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INITIAL AND RELATIVE LIMITING BEHAVIOUR OF TEMPERATURES ON A STRIP

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Abstract

Let u be a solution of the heat equation which can be written as the difference of two non-negative solutions, and let v be a non-negative solution. A study is made of the behaviour of $u(x,t)/v(x,t)$ as $t \to 0+$. The methods are based on the Gauss-Weierstrass integral representation of solutions on $R^n \times]0, a[$ and results on the relative differentiation of measures, which are employed in a novel way to obtain several domination, non-negativity, uniqueness and representation theorems.

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Let W denote the Gauss-Weierstrass kernel, defined, for all $(x,t) \in \mathbf{R}^n \times]0, \infty[$, by $W(x,t) = (4\pi t)^{-n/2}\exp(-\|x\|^2/4t)$, and let μ be a locally finite, signed Borel measure on $\mathbf{R}^n$. Then u, given by the convolution

$$u(x,t) = \int_{\mathbf{R}^n} W(x-y,t)\,d\mu(y), \tag{1}$$

is called the Gauss-Weierstrass integral of μ, provided that the integral exists. If the integral exists and is finite at a point (x_0, t_0), then u is a temperature, that is, a solution of the heat equation, on $\mathbf{R}^n \times]0, t_0[$. Conversely, if v is a temperature on a strip $\mathbf{R}^n \times]0, c[$, or on a half-space $\mathbf{R}^n \times]0, \infty[$, and v can be written as the difference of two non-negative temperatures, then v has a representation as the Gauss-Weierstrass integral of some signed measure ν. For details and references, see [14]. We write $u = W\mu$ if u and μ are related by (1), and always assume that such integrals are finite on some strip or half-space $\mathbf{R}^n \times]0, c[$, where $0 < c \leqslant \infty$.

In [5, Theorem 5.2], Doob proved that, if $u = W\mu$ and $v = W\nu$, then

$$\lim_{t \to 0} \frac{u(x,t)}{v(x,t)}$$

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exists a.e. $[|\nu|]$ on $\mathbf{R}^n$, and is then equal to the Radon-Nikodym derivative of μ with respect to ν. Similar results have been proved for harmonic functions, and in more general situations with different limits (see [3] for references), but further study of the behaviour of u/v, and application of the results about u/v, have apparently been neglected. In [3], Brelot mentioned one simple application of an analogous result for harmonic functions. In [19], new results and applications were given for Gauss-Weierstrass integrals, and the present paper contains further theorems, but generally of a different nature. We use the following basic result [19, Theorem 1].

Let $u = W\mu$ and $v = W\nu$, where ν is non-negative, and let $x \in \mathbf{R}^n$. If $\nu(B(x, r)) > 0$ for all closed balls $B(x, r)$ in $\mathbf{R}^n$ with positive radius r, then

$$\begin{aligned} \liminf_{r\to 0} \frac{\mu(B(x,r))}{\nu(B(x,r))} \leqslant \liminf_{t\to 0} \frac{u(x,t)}{v(x,t)} \leqslant \limsup_{t\to 0} \frac{u(x,t)}{v(x,t)} \\ \leqslant \limsup_{r\to 0} \frac{\mu(B(x,r))}{\nu(B(x,r))}. \end{aligned} \tag{2}$$

The first theorem of the present paper is concerned with the upper and lower limits of the quotient $\mu(B(x, r))/\nu(B(x, r))$ as $r \to 0$ and, in view of the above result, it has immediate application to the relative behaviour of temperatures. We are thus able to prove some new domination, non-negativity, uniqueness and representation theorems for temperatures. These results include a multi-variable version of a theorem of Gehring [6, Theorem 10], analogues of results for harmonic functions on a disc in the plane due to Bruckner, Lohwater and Ryan [4, Theorems 2 and 3], Hall [8, Theorem 4], and Lohwater [12], and a much more general version of a recent improvement for temperatures [15, Theorem 5] of a result of Krzyżański [11, Theorem 5].

In addition, we are able to compare the strengths of singularities of Gauss-Weierstrass integrals of singular and absolutely continuous measures. For example, it is well-known that, if $\mu(\{x\}) = \lambda \neq 0$ and $u = W\mu$, then $u(x, t) \sim (4\pi t)^{-n/2}\lambda$ as $t \to 0$, whereas if $\mu(\{x\}) = 0$ then $u(x, t) = o(t^{-n/2})$ as $t \to 0$. We shall show that, if ν is non-negative and absolutely continuous, $v = W\nu$, μ is non-negative and concentrated on the set where $v(x, t)$ is unbounded as $t \to 0$, and $u = W\mu$, then $v(x, t) = o(u(x, t))$ as $t \to 0$ for μ-almost every point x in $\mathbf{R}^n$.

