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COEFFICIENTS OF SYMMETRIC FUNCTIONS OF BOUNDED BOUNDARY ROTATION

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Let $V_m(k)$ denote the family of all functions of the form

$$f(z) = z + \sum_{p=1}^{\infty} a_{mp+1} z^{mp+1}$$

that are analytic in the unit disc $U, f'(z) \neq 0$ in U and f maps U onto a domain of boundary rotation at most $k\pi$. Recently Brannan, Clunie and Kirwan [**2**] and Aharonov and Friedland [**1**] have solved the problem of estimating $|a_{mp+1}|$ for all k, provided $m = 1$. The extremal function for $V_1(k)$ is defined by

$$f_1'(z) = \frac{(1+z)^{(k-2)/2}}{(1-z)^{(k+2)/2}}.$$

The following proposition is an immediate consequence of [**3**, Theorem 3.1]:

PROPOSITION. *$f(z) \in V_m(k)$ if and only if there is a function $g(z) \in V_1(k)$ such that $f'(z) = g'(z^m)^{1/m}$.*

Let $f_m(z)$ be the function in $V_m(k)$ defined by

$$f'(z) = (1 + z^m)^{(k-2)/2m}/(1 - z^m)^{(k+2)/2m}.$$

It is natural to conjecture that $|a_{mp+1}| \leqq A_{mp+1}$, where

$$f_m(z) = z + \sum_{p=1}^{\infty} A_{mp+1} z^{mp+1}.$$

In this note we obtain a partial solution to the problem of estimating $|a_n|$ and show that the conjecture is false in general if $m \geqq 2$. In addition, we determine the valency of functions in $V_m(k)$.

The following lemma is due implicitly to Umezawa [**9**].

LEMMA. *Let f be analytic in $|z| \leqq r$, with $f' \neq 0$ in $|z| \leqq r$. Let $z_1, \ldots, z_m$ be the roots of* $\operatorname{Re}\{1 + (zf''(z))/f'(z)\} = 0$ *on $|z| = r$. If*

$$\min_{1 \leqq i, j \leqq m} \int_{\theta_i}^{\theta_j} \operatorname{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} d\theta > -p\pi \quad (z = re^{i\theta}),$$

then f is at most p-valent in $|z| \leqq r$.

THEOREM 1. *Let $f(z) \in V_m(k)$. Then $f(z)$ is at most p-valent in U, where*

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$p = (k-2)/2m$ if $(k-2)/2m$ is an integer and $p = [(k-2)/2m] + 1$ if $(k-2)/2m$ is not an integer.

Proof. Let $r < 1$ be fixed and define

$$h(\theta) = \operatorname{Re}\left\{1 + \frac{re^{i\theta}f''(re^{i\theta})}{f'(re^{i\theta})}\right\}.$$

Let N be the largest non-negative integer for which $\theta_j - \theta_i \geqq 2\pi N/m$. Then since f is m-fold symmetric,

$$\begin{aligned}\int_{\theta_i}^{\theta_j} h(\theta)d\theta &= \int_{\theta_i}^{\theta_i+(2\pi N/m)} h(\theta)d\theta + \int_{\theta_i+(2\pi N/m)}^{\theta_j} h(\theta)d\theta \\ &= 2\pi N/m + \int_{\theta_i+(2\pi N/m)}^{\theta_j} h(\theta)d\theta \\ &> \frac{2\pi N}{m} + \left(1 - \frac{k}{2}\right)\frac{\pi}{m} \\ &\geqq \left(1 - \frac{k}{2}\right)\frac{\pi}{m}.\end{aligned}$$

The result now follows since $p < (k-2)/2m$ and p must be an integer.

Note. This result was proved in [**3**] for the case $m = 1$.

