FUNCTION SPACE TOPOLOGIES FOR CONNECTIVITY AND SEMI-CONNECTIVITY FUNCTIONS

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1. Introduction. Let X and Y be topological spaces. If Y is a uniform space then one of the most useful function space topologies for the class of continuous functions on X to Y (denoted by C) is the topology of uniform convergence. The reason for this usefulness is the fact that in this topology C is closed in Y^X (see Theorem 9, page 227 in [2]) and consequently, if Y is complete then C is complete. In this paper I shall show that a similar result is true for the function space of connectivity functions in the topology of uniform convergence and for the function space of semi-connectivity functions in the graph topology when $X \times Y$ is completely normal. In a subsequent paper the problem of connected functions will be discussed.

2. Connectivity Functions.

2.1. DEFINITION. The graph of a function $f: X \to Y$ is

$$G(f) = \{(x, f(x)) \mid x \in X\} \subset X \times Y.$$

For a subset $K \subset X$,

$$G(f \mid K) = \{(x, f(x)) \mid x \in K\}.$$

2.2. DEFINITION. A function $f: X \to Y$ is called a connectivity function if and only if for each connected subset $K \subset X$, $G(f \mid K)$ is connected. I shall denote by C^{-1} the class of all connectivity functions on X to Y. (This notation is due to Professor D.E. Sanderson.)

If Y is a uniform space with uniformity ν, then a basis for the uniformity of uniform convergence for Y^X is the collection $\{W(V) \mid V \in \nu\}$ where

$$W(V) = \{(f, g) \in Y^X \times Y^X \mid (f(x), g(x)) \in V \text{ for all } x \in X\}.$$

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For details see page 226 [2].

2.3. THEOREM. If Y is a uniform space with uniformity ν, then C^{-1} is closed in Y^X in the topology of uniform convergence.

Proof. Suppose f is a limit point of C^{-1} but $f \notin C^{-1}$. Then there exists a connected subset $K \subset X$ such that $G(f|K) = A_1 \cup A_2$ where $A_1 \neq \emptyset$, $A_2 \neq \emptyset$, $A_1 \cap \bar{A}_2 = \emptyset = \bar{A}_1 \cap A_2$. Let $D_1 = \{x \mid (x, f(x)) \in A_1\} \subset K$ and $D_2 = K - D_1$. Then D_1 and D_2 are not empty. Let W be an arbitrary element of ν and let V be a symmetric member of ν such that $V \circ V \circ V \subset W$. Since f is a limit point of C^{-1}, there exists a $g \in C^{-1}$ such that $g(x) \in V[f(x)]$ for all $x \in X$. Since V is symmetric, $f(x) \in V[g(x)]$ for all $x \in X$. Let $F_1 = G(g|D_1)$ and $F_2 = G(g|D_2)$. Since $g \in C^{-1}$, either $\bar{F}_1 \cap F_2 \neq \emptyset$ or $\bar{F}_2 \cap F_1 \neq \emptyset$. Suppose $\bar{F}_2 \cap F_1 \neq \emptyset$. Then there exists a set $\{p, p_n \; n \in T\}$, where T is a directed set such that $p \in D_1$, $p_n \in D_2$ for all $n \in T$, $\lim_{n \in T} p_n = p$ and $\lim_{n \in T} g(p_n) = g(p)$. So there exists an $m \in T$ such that for all $n \geq m$, $g(p_n) \in V[g(p)]$. So for $n \geq m$, $f(p_n) \in V[g(p_n)] \subset V \circ V[g(p)] \subset V \circ V \circ V[f(p)] \subset W[f(p)]$. So $\lim_{n \in T} f(p_n) = f(p)$. Thus $(p, f(p)) \in \bar{A}_2 \cap A_1$ which is a contradiction. So $f \in C^{-1}$.

Remark: In contrast to the above result it is well known that the limit of a uniformly convergent sequence of connected functions is not necessarily a connected function (see [4]).

2.4. COROLLARY. If Y is a complete uniform space then C^{-1} is complete in the topology of uniform convergence.

3. Semi-Connectivity Functions.

3.1. DEFINITION. A function $f: X \rightarrow Y$ is a semi-connectivity function if and only if for each component $K \subset X$, $G(f|K)$ is connected. Let Q denote the class of all semi-

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connectivity functions on X to Y.

3.2. DEFINITION. For each open set U in $X \times Y$, let $F_U = \{f \in Y^X \mid G(f) \subset U\}$. The collection $\{F_U \mid U$ open in $X \times Y\}$ is a basis for "Graph Topology" Γ. For properties of Γ see [3].

3.3. DEFINITION. A topological space is <u>completely normal</u> if and only if whenever M and K are two separated sets, there are disjoint open sets, one containing M and the other containing K (see page 42 [1]).

3.4. THEOREM. If $X \times Y$ is completely normal then Q is closed in Y^X in the graph topology Γ.

<u>Proof.</u> Suppose f is a limit point of Q but $f \notin Q$. Then there is a component $K \subset X$ such that $G(f|K)$ is not connected in $X \times Y$. Then $G(f|K) \subset A_1 \cup A_2$ where A_1 and A_2 are disjoint non-empty open subsets of $X \times Y$, (see 3.3). Now K, being a component of X, is a closed subset of X and so X - K is open. Also $G(f) \subset A_1 \cup A_2 \cup (X - K) \times Y$. Since f is a limit point of Q, there exists a $g \in Q$ such that $G(g) \subset A_1 \cup A_2 \cup (X - K) \times Y$. Clearly $G(g|K) \subset A_1 \cup A_2$, a contradiction. So $f \in Q$ and Q is closed in Y^X.

If X and Y are linearly orderable then $C^{-1} = Q$ and we have the following corollary.

3.4. COROLLARY. If X and Y are linearly orderable spaces such that $X \times Y$ is completely normal then C^{-1} is closed in the graph topology Γ.

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