We also give two theorems which show that we can sometimes deduce from the behaviour of u/v that μ or ν must be concentrated on some particular set.

Given $x = (x_1, \ldots, x_n) \in \mathbf{R}^n$ and $r > 0$, we put $\|x\| = (x_1^2 + \cdots + x_n^2)^{1/2}$ and $B(x, r) = \{y \in \mathbf{R}^n: \|x - y\| \leqslant r\}$. Every measure in this paper is a locally finite, signed Borel measure on $\mathbf{R}^n$. The letter m is used to denote Lebesgue measure on

$\mathbf{R}^n$. We shall call a measure ν *strictly positive* if $\nu(B(x, r)) > 0$ for all $x \in \mathbf{R}^n$ and $r > 0$. The positive, negative and total variations of a measure μ are denoted by μ^+, μ^- and $|\mu|$.

The following temperature occurs in several of our theorems. Given a number $\kappa \geqslant 0$, we let V_κ denote the Gauss-Weierstrass integral of the function $x \mapsto \exp(\kappa \|x\|^2)$, that is,

$$V_\kappa(x, t) = (1 - 4\kappa t)^{-n/2} \exp\{\kappa \|x\|^2 / (1 - 4\kappa t)\}$$

for all (x, t) in $\mathbf{R}^n \times]0, (4\kappa)^{-1}[$ if $\kappa > 0$, in $\mathbf{R}^n \times]0, \infty[$ if $\kappa = 0$. Of course, $V_0(x, t) = 1$.

Finally, if u is a temperature and v is a non-negative temperature such that $u \leqslant v$ on $\mathbf{R}^n \times]0, c[$, then v is called a *positive thermic majorant* of u on $\mathbf{R}^n \times]0, c[$. For details and references, see [18].

2. Relative differentiation of measures

In this section we present several results on the behaviour of $\mu(B(x, r))/\nu(B(x, r))$ as $r \to 0$, which we require later. The lemmas are all due to Besicovitch [1, 2], but one new theorem is also given.

LEMMA 1. *If μ and ν are non-negative measures on $\mathbf{R}^n$, then*

$$\lim_{r \to 0} \frac{\mu(B(x, r))}{\nu(B(x, r))}$$

exists and is finite for ν-almost all x in $\mathbf{R}^n$.

This result is proved, in the case $n = 2$, in [1, Theorem 2]. As with all the results in [1, 2], the proof carries over to the general case.

LEMMA 2. *Let μ and ν be non-negative measures on $\mathbf{R}^n$, and let Y be a Borel set such that $\mu(Y) = 0$. Then*

$$\lim_{r \to 0} \frac{\mu(B(x, r))}{\nu(B(x, r))} = 0$$

for ν-almost all x in Y.

See [1, Theorem 3].

LEMMA 3. *If μ is a non-negative measure, and if a family F of balls covers a Borel set E in such a way that, for each $x \in E$, there is a ball $B(x, r)$ in F with arbitrarily small r, then F contains a subfamily of disjoint balls whose union H has the property that $\mu(E \setminus H) = 0$.*

This is a special case of [2, Theorem 3].

We now come to a new theorem, which generalizes and strengthens a result which was stated, without proof and for the case $\nu = m$ only, by Rosenbloom in [13].

THEOREM 1. *Let μ and ν be measures on $\mathbf{R}^n$, ν being strictly positive. If*

$$\limsup_{r\to 0} \frac{\mu(B(x,r))}{\nu(B(x,r))} > -\infty \tag{3}$$

for all $x \in \mathbf{R}^n$, and

$$\limsup_{r\to 0} \frac{\mu(B(x,r))}{\nu(B(x,r))} \geqslant 0 \tag{4}$$

for ν-almost all $x \in \mathbf{R}^n$, then μ is non-negative.

PROOF. For each non-negative integer k, let P_k denote the set of all x for which

$$\limsup_{r\to 0} \frac{\mu(B(x,r))}{\nu(B(x,r))} \geqslant -k.$$

Then (3) implies that

$$\bigcup_{k=0}^{\infty} P_k = \mathbf{R}^n \tag{5}$$

and (4) shows that

$$\nu(\mathbf{R}^n \setminus P_0) = 0. \tag{6}$$

Let $\varepsilon > 0$. To each x in P_0 there corresponds a positive null sequence $\{r_i\}$ such that

$$\mu(B(x, r_i)) \geqslant -\varepsilon\nu(B(x, r_i)) \tag{7}$$

for all i. For each $k > 0$ we have $\nu(P_k \setminus P_{k-1}) = 0$, by (6), so that there is an open set $V_k \supseteq P_k \setminus P_{k-1}$ such that

$$\nu(V_k) < 2^{-k}\varepsilon.$$

To each $x \in P_k \setminus P_{k-1}$ there corresponds a positive null sequence $\{r_i\}$ such that

$$B(x, r_i) \subseteq V_k \quad \text{and} \quad \mu(B(x, r_i)) \geqslant -(k+\varepsilon)\nu(B(x, r_i)) \tag{8}$$

for all i.

