BULL. AUSTRAL. MATH. SOC. 42A04, 43A15, 43A40.
VOL. 11 (1974), 315-316.

Bernstein's inequality for locally compact groups

Walter R. Bloom

An extended form of a famous inequality of S.N. Bernstein states that for a trigonometric polynomial T_n of degree at most n,

$$\|T_n'\|_p \le n\|T_n\|_p ,$$

where T_n' denotes the derivative of T_n and $1 \le p \le \infty$. There is also a corresponding statement for functions on the real line which are extendible into the complex plane to functions of exponential type.

This thesis is concerned with versions of Bernstein's inequality for pth-power integrable functions on locally compact Hausdorff abelian and compact Hausdorff groups. The inequalities obtained are of the form

$$(1) \qquad \|{}_af-f\|_p \le \omega(a)\|f\|_p ,$$

where $f \in L^p(G)$ has spectrum $\Sigma(f)$ contained in a given relatively compact set Υ, ${}_af$ denotes the left a-translate of f, and $\lim_{a\to 0} \omega(a) = 0$. Chapter 1 deals with this problem for various choices of G and Υ. In Chapter 2 it is shown that if S is a translation-invariant subspace of $L^p(G)$ with every $f \in S$ satisfying (1), then $\cup\{\Sigma(f) : f \in S\}$ is relatively compact. This result can be considered as a converse to that dealt with in the first chapter. A more classical version of the inequality is examined in Chapter 3, namely

$$(2) \qquad \|D_\rho^p f\|_p \le \kappa\|f\|_p ;$$

Received 26 February 1974. Thesis submitted to the Australian National University, February 1974. Degree approved, July 1974. Supervisor: Professor R.E. Edwards.

here ρ is a continuous homomorphism of $\mathbb{R}$ into G and

$$D_\rho^p f = \lim_{r\to 0} r^{-1}\left({}_{-\rho(r)}f-f\right)$$

(where the limit is taken in the L^p-sense) is the L^p-derivative of f along ρ. Initially it is proved that if $f \in L^p(G)$ and $\Sigma(f) \subset \mathrm{T}$ (where T is relatively compact) then f is L^p-differentiable along ρ for any ρ. Inequality (2) then follows readily with $\kappa = \kappa(\rho, \mathrm{T})$. There is also a converse result.