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Uniqueness of Preduals in Spaces of Operators

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Abstract. We show that if E is a separable reflexive space, and L is a weak-star closed linear subspace of $L(E)$ such that $L \cap K(E)$ is weak-star dense in L, then L has a unique isometric predual. The proof relies on basic topological arguments.

1 Introduction

A Banach space X is called *unique predual* if a Banach space Y is isometric to X as soon as its dual Y^* is isometric to X^*. This property is frequently satisfied, although classical spaces such as $C(K)$ spaces (with K an infinite compact space) fail it. We refer to [3] for a survey of this topic as it was in 1989.

It has been shown by Ruan [8] that an operator algebra that has a weak-star dense subalgebra of compact operators has a unique operator space predual and a unique Banach space predual as well. Along these lines, Effros, Ozawa, and Ruan [2] have shown that W*TRO's (commonly known as corners of von Neumann algebras) have unique preduals. More recently, Pfitzner [6] showed that separable L-embedded spaces are unique preduals.

A simpler proof of Ruan's result was provided, among other results, in [1, Theorem 4.2]. This proof relies on Hilbertian methods, as expected, since it adresses weak-star closed subalgebras of the space $L(H)$ of bounded operators on the separable Hilbert space H.

The purpose of this short note is to show that elementary topological methods, relying ultimately on Baire's lemma, provide the extension of Ruan's result to all separable reflexive spaces (even when no approximation property is available) and to all weak-star closed subspaces, instead of subalgebras (see Theorem 2.1). Such techniques have been used before (see [3]). However, our main Lemma 2.2 provides a significant short cut to all available arguments.

2 Results

We recall that if E is a reflexive space, the space $L(E)$ of bounded linear operators from E to itself equipped with the operator norm is a dual space, and its isometric predual $E\widehat{\otimes}E^*$ is unique [4]. In this note, we always equip spaces of operators with the operator norm, and their subspaces with the restriction of this norm. The weak-star

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topology on $L(E)$, with E reflexive, refers to its predual $E\widehat{\otimes}E^*$. Let us mention that this weak-star topology coincides with the weak operator topology on bounded subsets of $L(E)$. Elementary duality arguments show that a weak-star closed subspace L of $L(E)$ is isometric to the dual of the quotient of $E\widehat{\otimes}E^*$ by its pre-orthogonal space $L_\perp$.

Our main result is the following generalization of [1, Theorem 4.2]. We note that what is actually shown is that the predual of L is *strongly* unique (see [1]), from which it follows in particular that every invertible isometry of L is weak-star to weak-star continuous.

Theorem 2.1 *Let E be a separable reflexive space, and let L be a weak-star closed linear subspace of $L(E)$ such that $L \cap K(E)$ is weak-star dense in L. Then L has a unique isometric predual.*

Proof The predual L_* of L is a quotient space of $E\widehat{\otimes}E^*$. Since E and E^* are separable, this predual is separable as well.

We claim that L_* is isometric to the dual of the space $L \cap K(E)$. Indeed, since E is reflexive, every compact operator attains its operator norm on the unit ball of E, which implies that it attains its norm as well as a linear form on $E\widehat{\otimes}E^*$. Now, by [5], if X is a separable Banach space and Z is a separating norm-closed subspace of X^* consisting of norm-attaining linear functionals, then Z is an isometric predual of X.