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Let E be any bounded open set in $\mathbf{R}^n$. Consider the family F of all balls $B(x, r_i) \subseteq E$ such that either $x \in E \cap P_0$ and (7) holds, or $x \in E \cap (P_k \setminus P_{k-1})$ and (8) holds. In view of (5) the family F covers E, and for each $x \in E$ there is a ball $B(x, r)$ in F with arbitrarily small r. Therefore, by Lemma 3, there is a sequence $\{C_j\}$ of disjoint members of F such that

$$|\mu|\left(E \setminus \left(\bigcup_j C_j\right)\right) = 0.$$

For each $k \geqslant 0$, let $\{\Gamma_{kj}\}$ denote the (possibly finite or empty) subsequence consisting of those C_j whose centres lie in $P_k \setminus P_{k-1}$ if $k > 0$, in P_0 if $k = 0$. Then

$$\begin{aligned}\mu(E) = \mu\left(\bigcup_j C_j\right) &= \sum_{k=0}^{\infty}\left(\sum_j \mu(\Gamma_{kj})\right)\\ &\geqslant -\sum_{k=0}^{\infty}(k+\varepsilon)\left(\sum_j \nu(\Gamma_{kj})\right)\\ &\geqslant -\varepsilon\nu(E) - \sum_{k=1}^{\infty}(k+\varepsilon)\nu(V_k)\\ &\geqslant -\varepsilon\left(\nu(E) + \sum_{k=1}^{\infty} 2^{-k}(k+\varepsilon)\right).\end{aligned}$$

Since ε is arbitrary, it follows that $\mu(E) \geqslant 0$.

Therefore $\mu^+(E) \geqslant \mu^-(E)$ for all bounded open sets E. Using the regularity properties of μ^+ and μ^-, we deduce that $\mu^+(S) \geqslant \mu^-(S)$ for every μ-measurable set S. This proves the theorem.

Corollary. *Let μ and ν be measures on $\mathbf{R}^n$ such that ν is strictly positive. If*

$$\lim_{r \to 0} \frac{\mu(B(x, r))}{\nu(B(x, r))} \tag{9}$$

is finite whenever it exists, and is zero ν-almost everywhere, then $\mu = 0$.

Proof. By Lemma 1, the limit in (9) exists and is finite ν-almost everywhere. Therefore the hypotheses of Theorem 1 are satisfied with μ itself, and also with μ replaced by $-\mu$ throughout. Hence both μ and $-\mu$ are non-negative, and the corollary is proved.

3. Some applications of Besicovitch's results

The results presented here are all consequences of the above lemmas and the fundamental inequalities in (2).

THEOREM 2. *Let μ and ν be non-negative measures on $\mathbf{R}^n$, and let Y be a Borel set such that $\mu(Y) = 0$. If $u = W\mu$ and $v = W\nu$ on $\mathbf{R}^n \times]0, c[$, then*

$$u(x, t) = o(v(x, t)) \quad \text{as } t \to 0 \tag{10}$$

for ν-almost all $x \in Y$. In particular, if μ and ν are mutually singular, then (10) *holds for ν-almost every $x \in \mathbf{R}^n$.*

PROOF. By (2) and Lemma 2, we have

$$\lim_{t \to 0} \frac{u(x, t)}{v(x, t)} = \lim_{r \to 0} \frac{\mu(B(x, r))}{\nu(B(x, r))} = 0$$

for ν-almost all $x \in Y$. This proves the first part, and the second now follows by taking Y to be any Borel set such that $\mu(Y) = 0$ and $\nu(\mathbf{R}^n \setminus Y) = 0$.

We now use Theorem 2 to show that the initial singularities of $W\mu$, where μ is absolutely continuous with respect to m, are milder than those of a corresponding $W\nu$ with ν singular with respect to m, at least ν-a.e.

THEOREM 3. *Let $u = W\mu$, where μ is non-negative and absolutely continuous with respect to m, and put*

$$Z = \left\{x \colon \limsup_{t \to 0} u(x, t) = \infty\right\}.$$

If ν is a non-negative measure concentrated on Z, and $v = W\nu$, then

$$u(x, t) = o(v(x, t)) \quad \text{as } t \to 0$$

for ν-almost every $x \in \mathbf{R}^n$.

