\xi_\alpha}\xi_\beta=\xi_\alpha(\sigma)\xi_\beta$.

Using this and other results of section 3 and also lemma 3, we compute the following:

$$\begin{aligned}(\nabla_{(\xi_\alpha,0)}\tilde{g})((\xi_\beta,0),(\xi_\gamma,0)) &= \tilde{\nabla}_{\xi_\alpha}(\phi_1^2 g(\xi_\beta,\xi_\gamma))\\ &\quad -\tilde{g}((\nabla_{\xi_\alpha}\xi_\beta-g(\xi_\alpha,\xi_\gamma)\text{grad }\phi_1^2,0),(\xi_\gamma,0))\\ &\quad -\tilde{g}((\xi_\beta,0),(\nabla_{\xi_\alpha}\xi_\gamma-g(\xi_\alpha,\xi_\gamma)\text{grad }\phi_1^2,0))\\ &= \xi_\alpha(\phi_1^2\sigma)-2\xi_\alpha(\sigma)\sigma=(\xi_\alpha(\sigma))(1-2\sigma)=0.\end{aligned}$$

Hence, $\xi_\alpha(\sigma)=0$ as $\sigma\neq\frac{1}{2}$ at any point of V_4.

Now, using (a) and lemmas 1 and 2, we show that (b) holds.

$$\begin{aligned}(\mathcal{L}_{\xi_\alpha}F)(X, Y) &= \xi_\alpha F(X, Y) - F([\xi_\alpha, X], Y) - F(X, [\xi_\alpha, Y])\\ &= \xi_\alpha g(X, JY) - g([\xi_\alpha, X], JY) - g(X, [\xi_\alpha, JY])\\ &= (\mathcal{L}_{\xi_\alpha}g)(X, JY) = 0,\end{aligned}$$

where we have used lemma 2 and lemma 1(i). Also from (a) above and lemma 1(ii), we get

$$\begin{aligned}(\mathcal{L}_{\xi_\alpha}g)(\xi_\beta, \xi_\gamma) &= \xi_\alpha(\sigma)\delta_{\beta\gamma} - g([\xi_\alpha, \xi_\beta], \xi_\gamma) - g(\xi_\alpha, [\xi_\beta, \xi_\gamma])\\ &= \xi_\alpha(\sigma) = 0\end{aligned}$$

Thus, we conclude that $\mathcal{L}_{\xi_\alpha}g = 0$ since $\xi_{\alpha,s}$ generate L and in the first part of (b) we have shown that $(\mathcal{L}_{\xi_\alpha}g)(X, Z) = 0$ for all X and for all Z in $\tilde{L}$.

Case 2. (ϕ_1 is imaginary). Proceeding exactly as in Case 1, we say that V_4 can be endowed with a K-contingent structure $(J, \xi_{\alpha^*}, \eta^{\alpha^*}, \phi_1, g, \sigma')$ for $\alpha^* = 3, 4$, where $\sigma_3 = -\sigma_4 = \sigma' > 0$.

$$l = \frac{\xi_3 + \xi_4}{\sqrt{2\sigma'}}, \qquad n = \frac{\xi_3 - \xi_4}{\sqrt{2\sigma'}} \tag{4.8}$$

and ϕ_1 is constant along ξ_{α^*}-curves. This leads to the following theorem (the proof follows the pattern of the proof of Theorem 4.1).

Theorem 4.2. *Let F be a non-null electromagnetic field with source and g a metric of signature $(+---)$ of the space-time V_4. Let V_4 be endowed with a K-contingent structure $(J, \xi_{\alpha^*}, \eta^{\alpha^*}, g, \phi_1, \sigma')$ satisfying* (2.4)~(2.10), (4.5) *and* (4.8) *for $\alpha^* = 3, 4$. If V_4 is embedded in $M_6 = V_4 \times R^2$ so that the respective connections $\tilde{\nabla}$ and ∇ of M_6 and V_4 are related by* (3.3), (3.4), *then*

(c) *σ' is constant along the ξ_{α^*}-curves if $\sigma' \neq \frac{1}{2}$ at any point of V_4.*

(d) *ξ_3 and ξ_4 are time like and space like Killing vector fields respectively.*

Remark. It is often assumed, while studying the Einstein–Maxwell equations, that V_4 admits one or more Killing vector fields ξ_a i.e. $\mathcal{L}_{\xi_a}g = 0$. If $\mathcal{L}_{\xi_a}F = 0$ then we say that the space time has symmetry property. In 1973, Woolley [10] showed that $\mathcal{L}_{\xi_a}g = 0 \Rightarrow (\mathcal{L}_{\xi_a}F = k(a)^*F)$, where $^*F$ is a dual of F and $k(a)$ are some scalar quantities. Extending this result, Michalski and Wainwright [7] have recently obtained conditions (i.e. for $k(a)$ to vanish) so that $\mathcal{L}_{\xi_a}g = 0 \Rightarrow \mathcal{L}_{\xi_a}F = 0$. In this paper, as a byproduct of developing a geometric proof for $\mathcal{L}_{\xi_a}g = 0$, we have obtained conditions under which the converse holds (i.e. $\mathcal{L}_{\xi_a}F = 0 \Rightarrow \mathcal{L}_{\xi_a}g = 0$). Thus, under certain geometric conditions, we have shown that the symmetry property of the space time is equivalent to the existence of certain Killing vector fields.

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DEPARTMENT OF MATHEMATICS,
UNIVERSITY OF WINDSOR
WINDSOR, ONTARIO N9B 3P4
CANADA