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ON APPROXIMATE SOLUTIONS OF SOME DIFFERENCE EQUATIONS

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Abstract

In this paper we present a simple (fixed point) method that yields various results concerning approximate solutions of some difference equations. The results are motivated by the notion of Ulam stability.

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1. Introduction

Let (M, ρ) be a metric space, $\mathbb{J}$ be either $\mathbb{N}$ (positive integers) or $\mathbb{Z}$ (integers) and $p \in \mathbb{N}$. We study the approximate solutions in M of the difference equations

$$x_{n+p} = T_n(x_n, x_{n+1}, \dots, x_{n+p-1}), \quad n \in \mathbb{J}, \tag{1.1}$$

$$x_n = T_n(x_{n+1}, x_{n+2}, \dots, x_{n+p}), \quad n \in \mathbb{J}, \tag{1.2}$$

or, in other words, we investigate solutions $(x_n)_{n\in\mathbb{J}} \in M^{\mathbb{J}}$ ($M^{\mathbb{J}}$ denotes, as usual, the family of all sequences $(x_n)_{n\in\mathbb{J}}$ in M) of the inequalities

$$\rho(z_{n+p}; T_n(z_n, z_{n+1}, \dots, z_{n+p-1})) \leqslant \delta_{n+p}, \quad n \in \mathbb{J}, \tag{1.3}$$

$$\rho(z_n; T_n(z_{n+1}, z_{n+2}, \dots, z_{n+p})) \leqslant \delta_n, \quad n \in \mathbb{J}, \tag{1.4}$$

for $T_n : M^p \to M$ and $\delta_n \in \mathbb{R}_+$ (positive reals). Such investigations are connected with the issue of Ulam stability, which has been a very popular subject for many years and covers a broad variety of mathematical objects (for example, differential, difference, functional, integral and operator equations); we refer to [2, 6] for further information and some recent related results. This type of stability is connected with the following natural question: *When is an approximate (in some sense) solution of an equation somehow close to a solution of the equation?*

Such questions appear in natural ways. For instance, if we cannot determine a suitable description of solutions to an equation, then we can try to find functions of simpler forms satisfying the equation only approximately (with a particular error) and show that each such function is close (in some sense) to a solution of the equation. The theory of Ulam stability provides convenient tools for such investigations. The case $p = 1$ and $\mathbb{J} = \mathbb{N}$ has been studied for (1.1) in [3] (cf. [1, 7–9]). In particular, it has been proved in [3, Theorem 2] that if there is $(\alpha_n)_{n\in\mathbb{N}} \in \mathbb{R}_+^{\mathbb{J}}$ with

$$\rho(T_n(x), T_n(y)) \leqslant \alpha_n \rho(x, y), \quad n \in \mathbb{N}, \quad \limsup_{n\to\infty} \frac{\delta_n \alpha_n}{\delta_{n+1}} < 1, \tag{1.5}$$

then there are a sequence $(x_n)_{n\in\mathbb{N}}$ in M and a positive real $\mu \in \mathbb{R}_+$ with $x_{n+1} = T_n(x_n)$ and $d(z_n, x_n) \leqslant \mu\delta_n$ for $n \in \mathbb{N}$. Moreover, [3, Theorem 1] contains the following result.

Theorem 1.1. *Let $(X, +)$ be an abelian group, d be a complete and invariant metric in X, $a_n : X \to X$ be a continuous isomorphism for every $n \in \mathbb{N}$, and let $(\delta_n)_{n\in\mathbb{N}} \in \mathbb{R}_+^{\mathbb{N}}$, $(\alpha_n)_{n\in\mathbb{N}} \in [0, +\infty)^{\mathbb{N}}$ and $(z_n)_{n\in\mathbb{N}}, (b_n)_{n\in\mathbb{N}} \in X^{\mathbb{N}}$. Suppose that*

$$d(z_{n+1}, a_n(z_n) + b_n) \leqslant \delta_{n+1}, \quad n \in \mathbb{N}, \quad \liminf_{n\to\infty} \frac{\delta_n \alpha_n}{\delta_{n+1}} > 1 \tag{1.6}$$

and $d(a_n(x), a_n(y)) \geqslant \alpha_n d(x, y)$ for $x, y \in X$, $n \in \mathbb{N}_0$. Then there is a unique $(x_n)_{n\in\mathbb{N}} \in X^{\mathbb{N}}$ with $x_{n+1} = a_n(x_n) + b_n$ for $n \in \mathbb{N}$ and $\sup_{n\in\mathbb{N}} d(z_n, x_n)/\delta_n < \infty$.