PROOF. Since $u(x, t)$ tends to a finite limit as $t \to 0$ for m-almost every x in $\mathbf{R}^n$, we see that $m(Z) = 0$ and hence that $\mu(Z) = 0$. Since ν is concentrated on Z, we deduce that μ and ν are mutually singular, and the result now follows from Theorem 2.

The next theorem is analogous to certain results of Brelot [3] on various limits of quotients of positive harmonic or superharmonic functions.

THEOREM 4. *Let $u = W\mu$ and $v = W\nu$, where ν is non-negative on $\mathbf{R}^n$. The limit*

$$\lim_{t\to 0} \frac{v(x,t)}{u(x,t)} \tag{11}$$

exists and is non-zero ν-a.e. In particular

$$\lim_{t\to 0} v(x,t)$$

exists and is strictly positive ν-a.e.

PROOF. Let $N = \{x\colon \nu(B(x,r)) = 0 \text{ for some } r > 0\}$. Then N is an open set and $\nu(N) = 0$. Since the inequalities in (2) are applicable to any x in $\mathbf{R}^n \setminus N$, it follows from (2) and Lemma 1 that

$$\lim_{t\to 0} \frac{u(x,t)}{v(x,t)} = \lim_{r\to 0} \frac{\mu(B(x,r))}{\nu(B(x,r))}$$

exists and is finite for ν-almost all x in $\mathbf{R}^n$. Hence the limit in (11) exists and is non-zero ν-a.e. in $\mathbf{R}^n$. The second part of the theorem follows from the first by taking $u = 1$.

COROLLARY. *Let $u = W\mu$ and $v = W\nu$, where ν is non-negative on $\mathbf{R}^n$. The set of x for which*

$$\liminf_{t\to 0} \frac{v(x,t)}{u(x,t)} = 0$$

has ν-measure zero. In particular,

$$\nu\left(\left\{x\colon \liminf_{t\to 0} v(x,t) = 0\right\}\right) = 0.$$

Our final result in this section is a generalization of [15, Corollary, page 278], which corresponds to the case where $\nu = m$ and $S = \varnothing$. In view of Theorem 2, it is essentially a sharpened form of the above Corollary for the case where μ and ν are mutually singular.

THEOREM 5. *Let $u = W\mu$ and $v = W\nu$, where μ is non-negative and ν is strictly positive, and put*

$$E = \{x\colon u(x,t)/v(x,t) \text{ tends to a finite limit as } t \to 0\}$$

and

$$S = \{x\colon u(x,t)/v(x,t) \text{ tends to } \infty \text{ as } t \to 0\}.$$

If $u(x,t) = o(v(x,t))$ as $t \to 0$, for ν-almost all x in E, then μ is concentrated on S.

PROOF. If $x \notin S$, then either

(i) $u(x, t)/v(x, t)$ tends to zero as $t \to 0$, or

(ii) $u(x, t)/v(x, t)$ tends to a finite, non-zero limit as $t \to 0$, or

(iii) $u(x, t)/v(x, t)$ tends neither to a limit nor to infinity.

Let A, B and C denote the sets where (i), (ii) and (iii) hold respectively. By the Corollary to Theorem 4, $\mu(A) = 0$. By hypothesis, $\nu(B) = 0$. By [5, Theorem 5.2], $\nu(\mathbf{R}^n \setminus E) = 0$ and hence $\nu(C) = 0$. Therefore $\nu(B \cup C) = 0$, and hence Lemma 2 implies that

$$\lim_{r \to 0} \frac{\nu(B(x, r))}{\mu(B(x, r))} = 0$$

for μ-almost all $x \in B \cup C$. The inequalities in (2) now show that

$$\lim_{t \to 0} \frac{u(x, t)}{v(x, t)} = \lim_{r \to 0} \frac{\mu(B(x, r))}{\nu(B(x, r))} = \infty$$

μ-a.e. on $B \cup C$. The definitions of B and C now imply that $\mu(B \cup C) = 0$, and hence $\mu(\mathbf{R}^n \setminus S) = 0$, as required.

4. Domination, non-negativity and uniqueness theorems for temperatures

We now present some immediate consequences of Theorem 1.

THEOREM 6. *Let $u = W\mu$ and $v = W\nu$ on $R^n \times]0, c[$, where ν is strictly positive and $0 < c \leqslant \infty$. If*

$$\limsup_{t \to 0} \frac{u(x, t)}{v(x, t)} > -\infty \tag{12}$$

for all $x \in \mathbf{R}^n$, and

$$\limsup_{t \to 0} \frac{u(x, t)}{v(x, t)} \geqslant A \tag{13}$$

for ν-almost every $x \in \mathbf{R}^n$, then $u \geqslant Av$ on $R^n \times]0, c[$.