Let us show for convenience how to deduce the result of [5] from Simons' inequality [9]. We call B the restriction to Z of the closed unit ball B_X of X. If $\overline{B} \neq B_{Z^*}$, there exists $F \in B_{Z^{**}}$ and $z_0^* \in B_{Z^*}$ such that $F(z_0^*) > \sup F(B)$. Let $\sup F(B) < \alpha < F(z_0^*)$. Let $C = \{z \in B_Z : z_0^*(z) > \alpha\}$. Clearly $F \in \overline{C}^{w^*}$. Since B is separable, there exists a sequence $\{z_n\} \subseteq C$ such that $\lim_{n\to\infty} z^*(z_n) = F(z^*)$ for all $z^* \in B$. Our assumption states that B is a James boundary of B_{Z^*}, that is, every $z \in Z$ attains its norm at some point of B. We can therefore apply Simons' inequality [9] to the sequence $\{z_n\}$, from which it follows that there is $z \in co(\{z_n\}) \subseteq C$ such that $\alpha > \sup z(B) = \|z\|$. But this contradicts $z_0^*(z) > \alpha$. Hence, $\overline{B} = B_{Z^*}$, and since Z separates X, this shows that Z is an isometric predual of X.

We can apply this result to $X = L_*$ and $Z = K(E) \cap L$, since our assumption implies that Z separates X. Therefore $L_* = (K(E) \cap L)^*$ is isometric to a dual space.

Every separable dual has the Radon–Nikodym property, and this would suffice to conclude that L_* is unique predual (see [3]). However, we now provide a short and self-contained argument. Our main lemma follows.

Lemma 2.2 *Let X be a Banach space. If X is not the unique isometric predual of its dual, there exist $u \in X^{***}$, $\alpha > 0$ and a nonempty weak-star open slice U of $B_{X^{**}}$ such that the sets $\{z \in U : u(z) > \alpha\}$ and $\{z \in U : u(z) < -\alpha\}$ are both weak-star dense in U.*

Proof Let Y be an isometric predual of X^* distinct of X, where both X and Y are considered as subspaces of X^{**}. Then $X \not\subset Y$, since if it is not, there is $y \in Y^* = X^*$ with $y \neq 0$ and $y = 0$ on X, a contradiction. Hence, $Y^\perp \not\subset X^\perp$.

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Let $\Pi_X \colon X^{***} \to X^*$ be the canonical projection defined by the restriction to X. Pick $u \in Y^\perp \backslash X^\perp$. Then $\|\Pi_X(u)\| > 0$, and we pick $\alpha \in \left]0, \|\Pi_X(u)\|\right[$. The set

$$U = \{z \in B_{X^{**}} : z(\Pi_X(u)) > \alpha\}$$

is a nonempty weak-star open slice of $B_{X^{**}}$. Since u coincide with $\Pi_X(u)$ on X, one has $u > \alpha$ on the weak-dense subset $(U \cap X)$ of U. On the other hand, $(U \cap Y)$ is weak-star dense in U, since Y is a predual, and we have $u = 0$ on $(U \cap Y)$, since $u \in Y^\perp$. Using linearity of u, it is easy to deduce from the denseness of $\{z \in U : u(z) > \alpha\}$ and $\{z \in U : u(z) = 0\}$ in the convex open set U that the set $\{z \in U : u(z) < -\alpha\}$ is also dense in U. This proves Lemma 2.2. ■

To complete the proof of Theorem 2.1, we recall that L_* is a separable dual space. It suffices therefore to check that if $X = Z^*$ is a separable dual, it does not satisfy the conclusions of Lemma 2.2. This can be done by a classical Baire category argument. Let $U = \{z \in B_{X^{**}} : z(y) > \beta\}$ (for some $y \in X^*$ and $\beta \geq 0$) be a nonempty weak-star open slice of $B_{X^{**}}$, $u \in X^{***}$ and $\alpha > 0$. We can and do assume that $\|u\| = 1$.