Those results have been obtained in a quite involved way. We present a simple fixed point method that yields various more general results of the same type. In particular, we generalise, complement and extend the results in [3] (see Theorems 2.1 and 2.3 and Remark 2.4). Namely, we consider a more general difference equation (1.1) (for any $p \in \mathbb{N}$ and without assuming any group structure in M), prove analogous results for (1.2) and consider simultaneously also the case where $\mathbb{N}$ is replaced by $\mathbb{Z}$. Since we also study the case $\mathbb{J} = \mathbb{Z}$, we assume conditions somewhat stronger than (1.5) and (1.6), but in this way we obtain more precise results.

A kind of completeness is generally necessary to obtain such stability results (cf. [2]); for example, completeness of d in Theorem 1.1. We show, in particular, that even if such completeness is lacking, then this method allows us to obtain a suitable substitute of such stability; namely, there are sequences satisfying the considered equation with 'arbitrarily small error' (in some sense).

Stability of (1.1) for arbitrary p was considered in [4], but in a Banach space and with $T_n(x_1, \ldots, x_p) \equiv \sum_{i=1}^{p} \xi_i x_i + \beta_0$ and $\delta_n = \delta$ for $n \in \mathbb{N}$ and some $\delta \in \mathbb{R}_+$, where ξ_i are fixed scalars and β_0 a given element of the Banach space. So, our results also generalise and complement those in [4] to some extent (see also [1]).

In what follows, as before, (M, ρ) is a metric space, $\mathbb{J}$ is either $\mathbb{N}$ or $\mathbb{Z}$ and $p \in \mathbb{N}$. Moreover, $T_n : M^p \to M$ and $\delta_n \in \mathbb{R}^+$ for $n \in \mathbb{N}$ are given. To avoid any confusion in what follows, let us explain that, given $(a_n)_{n\in\mathbb{J}} \in M^{\mathbb{J}}$, the symbol $(a_{n-1})_{n\in\mathbb{J}}$ denotes the sequence $(b_n)_{n\in\mathbb{J}}$ in M given by $b_n := a_{n-1}$ for $n \in \mathbb{J}$.

2. Main results

THEOREM 2.1. *Let* $(z_n)_{n\in\mathbb{J}} \in M^{\mathbb{J}}$, $\Theta_n : \mathbb{R}_+^p \to \mathbb{R}_+$ *for* $n \in \mathbb{N}$, $\vartheta \in (0,1)$,

$$\rho(T_n(\overline{y}), T_n(\overline{w})) \leqslant \Theta_n(\rho(y_1, w_1), \dots, \rho(y_p, w_p)) \quad (2.1)$$
$$\text{for } \overline{y} = (y_1, \dots, y_p), \overline{w} = (w_1, \dots, w_p) \in M^p, n \in \mathbb{J},$$

$$\sup_{i\in\mathbb{J}} \frac{\Theta_i(a_i, \dots, a_{i+p-1})}{\delta_{p+i}} < \vartheta \sup_{i\in\mathbb{J}} \frac{a_i}{\delta_i}, \quad (a_n)_{n\in\mathbb{J}} \in \mathbb{R}_+^{\mathbb{J}} \quad (2.2)$$

and suppose that (1.3) holds. Then, for each $\varepsilon \in (0,1)$, *there is* $x = (x_n)_{n\in\mathbb{J}} \in M^{\mathbb{J}}$ *with*

$$\rho(x_{n+p}, T_n(x_n, \dots, x_{n+p-1})) \leqslant \varepsilon\delta_{n+p}, \quad \rho(z_n, x_n) \leqslant \frac{\delta_n}{1-\vartheta}, \ n \in \mathbb{J}. \quad (2.3)$$

Next, if the metric ρ *is complete, then there is a solution* $u = (u_n)_{n\in\mathbb{J}} \in M^{\mathbb{J}}$ *of (1.1) such that* $\sigma := \sup_{n\in\mathbb{J}} \rho(z_n, u_n)/\delta_n \leqslant 1/(1-\vartheta)$; *moreover, if* $\mathbb{J} = \mathbb{Z}$, *then there exists only one solution* $u = (u_n)_{n\in\mathbb{J}} \in M^{\mathbb{J}}$ *of (1.1) with* $\sigma < \infty$.