PROOF. We may suppose that $A = 0$, since we could replace u by $u - Av$ throughout. By (2),

$$\limsup_{t \to 0} \frac{u(x, t)}{v(x, t)} \leqslant \limsup_{r \to 0} \frac{\mu(B(x, r))}{\nu(B(x, r))}$$

for all $x \in \mathbf{R}^n$, so that (12) and (13) imply that the hypotheses of Theorem 1 are satisfied. Hence $\mu \geqslant 0$, and therefore $u \geqslant 0$.

As a consequence of Theorem 6, we can extend a result of Gehring [6, Theorem 10] to the case of an arbitrary n, and thus sharpen [18, Theorem 3] and extend [17, Theorem 5] to $\mathbf{R}^n$ for all n.

THEOREM 7. *Let $u = W\mu$ on $\mathbf{R}^n \times]0, c[$. If*

$$\lim_{t\to 0} u(x,t) > -\infty$$

for all x at which the limit exists, and

$$\lim_{t\to 0} u(x,t) \geqslant A$$

for m-almost every $x \in \mathbf{R}^n$, then $u \geqslant A$ on $\mathbf{R}^n \times]0, c[$.

PROOF. Take $\nu = m$ in Theorem 6.

Theorem 6 also gives rise to the following uniqueness result.

THEOREM 8. *Let $u = W\mu$ and $v = W\nu$ on $\mathbf{R}^n \times]0, c[$, where ν is strictly positive. If*

$$\liminf_{t\to 0} \frac{|u(x,t)|}{v(x,t)} < \infty$$

for all $x \in \mathbf{R}^n$, and

$$\liminf_{t\to 0} \frac{u(x,t)}{v(x,t)} = 0 \tag{14}$$

for ν-almost every $x \in \mathbf{R}^n$, then $u = 0$ throughout $\mathbf{R}^n \times]0, c[$.

PROOF. By applying Theorem 6 to $-u$ and v, we deduce that $u \leqslant 0$. Hence (14) implies that

$$\lim_{t\to 0} \frac{u(x,t)}{v(x,t)} = 0$$

for ν-almost every $x \in \mathbf{R}^n$. Another application of Theorem 6 now shows that $u \geqslant 0$, and the result is proved.

If we put $\nu = m$ in Theorem 8, we obtain a strengthened form of a result which was announced, without proof, in [13], and incorrectly demonstrated in [7]. See [15, page 278] for further details. The result is also analogous to one due to Lohwater [12, Corollary] for harmonic functions on a disc in the plane.

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THEOREM 9. *Let* $u = W\mu$ *on* $\mathbf{R}^n \times]0, c[$. *If*

$$\liminf_{t \to 0} |u(x,t)| < \infty \tag{15}$$

for all $x \in \mathbf{R}^n$, *and*

$$\liminf_{t \to 0} u(x,t) = 0$$

for m-almost every $x \in \mathbf{R}^n$, *then* $u = 0$ *throughout* $\mathbf{R}^n \times]0, c[$.

Another interesting consequence of Theorem 6 is motivated by analogy with recent work of Kuran [10]. It implies that condition (15) in Theorem 9 can be weakened in a particular way, without affecting the conclusion of the theorem (cf. the proof of Theorem 8).

We first recall [16, Theorem 11]. If $Z \subseteq \mathbf{R}^n$ and $m(Z) = 0$, then there exists a positive temperature v on $\mathbf{R}^n \times]0, \infty[$ such that $v(x,t) \to \infty$ as $(x,t) \to (y,0)$ for all $y \in Z$. We can obviously suppose that $v \geqslant 1$, since $v + 1$ has similar properties.

THEOREM 10. *Let* $u = W\mu$ *on* $R^n \times]0, c[$, *and suppose that*

$$\liminf_{t \to 0} u(x,t) \leqslant A \tag{16}$$

for all $x \in \mathbf{R}^n \setminus Z$, *where* $m(Z) = 0$. *Let* v *be a temperature such that* $v \geqslant 1$ *on* $\mathbf{R}^n \times]0, c[$ *and* $v(x,t) \to \infty$ *as* $t \to 0$ *for all* $x \in Z$. *If*

$$\liminf_{t \to 0} \frac{u(x,t)}{v(x,t)} \leqslant 0 \tag{17}$$

for all $x \in Z$, *then* $u \leqslant A$ *on* $\mathbf{R}^n \times]0, c[$.

PROOF. There is a non-negative measure ν on $\mathbf{R}^n$ such that $v = W\nu$ on $\mathbf{R}^n \times]0, c[$. Since $v \geqslant 1$, we have $\nu \geqslant m$ and hence ν is strictly positive. We can suppose that $A = 0$, since we could replace u by $u - A$ throughout. It follows from (16) and (17) that

$$\liminf_{t \to 0} \frac{u(x,t)}{v(x,t)} \leqslant 0$$

for all $x \in \mathbf{R}^n$, so that Theorem 6 gives the desired result.