The set $V = U \cap B_X$ is a nonempty weakly open slice of B_X. We equip the dual space $X = Z^*$ with its weak-star topology. Let K be the weak-star closure of V in B_X. The set K is weak-star compact. Since X is separable, K is contained in the union of countably many closed balls B_n of radius $\alpha/2$. By the Baire category lemma, there is n_0 such that $(K \cap B_{n_0})$ has nonempty weak-star interior in K. Hence there is a weak-star open subset W of B_X such that $(K \cap W)$ is nonempty and has (norm) diameter $d \leq \alpha$. The set $(V \cap W)$ is nonempty, since K is the weak-star closure of V, it is weakly open in B_X and has the same diameter d. Now there is a weak-star open subset A of $B_{X^{**}}$ of diameter d with $A \subset U$ such that $V \cap W = A \cap B_X$. In particular, U contains a nonempty weak-star open subset A of (norm) diameter $d \leq \alpha$. Since $\|u\| = 1$, at least one of the sets $\{u > \alpha\}$ and $\{u < -\alpha\}$ does not intersect A. Hence the conclusions of Lemma 2.2 fail, and therefore the space X is unique isometric predual. ■

Remarks (1) We assume that X is not unique isometric predual, and use the notation of Lemma 2.2. Using weak-star approximation of the oscillating linear form u by elements of X^*, it is easy to construct by induction a sequence in X^* that is equivalent to the canonical basis of l_1 (see [7]). This provides a simple and self-contained proof of the known result (see [3]): if X is not unique isometric predual, then X^* contains l_1 isomorphically. Let us outline the argument: we may and do assume that $\|u\| = 1$. We first pick t_0 and t_1 in U with $u(t_0) > \alpha$ and $u(t_1) < -\alpha$. Since u belongs to the weak-star closure of B_{X^*}, there is $x_1 \in B_{X^*}$ such that $x_1(t_0) > \alpha$ and $x_1(t_1) < -\alpha$. By weak-star continuity of x_1, there exist weak-star open neighbourhoods U_i of $t_i (i \in \{0,1\})$ contained in U such that $x_1 > \alpha$ on U_0 and $x_1 < -\alpha$ on U_1. Our assumption on u provides t_{i0} and t_{i1} in $U_i (i \in \{0,1\})$ such that $u(t_{i0}) > \alpha$ and $u(t_{i1}) < -\alpha$. As before, we use an approximation of u on the set $\{t_{00}, t_{01}, t_{10}, t_{11}\}$ by $x_2 \in B_{X^*}$ to find weak-star open subsets U_{00} and U_{01} of U_0, and similarly U_{10} and U_{11} of U_1, such that $x_2 > \alpha$ on U_{00} and U_{10}, and $x_2 < -\alpha$ on U_{01} and U_{11}. Note that

$$\alpha\big(|\lambda_1| + |\lambda_2|\big) \leq \|\lambda_1 x_1 + \lambda_2 x_2\| \leq |\lambda_1| + |\lambda_2|$$

for all scalars (λ_1, λ_2). Continuing this Cantor-like construction in the obvious manner provides a sequence $(x_n) \subset B_{X^*}$ that is α-equivalent to the unit vector basis of l_1.

(2) The proof of Lemma 2.2 actually yields to a quantitative version of the lack of uniqueness of preduals. Indeed, assume that there is a 1-norming subspace Y of X^{**} such that

$$\sup\{\operatorname{dist}(x, Y) : x \in B_X\} = \lambda > 0.$$

Then the argument of Lemma 2.2 provides for all $0 < \alpha < \lambda$ a functional $u \in B_{X^{***}}$ and a weak-star open slice of $B_{X^{**}}$ on which u has the behavior described in the conclusion of Lemma 2.2. Hence if, conversely, every element of the third dual enjoys some kind of quantitative weak-star regularity, then λ cannot be too big, which means that every 1-norming subspace of X^{**} "nearly contains" X, and in particular that the "angle" between two isometric preduals is bound to be small. Let us give precise statements along these lines: let X be a Banach space and $\lambda > 0$. We assume that every open slice of the unit ball B_X contains a nonempty weakly open set of norm diameter $d \leq \lambda$. If $Y \subset X^{**}$ is a 1-norming subspace, then $\sup\{\operatorname{dist}(x, Y) : x \in B_X\} \leq \lambda/2$. If, moreover, Y is an isometric predual of X^*, then the Hausdorff distance between B_X and B_Y is at most $\lambda/2$.

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