PROOF. Write $\rho_\infty(u, w) := \sup_{n\in\mathbb{J}} \rho(u_n, w_n)/\delta_n$ for $u = (u_n)_{n\in\mathbb{J}}$, $w = (w_n)_{n\in\mathbb{J}} \in M^{\mathbb{J}}$ and $\mathcal{M} := \{y = (y_n)_{n\in\mathbb{J}} \in M^{\mathbb{J}} : \rho_\infty(y, z) < \infty\}$ and set

$$\mathcal{T}(y) := \begin{cases} (T_{n-p}(y_{n-p}, \dots, y_{n-1}))_{n\in\mathbb{J}} & \text{if } \mathbb{J} = \mathbb{Z}; \\ (z_1, \dots, z_p, T_1(y_1, \dots, y_p), T_2(y_2, \dots, y_{p+1}), \dots) & \text{if } \mathbb{J} = \mathbb{N}; \end{cases}$$

for every $y = (y_n)_{n\in\mathbb{J}} \in \mathcal{M}$. Then $(\mathcal{M}, \rho_\infty)$ is a metric space, $\rho_\infty(z, \mathcal{T}(z)) \leqslant 1$ and

$$\begin{aligned} \rho_\infty(\mathcal{T}(y), \mathcal{T}(w)) &= \sup_{i\in\mathbb{J}} \frac{\rho(T_i(y_i, \dots, y_{i+p-1}), T_i(w_i, \dots, w_{i+p-1}))}{\delta_{p+i}} \\ &\leqslant \sup_{i\in\mathbb{J}} \frac{\Theta_i(\rho(y_i, w_i), \dots, \rho(y_{i+p-1}, w_{i+p-1}))}{\delta_{p+i}} \end{aligned} \quad (2.4)$$

for every $y = (y_n)_{n\in\mathbb{J}}, w = (w_n)_{n\in\mathbb{J}} \in \mathcal{M}$. Next, note that condition (2.2) implies $\sup_{i\in\mathbb{J}} \Theta_i(\rho(y_i, w_i), \dots, \rho(y_{i+p-1}, w_{i+p-1}))/\delta_{p+i} \leqslant \vartheta\rho_\infty(y, w)$ for all sequences $y = (y_n)_{n\in\mathbb{J}}$, $w = (w_n)_{n\in\mathbb{J}} \in \mathcal{M}$, whence, from (2.4), we deduce that $\rho_\infty(\mathcal{T}(y), \mathcal{T}(w)) \leqslant \vartheta\rho_\infty(y, w)$ for every $y = (y_n)_{n\in\mathbb{J}}, w = (w_n)_{n\in\mathbb{J}} \in \mathcal{M}$. So, $\mathcal{T} : \mathcal{M} \to \mathcal{M}$ is a contraction with the constant ϑ. Hence, $\rho_\infty(\mathcal{T}^n(z), \mathcal{T}^{n+1}(z)) \leqslant \vartheta^n\rho_\infty(z, \mathcal{T}(z))$ for $n \in \mathbb{N}$. Fix $\varepsilon \in (0,1)$. Clearly, $\vartheta^{n_0}\rho_\infty(z, \mathcal{T}(z)) \leqslant \varepsilon$ for some $n_0 \in \mathbb{N}$. Take $x := \mathcal{T}^{n_0}(z)$. Then it is easily seen that $\rho_\infty(z, x) \leqslant \sum_{i=0}^{n_0-1} \rho_\infty(\mathcal{T}^i(z), \mathcal{T}^{i+1}(z)) \leqslant \sum_{i=0}^{n_0-1} \vartheta^i\rho_\infty(z, \mathcal{T}(z)) \leqslant 1/(1-\vartheta)$ and $\rho_\infty(x, \mathcal{T}(x)) \leqslant \varepsilon$, which means that (2.3) holds.