In the next section we shall use Theorem 10 to prove some new representation theorems for temperatures.

5. Representation theorems

The theorems of this section feature a countable set C. We allow this set to be finite or empty, but retain the notation for a countably infinite set.

The first result is analogous to one due to Bruckner, Lohwater and Ryan [4, Theorem 3] for harmonic functions on the unit disc in $\mathbf{R}^2$, at least when $A = 0$. Another special case, in which $C = \varnothing$, parallels [4, Theorem 2].

THEOREM 11. *Let* $u = W\mu$ *on* $\mathbf{R}^n \times]0, c[$, *and let* $C = \{x_j\}_{j \geqslant 1}$ *be a sequence of points in* $\mathbf{R}^n$. *If there is a real constant* A, *and a non-negative constant* κ, *such that*

(18) $$\liminf_{t \to 0} u(x, t) \leqslant A \exp(\kappa \| x \|^2)$$

for m*-almost all* $x \in \mathbf{R}^n$, *and*

(19) $$\liminf_{t \to 0} u(x, t) < \infty$$

for all $x \in \mathbf{R}^n \setminus C$, *then* u *can be written in the form*

$$u(x, t) = AV_\kappa(x, t) - h(x, t) + \sum_{j=1}^{\infty} \mu^+(\{x_j\}) W(x - x_j, t)$$

on $\mathbf{R}^n \times]0, \min\{c, (4\kappa)^{-1}\}[$ *if* $\kappa > 0$, *on* $\mathbf{R}^n \times]0, c[$ *if* $\kappa = 0$, *where* h *is a non-negative temperature and* V_κ *is as defined in Section* 1.

PROOF. If we put $u^* = u - AV_\kappa$, then (18) becomes

$$\liminf_{t \to 0} u^*(x, t) \leqslant 0$$

for m-almost all $x \in \mathbf{R}^n$, and (19) holds with u^* in place of u. If we prove the result for u^*, then the result for u will follow immediately. We may therefore suppose that $A = 0$ and $\kappa = 0$.

Let $\varepsilon > 0$. For each j, put $\lambda_j = \mu^+(\{x_j\}) + \varepsilon 2^{-j}$, and let

$$w(x, t) = u(x, t) - \sum_{j=1}^{\infty} \lambda_j W(x - x_j, t)$$

for all $(x, t) \in \mathbf{R}^n \times]0, c[$. Since

(20) $$\begin{aligned} \sum_{j=1}^{\infty} \lambda_j W(x - x_j, t) &\leqslant \sum_{j=1}^{\infty} \mu^+(\{x_j\}) W(x - x_j, t) + \varepsilon (4\pi t)^{-n/2} \sum_{j=1}^{\infty} 2^{-j} \\ &\leqslant \int_{\mathbf{R}^n} W(x - y, t) \, d\mu^+(y) + \varepsilon (4\pi t)^{-n/2} < \infty \end{aligned}$$

for all $(x, t) \in \mathbf{R}^n \times]0, \infty[$, it follows from [14, Lemma 1] that w is a temperature.

Let Z denote the set of points where (18) fails to hold, so that $m(Z) = 0$. Let v be a temperature such that $v \geqslant 1$ on $\mathbf{R}^n \times]0, c[$ and $v(x, t) \to \infty$ as $t \to 0$ for all $x \in Z$. Since $w \leqslant u$, for all $x \in \mathbf{R}^n \setminus Z$ we have

$$\liminf_{t \to 0} w(x, t) \leqslant 0. \tag{21}$$

Next, for each j let δ_j denote the Dirac δ-measure concentrated at x_j. Then $w = W\eta$, where $\eta = \mu - \sum_{j=1}^{\infty} \lambda_j \delta_j$, and for each j we have $\eta(\{x_j\}) = \mu(\{x_j\}) - \lambda_j < 0$. Therefore $w(x_j, t) \sim \eta(\{x_j\})(4\pi t)^{-n/2}$ as $t \to 0$, in view of [19, Examples 1 and 2]. Thus we see that

$$\lim_{t \to 0} w(x, t) = -\infty \tag{22}$$

for all $x \in C$. Finally, if $x \in Z \setminus C$ we have

$$\liminf_{t \to 0} w(x, t) \leqslant \liminf_{t \to 0} u(x, t) < \infty$$

by (19), so that

$$\liminf_{t \to 0} \frac{w(x, t)}{v(x, t)} \leqslant 0. \tag{23}$$

It follows from (21), (22), (23) and Theorem 10 that $w \leqslant 0$ on $\mathbf{R}^n \times]0, c[$.