THEOREM 2. *Let $f(z) = z + \sum_{p=1}^{\infty} a_{mp+1}z^{mp+1} \in V_m(k)$ with $k \geqq 2m + 2$. Then $|a_{mp+1}| \leqq A_{mp+1}$.*

Proof. Let $g(z) \in V_1(k)$ be defined by $g'(z^m)^{1/m} = f'(z)$. By a result due to Noonan [**7**],

$$zg'(z) = aQ(z)^{\beta}S(z)$$

where $\operatorname{Re} Q(z) > 0$, $S(z)$ is starlike, $|a| = 1$, and $\beta = k/2 - 1$. Therefore

(1) $$zf'(z) = a^{1/m}Q(z^m)^{\beta/m}S(z^m)^{1/m}.$$

Since $|a^{1/m}| = 1$, $\operatorname{Re} Q(z^m) > 0$ and since $S(z^m)^{1/m}$ is an m-fold symmetric starlike function, it follows from [**1**] and [**2**] that if $\beta/m \geqq 1$, the coefficients of $Q(z^m)^{\beta/m}$ and $S(z^m)^{1/m}$ are simultaneously maximal when

$$f'(z) = (1 + z^m)^{(k-2)/4}/(1 - z^m)^{(k+2)/4}.$$

Thus the result follows if $(k/2 - 1)/m \geqq 1$ or $k \geqq 2m + 2$.

We note that the proof actually holds for the larger class of m-fold symmetric functions that are close-to-convex of order $(k-2)/2m \geqq 1$.

The following theorem is of interest only when $k < 2m + 2$. It was proved by Lehto [**6**] if $m = 1$ and the author [**5**] if $m = 2$. The technique is essentially due to Lehto.

THEOREM 3. *Let* $f(z) = z + \sum_{p=1}^{\infty} a_{mp+1}z^{mp+1} \in V_m(k)$. *Then:*

$$\text{(i)}\quad |a_{m+1}| \leqq \frac{k}{m(m+1)} \qquad k \geqq 2$$

$$\text{(ii)}\quad |a_{2m+1}| \leqq \frac{k^2 + 2m}{2m^2(2m+1)} \qquad k \geqq 2m$$

$$\text{(iii)}\quad |a_{2m+1}| \leqq \frac{4mk + 6k + 4}{(4m + 2 - k)(2m)(2m+1)} \qquad 2 \leqq k < 2m.$$

All of the results are sharp for the indicated range of k.

Proof. By a result due to Lehto [**6**]

$$(2)\quad a_{mp+1} = \frac{1}{(mp+1)(mp)} \sum_{j=0}^{p-1} (mj+1)a_{mj+1} \int_0^{2\pi} e^{-(p-j)\,im\theta} d\mu(\theta),$$

where $\mu(\theta)$ is of bounded variation on $[0, 2\pi]$ with

$$\int_0^{2\pi} d\mu(\theta) = 2 \quad \text{and} \quad \int_0^{2\pi} |d\mu(\theta)| \leqq k.$$

From (2),

$$|a_{m+1}| \leqq \frac{1}{(m+1)m} \int_0^{2\pi} |e^{-im\theta} d\mu(\theta)| \leqq \frac{k}{m(m+1)},$$

which proves (i). From (2),

$$(2m)(2m+1)a_{2m+1} = (m+1)a_{m+1} \int_0^{2\pi} e^{-im\theta} d\mu(\theta) + \int_0^{2\pi} e^{-2im\theta} d\mu(\theta)$$

$$= \frac{1}{m}\left[\int_0^{2\pi} e^{-im\theta} d\mu(\theta)\right]^2 + \int_0^{2\pi} e^{-2im\theta} d\mu(\theta).$$

We may suppose without loss of generality that $a_{2m+1} \geqq 0$ since if not we consider $e^{-i\alpha}f(e^{i\alpha}z)$ for suitably chosen α. Then

$$(2m)(2m+1)a_{2m+1} = \frac{1}{m}\left(\int_0^{2\pi} \cos m\theta d\mu(\theta)\right)^2 - \frac{1}{m}\left(\int_0^{2\pi} \sin m\theta d\mu(\theta)\right)^2 + \int_0^{2\pi} \cos 2m\theta d\mu(\theta)$$

$$\leqq \frac{1}{m}\left(\int_0^{2\pi} \cos m\theta d\mu(\theta)\right)^2 + \int_0^{2\pi} \cos 2m\theta d\mu(\theta).$$

Suppose first that $\mu(\theta)$ is a step function with at most N jumps d_j at θ_j. Then

$$(3)\quad (2m)(2m+1)a_{2m+1} \leqq \frac{1}{m}\left(\sum_{j=1}^{N} \cos m\theta_j d_j\right)^2 + \sum_{j=1}^{N} \cos 2m\theta_j d_j$$
$$= \frac{1}{m}\left(\sum_{j=1}^{N} \cos m\theta_j d_j\right)^2 + 2\sum_{j=1}^{N} \cos^2 m\theta_j d_j - 2.$$