Assume that the metric space (M, ρ) is complete. Then so is $(\mathcal{M}, \rho_\infty)$ and by the Banach contraction principle there is a unique fixed point $u \in \mathcal{M}$ of $\mathcal{T}$ and $\rho_\infty(u, z) \leqslant \rho_\infty(z, \mathcal{T}(z))/(1-\vartheta) \leqslant 1/(1-\vartheta)$. Since $u = \mathcal{T}(u)$, u is a solution to (1.1).

Finally, let $\mathbb{J} = \mathbb{Z}$ and $v = (v_n)_{n\in\mathbb{J}} \in \mathcal{M}$ be a solution to (1.1). Then $v = \mathcal{T}(v)$, whence v is a fixed point of $\mathcal{T}$ and consequently $u = v$. □

It follows from the proof that $u = (u_n)_{n\in\mathbb{J}}$ in Theorem 2.1 can be chosen, for $\mathbb{J} = \mathbb{N}$, with $u_i = z_i$ for $i = 1, \dots, p$. Clearly, there is only one such solution, but generally the uniqueness property, as for $\mathbb{J} = \mathbb{Z}$, does not hold for $\mathbb{J} = \mathbb{N}$.

Remark 2.2. Let $\tau_{n,i} \in [0, \infty)$ for $n, i \in \mathbb{Z}$ and $\Theta_n(\overline{a}) := \max_{i=1,\dots,p} \tau_{n,i} a_i$ for $n \in \mathbb{J}$, $\overline{a} = (a_1, \dots, a_p) \in \mathbb{R}_+^p$. Then, for example, (2.2) follows with $\vartheta := \vartheta_0$ from the condition

$$\vartheta_0 := \sup_{i \in \mathbb{J}} \max_{k=0,\dots,p-1} \frac{\delta_{i+k}\tau_{i,i+k}}{\delta_{p+i}} < 1, \tag{2.5}$$

because $\tau_{i,i+k}/\delta_{p+i} \leqslant \vartheta_0/\delta_{i+k}$ for $i \in \mathbb{J}, k = 0, \dots, p-1$ and consequently

$$\sup_{i \in \mathbb{J}} \max_{k=0,\dots,p-1} \frac{\tau_{i,i+k} a_{i+k}}{\delta_{p+i}} \leqslant \sup_{i \in \mathbb{J}} \max_{k=0,\dots,p-1} \vartheta_0 \frac{a_{i+k}}{\delta_{i+k}} = \vartheta_0 \sup_{i \in \mathbb{J}} \frac{a_i}{\delta_i}, \quad (a_n)_{n \in \mathbb{J}} \in \mathbb{R}_+^{\mathbb{J}}.$$

If $\Theta_n(\overline{a}) = \sum_{i=1}^p \tau_{n,i} a_i$ for $n \in \mathbb{J}$ and $\overline{a} = (a_1, \dots, a_p) \in \mathbb{R}_+^p$, then analogously it is easy to show that (2.2) holds for instance with $\theta = p\theta_0$ when $\theta_0 < 1/p$ is defined by (2.5). Clearly, for $p = 1$, condition (2.5) corresponds to the second inequality in (1.5), because (with $\tau_j := \tau_{j,j}$ for $j \in \mathbb{J}$) it has the form

$$\vartheta := \sup_{j \in \mathbb{J}} \frac{\delta_j \tau_j}{\delta_{j+1}} < 1. \tag{2.6}$$

Theorem 2.3. *Let $(z_n)_{n \in \mathbb{J}} \in M^{\mathbb{J}}$, $\Theta_n : \mathbb{R}_+^p \to \mathbb{R}_+$ for $n \in \mathbb{N}$, (1.4) and (2.1) hold and*

$$\sup_{i \in \mathbb{J}} \frac{\Theta_i(a_{i+1}, \dots, a_{i+p})}{\delta_i} \leqslant \vartheta \sup_{i \in \mathbb{J}} \frac{a_i}{\delta_i}, \quad (a_n)_{n \in \mathbb{J}} \in \mathbb{R}_+^{\mathbb{J}}, \tag{2.7}$$

with some $\vartheta \in (0, 1)$. Then, for each $\varepsilon \in (0, 1)$, there is $x = (x_n)_{n \in \mathbb{J}} \in M^{\mathbb{J}}$ with

$$\rho(x_n, T_n(x_{n+1}, \dots, x_{n+p})) \leqslant \varepsilon \delta_n, \quad \rho(z_n, x_n) \leqslant \frac{\delta_n}{1 - \vartheta}, \ n \in \mathbb{J}.$$