Therefore, in view of (20),

$$u(x, t) \leqslant \sum_{j=1}^{\infty} \mu^+ (\{x_j\}) W(x - x_j, t) + \varepsilon(4\pi t)^{-n/2}$$

for all $(x, t) \in \mathbf{R}^n \times]0, c[$ and all $\varepsilon > 0$. Making $\varepsilon \to 0$, we obtain

$$u(x, t) \leqslant \sum_{j=1}^{\infty} \mu^+ (\{x_j\}) W(x - x_j, t).$$

The sum on the right is therefore a positive thermic majorant of u on $\mathbf{R}^n \times]0, c[$, and hence majorizes the least such majorant. Hence, by [18, Theorem 2],

$$\int_{\mathbf{R}^n} W(x - y, t) d\mu^+ (y) \leqslant \sum_{j=1}^{\infty} \mu^+ (\{x_j\}) W(x - x_j, t),$$

so that

$$u(x, t) - \sum_{j=1}^{\infty} \mu^+ (\{x_j\}) W(x - x_j, t) \leqslant -\int_{\mathbf{R}^n} W(x - y, t) d\mu^-(y) \leqslant 0,$$

and the result is proved.

Theorem 11 gives rise to another representation theorem, as follows.

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THEOREM 12. *Let* $u = W\mu$ *on* $\mathbf{R}^n \times]0, c[$, *and let* $C = \{x_j\}_{j \geq 1}$ *be a sequence in* $\mathbf{R}^n$. *If there exist non-negative constants* A *and* κ *such that*

$$\liminf_{t \to 0} |u(x, t)| \leq A \exp(\kappa \|x\|^2)$$

m-a.e. on $\mathbf{R}^n$, *and*

$$\liminf_{t \to 0} |u(x, t)| < \infty$$

for all $x \in \mathbf{R}^n \setminus C$, *then* u *can be written in the form*

$$u(x, t) = h(x, t) + \sum_{j=1}^{\infty} \mu(\{x_j\}) W(x - x_j, t) \tag{24}$$

on $\mathbf{R}^n \times]0, \min\{c, (4\kappa)^{-1}\}[$ *if* $\kappa > 0$, *on* $\mathbf{R}^n \times]0, c[$ *if* $\kappa = 0$, *where* h *is a temperature which satisfies*

$$|h| \leq A V_\kappa. \tag{25}$$

PROOF. Applying Theorem 11 to u we obtain

$$u(x, t) \leq A V_\kappa(x, t) + \sum_{j=1}^{\infty} \mu^+(\{x_j\}) W(x - x_j, t),$$

so that u has a positive thermic majorant given by the expression on the right. This expression therefore majorizes the least positive thermic majorant of u, so that by [18, Theorem 2],

$$\int_{\mathbf{R}^n} W(x - y, t) d\mu^+(y) \leq A V_\kappa(x, t) + \sum_{j=1}^{\infty} \mu^+(\{x_j\}) W(x - x_j, t).$$

Therefore

$$0 \leq \int_{\mathbf{R}^n} W(x - y, t) d\mu^+(y) - \sum_{j=1}^{\infty} \mu^+(\{x_j\}) W(x - x_j, t) \leq A V_\kappa(x, t),$$

and a similar argument applied to $-u$ gives

$$0 \leq \int_{\mathbf{R}^n} W(x - y, t) d\mu^-(y) - \sum_{j=1}^{\infty} \mu^-(\{x_j\}) W(x - x_j, t) \leq A V_\kappa(x, t).$$

It follows that

$$-A V_\kappa(x, t) \leq u(x, t) - \sum_{j=1}^{\infty} \mu(\{x_j\}) W(x - x_j, t) \leq A V_\kappa(x, t),$$

which shows that $|h| \leq A V_\kappa$, as required.

There is a known representation theorem for a temperature h which satisfies (25). For $n = 1$, it is proved in [9, page 206]. Combining this with Theorem 12, we obtain a more explicit representation of u.

COROLLARY 1. *If u satisfies the hypotheses of Theorem* 12, *then there exists a function f on $\mathbf{R}^n$ such that*

$$|f(x)| \leqslant A\exp(\kappa\|x\|^2)$$

for all x, and

$$u(x,t) = \int_{\mathbf{R}^n} W(x-y,t)f(y)\,dy + \sum_{j=1}^{\infty} \mu(\{x_j\})W(x-x_j,t)$$

on $\mathbf{R}^n \times]0, \min\{c, (4\kappa)^{-1}\}[$ *if* $\kappa > 0$, *on* $\mathbf{R}^n \times]0, c[$ *if* $\kappa = 0$.