First assume that the maximum of (3) occurs at a point where r of the $|\cos m\theta_j| \neq 1$. We may assume $|\cos m\theta_j| \neq 1$ for $1 \leqq j \leqq r$. Then a differentiation of (3) with respect to $\cos m\theta_h$, $1 \leqq h \leqq r$ yields

$$(4)\quad \frac{2}{m}\sum_{j=1}^{N} \cos m\theta_j d_j = -4\cos m\theta_h \quad 1 \leqq h \leqq r$$
$$\equiv -4\cos m\alpha.$$

Substituting in (3), we obtain

$$\frac{1}{m}(4m^2\cos^2 m\alpha) + 2\sum_{1}^{r} \cos^2 m\alpha \cdot d_j + 2\sum_{r+1}^{N} d_j - 2.$$

Since $\sum_1^N d_j = 2$ and $\sum_1^N |d_j| \leqq k$, $\sum_1^r d_j \geqq 1 - k/2$ and $\sum_{r+1}^N d_j \leqq 1 + k/2$, it follows from (3) that

$$2m\cos m\alpha = -\cos m\alpha \sum_{1}^{r} d_j - \sum_{r+1}^{N} \cos m\theta_j d_j$$
$$|\cos m\alpha| = \left|\sum_{r+1}^{N} \cos m\theta_j d_j\right| \left(\left|2m + \sum_{1}^{r} d_j\right|\right)^{-1}$$
$$\leqq \frac{1 + k/2}{2m + 1 - k/2}.$$

If $k \geqq 2m$, $(k+2)/(4m+2-k) \geqq 1$ and hence there is no restriction on $|\cos m\alpha|$. Thus for $k \geqq 2m$, (3) is less than or equal to

$$\max\{2(1 + k/2) - 2, 2m + 2\} = k.$$

If $2 \leqq k < 2m$, $|\cos m\alpha| \leqq (k+2)/(4m+2-k) < 1$ and thus the maximum of (3) is

$$\frac{(k+2)^2}{(4m+2-k)^2}\left[4m + 2\left(1 + \frac{k}{2}\right)\right] + 2\left(1 + \frac{k}{2}\right) - 2 = \frac{4mk + 6k + 4}{4m + 2 - k}.$$

It remains to consider the case where all $|\cos m\theta_j| = 1$. Then

$$(2m)(2m+1)a_{2m+1} \leqq \frac{1}{m}\left[\sum_{1}^{N} \cos m\theta_j d_j\right]^2 + 2\sum_{1}^{N} d_j - 2$$
$$\leqq \frac{k^2}{m} + 2.$$

An elementary calculation shows that

$$\frac{k^2}{m} + 2 < \frac{4mk + 6k + 4}{4m + 2 - k}$$

if $2 < k < 2$m.

Since step functions are dense in the class of functions of bounded variation, the result follows. The function f_m shows that (i) and (ii) are sharp. To show that (iii) is sharp we construct a step function with jumps at $\cos m\alpha$ in a manner similar to [**5**].

Since $(2m)(2m + 1)A_{2m+1} = k + 2$, the conjecture is false if $k < 2m$ and $p = 2$. The coefficient problem remains to be solved in the case $k < 2m + 2$ for large values of $mp + 1$. To this end we have the following

THEOREM 4. *Let* $f(z) = z + \sum_{p=1}^{\infty} a_{mp+1}z^{mp+1} \in V_m(k)$, *where* $k > 2m - 2$. *Then if* $f(z) \neq e^{-i\theta}f_m(e^{i\theta}z)$, *there is an integer* p_0 *depending on* f *such that if* $p > p_0$,

$$|a_{mp+1}| < A_{mp+1}.$$

Proof. Since $(k + 2)/2m > 1$, the methods of [**5**, Theorem 4.3] show that there is a θ_0 such that

$$\lim_{r\to 1} (1 - r)^{(k+2)/2m}|f'(re^{i\theta_0})| = \lim_{r\to 1} (1 - r)^{(k+2)/2m}M(r, f')$$
$$= \alpha,$$

where α is maximal only for $f(z) = e^{-i\theta}f_m(e^{i\theta}z)$.

The result now follows using the major-minor arc technique of Hayman [**4**, Theorem 5.7] as modified by Noonan [**8**]. (See also [**5**].)

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