Next, if ρ is complete, then there is a unique solution $u = (u_n)_{n \in \mathbb{J}} \in M^{\mathbb{J}}$ of (1.2) with $\sigma := \sup_{n \in \mathbb{J}} \rho(z_n, u_n)/\delta_n < \infty$; moreover, $\rho(z_n, u_n) \leqslant \delta_n/(1 - \vartheta)$ for $n \in \mathbb{J}$.

Proof. Write $\rho_\infty(u, w) := \sup_{n \in \mathbb{J}} \rho(u_n, w_n)/\delta_n$ for $u = (u_n)_{n \in \mathbb{J}}$, $w = (w_n)_{n \in \mathbb{J}} \in M^{\mathbb{J}}$ and $\mathcal{M} := \{y = (y_n)_{n \in \mathbb{J}} \in M^{\mathbb{J}} : \rho_\infty(y, z) < \infty\}$ and set $\mathcal{T}(y) := (T_n(y_{n+1}, \dots, y_{n+p}))_{n \in \mathbb{J}}$ for $y = (y_n)_{n \in \mathbb{J}} \in \mathcal{M}$. Then $(\mathcal{M}, \rho_\infty)$ is a metric space, $\rho_\infty(z, \mathcal{T}(z)) \leqslant 1$ and

$$\begin{aligned}\rho_\infty(\mathcal{T}(y), \mathcal{T}(w)) &= \sup_{i \in \mathbb{J}} \frac{\rho(T_i(y_{i+1}, \dots, y_{i+p}), T_i(w_{i+1}, \dots, w_{i+p}))}{\delta_i} \\ &\leqslant \sup_{j \in \mathbb{J}} \frac{\Theta_i(\rho(y_{i+1}, w_{i+1}), \dots, \rho(y_{i+p}, w_{i+p}))}{\delta_i} \leqslant \vartheta \rho_\infty(y, w)\end{aligned}$$

for every $y = (y_n)_{n \in \mathbb{J}}, w = (w_n)_{n \in \mathbb{J}} \in \mathcal{M}$. So, $\mathcal{T} : \mathcal{M} \to \mathcal{M}$ is a contraction with the constant ϑ. The rest of the proof is analogous as for Theorem 2.1. □

Some consequences of Theorems 2.1 and 2.3 (in the case $p = 1$) are described in the subsequent remark.

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REMARK 2.4. Now consider a situation corresponding to Theorem 1.1. Namely, let $S_n : M \to M$ be surjective for $n \in \mathbb{J}$, $(\gamma_n)_{n\in\mathbb{J}}, (\eta_n)_{n\in\mathbb{J}} \in \mathbb{R}_+^{\mathbb{J}}$, $(w_n)_{n\in\mathbb{J}} \in M^{\mathbb{J}}$ and

$$\rho(w_{n+1}, S_n(w_n)) \leqslant \eta_{n+1}, \quad n \in \mathbb{J}, \tag{2.8}$$

$$\rho(S_n(x), S_n(y)) \geqslant \gamma_n \rho(x, y), \quad x, y \in M, n \in \mathbb{J}. \tag{2.9}$$

Then S_n must be bijective for each $n \in \mathbb{J}$ (by (2.9)). Write

$$T_n := S_{n+1}^{-1}, \quad z_n := S_n(w_n), \quad \delta_n := \eta_{n+1} \quad \text{and} \quad \alpha_n := \gamma_{n+1}^{-1} \quad \text{for } n \in \mathbb{J}.$$