PROOF. By Theorem 12, u has the representation (24). Define f on $\mathbf{R}^n$ by

$$f(x) = \limsup_{t \to 0} h(x,t).$$

Since $|h| \leqslant AV_\kappa$, it is obvious that h has a positive thermic majorant v such that

$$\limsup_{t\to 0} v(x,t) < \infty$$

for all x, and that $f(x) > -\infty$ for all x. The result now follows from [18, Theorem 1].

The special case of Theorem 12 in which $A = 0$ and $\kappa = 0$ gives us the following analogue of a theorem on harmonic functions on a disc in $\mathbf{R}^2$ due to Lohwater [12]. This corollary also contains, as the special case where μ is non-negative and C is a singleton, a recent improvement [15, Theorem 5] of a theorem of Krzyżański [11, Theorem 5].

COROLLARY 2. *Let* $u = W\mu$ *on* $\mathbf{R}^n \times]0, c[$, *let*

$$E = \left\{x \in \mathbf{R}^n\colon \lim_{t\to 0} u(x,t) \text{ exists}\right\},$$

and let $C = \{x_j\}_{j \geqslant 1}$ *be a sequence of points in E. If* $\lim_{t\to 0} u(x,t) = 0$ *m-a.e. on E, and* $\lim_{t \to 0} u(x,t)$ *is finite on* $E \setminus C$, *then*

$$u(x,t) = \sum_{j=1}^{\infty} \mu(\{x_j\})W(x-x_j,t)$$

for all $(x,t) \in \mathbf{R}^n \times]0, c[$.

PROOF. By [5, Theorem 5.2], $m(\mathbf{R}^n \setminus E) = 0$. It now follows that the hypotheses of Theorem 12 are satisfied, with $A = \kappa = 0$, so that u can be written in the form (24). Since $|h| \leqslant AV_\kappa = 0$, the corollary is proved.

Another consequence of Theorem 12 is roughly analogous to a result of Hall [8, Theorem 4] on holomorphic functions on a disc. His hypotheses allow approach to the boundary along arbitrary Jordan arcs, not just along radii, but require a uniform rate of growth where the values of the modulus are unbounded.

THEOREM 13. *Let* $u = W\mu$ *on* $R^n \times]0, c[$, *and suppose that there are non-negative constants* A *and* κ *such that*

(i) $\lim_{t\to 0} |u(x,t)| \leq A\exp(\kappa\|x\|^2)$ *m-a.e. on* $\mathbf{R}^n$,

(ii) $\lim_{t\to 0} |u(x,t)| = \infty$ *on a countable set* C, *and*

(iii) $\lim_{t\to 0} t^{n/2}u(x,t) = 0$ *for all* $x \in C$.

Then $|u| \leq AV_\kappa$ *on* $\mathbf{R}^n \times]0, \min\{c, (4\kappa)^{-1}\}[$ *if* $\kappa > 0$, *on* $\mathbf{R}^n \times]0, c[$ *if* $\kappa = 0$, *so that* u *has a representation in the form*

$$u(x,t) = \int_{\mathbf{R}^n} W(x-y,t)f(y)\,dy \tag{26}$$

for some function f *such that* $|f(y)| \leq A\exp(\kappa\|y\|^2)$ *for all* y.

PROOF. Hypotheses (i) and (ii), together with Theorem 12, imply that u has the representation (24), where $\{x_j\}_{j\geq 1} = C$ and (25) holds. Using (iii) and (25), we obtain

$$\lim_{t\to 0}\left(t^{n/2}\sum_{j=1}^{\infty}\mu(\{x_j\})W(x-x_j,t)\right) = \lim_{t\to 0} t^{n/2}(u(x,t) - h(x,t)) = 0 \tag{27}$$

for all $x \in C$. For each non-negative integer i, we can write

$$\sum \mu(\{x_j\})W(x-x_j,t) = \int_{\mathbf{R}^n} W(x-y,t)\,d\nu_i(y),$$

where the summation is taken over all $j \neq i$ and where, if δ_j denotes the Dirac δ-measure concentrated at x_j,

$$\nu_i = \sum \mu(\{x_j\})\delta_j.$$

Since $\nu_i(\{x_j\}) = 0$, [19, Example 1] shows that

$$t^{n/2}\int_{\mathbf{R}^n} W(x-y,t)\,d\nu_i(y) \to 0$$

as $t \to 0$. It therefore follows from (27) that

$$\mu(\{x_i\}) = (4\pi t)^{n/2}\mu(\{x_i\})W(x_i - x_i, t) \to 0.$$

Hence $\mu(\{x_i\}) = 0$ for each i. It follows that $u = h$, and hence that $|u| \leq AV_\kappa$. The representation (26) now follows by an argument similar to the one used to prove Corollary 1 of Theorem 12.

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