Then, from (2.8) and (2.9),

$$\rho(z_n, T_n(z_{n+1})) = \rho(S_n(w_n), T_n(S_{n+1}(w_{n+1}))) = \rho(S_n(w_n), w_{n+1}) \leqslant \eta_{n+1} = \delta_n$$

and $\rho(T_n(z), T_n(w)) \leqslant \alpha_n \rho(z, w)$ for $n \in \mathbb{J}$, $z, w \in M$, whence (1.4) and (2.1) hold for $p = 1$ and $\Theta_n(a) \equiv \alpha_n a$. If we assume additionally that $\sup_{n\in\mathbb{J}} \eta_{n+1}/(\eta_n\gamma_n) < 1$, then $\sup_{n\in\mathbb{J}}(\delta_{n+1}\alpha_n)/\delta_n < 1$, which implies (2.7) for $p = 1$. So, we have reduced that situation to a particular case of Theorem 2.3. Thus, we obtain a generalisation of Theorem 1.1. An analogous result can be derived from Theorem 2.1 when (2.8) is replaced by the condition $\rho(w_n, S_n(w_{n+1})) \leqslant \eta_n$ for $n \in \mathbb{J}$ and (2.9) holds; thus, we obtain a generalisation of [3, Theorem 2].

3. Final remarks

The next example shows that $\vartheta = 1$ cannot be admitted in (2.6) (that is, in (2.5) with $p = 1$) in the general situation. Namely, let X be a normed space with $\dim X \geqslant 2$ and $T_n : X \to X$ be a linear isometry for $n \in \mathbb{N}$. Then each T_n is a Lipschitz mapping with a constant $\vartheta = 1$. Assume that there is $w \in X$, which is a fixed point of each T_k; take w with $\|w\| = 1$.

Fix $\delta > 0$, $\gamma > 0$ and $m_0 \in \mathbb{N}$ with $\gamma\pi < m_0\delta$. Define $(z_n)_{n\in\mathbb{N}} \in X^{\mathbb{N}}$ by

$$z_n := 2\gamma \sin\Big(\frac{\pi}{2} \cdot \frac{n}{m_0}\Big) w, \quad n \in \mathbb{N}.$$

Then $\sup_{n\in\mathbb{N}} \|z_{n+1} - T_n(z_n)\| \leqslant \delta$, because

$$\begin{aligned} \|z_{n+1} - T_n(z_n)\| &= \Big\|2\gamma \sin\Big(\frac{\pi}{2} \cdot \frac{n+1}{m_0}\Big) \cdot w - 2\gamma \sin\Big(\frac{\pi}{2} \cdot \frac{n}{m_0}\Big) \cdot T_n\big(w\big)\Big\| \\ &= 2\gamma \Big|\sin\Big(\frac{\pi}{2} \cdot \frac{n+1}{m_0}\Big) - \sin\Big(\frac{\pi}{2} \cdot \frac{n}{m_0}\Big)\Big| \cdot \|w\| \\ &\leqslant 2\gamma \Big|\frac{\pi}{2} \cdot \frac{n+1}{m_0} - \frac{\pi}{2} \cdot \frac{n}{m_0}\Big| \leqslant 2\gamma \cdot \frac{\pi}{2} \cdot \frac{1}{m_0} < \delta, \quad n \in \mathbb{N}. \end{aligned}$$

Let $(x_n)_{n\in\mathbb{N}_0} \in X^{\mathbb{N}}$ and $x_{n+1} = T_n(x_n)$ for $n \in \mathbb{N}$. Then, for each $k \in \mathbb{N}$,

$$\|z_{2km_0} - x_{2km_0}\| \geqslant \Big|\, 2\gamma \Big| \sin\Big(\frac{\pi}{2} \cdot \frac{2km_0}{m_0}\Big)\Big| - \|x_0\| \Big| = \|x_0\|,$$

$$\|z_{(4k+1)m_0} - x_{(4k+1)m_0}\| \geqslant \Big|\, 2\gamma \Big| \sin\Big(\frac{\pi}{2} \cdot \frac{(4k+1)m_0}{m_0}\Big)\Big| - \|x_0\| \Big| = |\, 2\gamma - \|x_0\| \,|,$$

because $\|x_k\| = \|x_0\|$. This means that $\sup_{n\in\mathbb{N}} \|z_n - x_n\| \geqslant \gamma$.

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Similar nonstability results, as described above, have been obtained in [5] also in the case where $p = 1$ and $\lim_{n\to\infty} \delta_n\alpha_n/\delta_{n+1} = 1$. On the other hand, in several similar cases with $p > 1$, the stability results can be derived from [4] (cf. [2]).

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