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ON A MAXIMAL OUTER AREA PROBLEM FOR A CLASS OF MEROMORPHIC UNIVALENT FUNCTIONS

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For $0 < p < 1$, let S_p denote the class of functions $f(z)$ which are meromorphic and univalent in the unit disk U, with the normalisations $f(0) = 0$, $f'(0) = 1$ and $f(p) = \infty$, and let $S_p(a)$ denote the subclass of S_p consisting of those functions in S_p whose residue at the pole is equal to a. In this paper, we determine, for values of the residue a in a certain disk Δ_p, the greatest possible outer area over all functions in the class $S_p(a)$. We also determine additional information concerning extremal functions if the residue a does not lie in Δ_p.

1. Introduction.

For $0 < p < 1$, let S_p denote the class of functions $f(z)$ which are meromorphic and univalent in the unit disk $U = \{z : |z| < 1\}$, with the normalisations $f(0) = 0$, $f'(0) = 1$ and $f(p) = \infty$.

Many authors have considered this class in their research.

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K. Ladegast, in an extensive paper [3], derived many inequalities satisfied by functions belonging to S_p . Y. Komatu, in two successive papers [2], derived many inequalities which must hold for the coefficients of functions belonging to S_p . Z. Lewandowski and E. Złotkiewicz, [4] and [5], established several variational formulae for the class S_p and showed how they might be used to obtain results about functions in S_p which are solutions to certain extremal problems. More recently, Z. Lewandowski, R. J. Libera and E. Złotkiewicz [6] have used sophisticated conformal mapping techniques to study the complement of the range of functions belonging to certain subclasses of S_p . In the present paper, we study the complement of the range of functions belonging to certain subclasses of S_p .

Define the set

$$\Omega_p = \{a : a = \text{Res}_{z=p} f(z) \ , \ f \in S_p\} \ .$$

In [8], it has been determined that

$$\Omega_p = \{-p^2(1 - p^2)^{\varepsilon} : |\varepsilon| \le 1\} \ .$$

Now consider the subclasses $S_p(a)$ of S_p defined by

$$S_p(a) = \{f \in S_p : \text{Res}_{z=p} f(z) = a\} \ .$$

Every function $f(z)$ in $S_p(a)$ will have an expansion of the form

$$f(z) = \frac{a}{z - p} + \frac{a}{p} + \left(1 + \left(\frac{a}{p^2}\right)\right) z + \sum_{k=2}^{\infty} \alpha_k z^k$$

due to the normalisations described above.

The outer area $\overline{A}(f)$ of $f(z)$ (the area of the complement of the range of $f(z)$) is finite and can be expressed in terms of the residue a and the coefficients α_k . Indeed, with the aid of Green's Formula, we have

(1)
$$\overline{A}(f) = \lim_{r\to 1} \frac{-1}{2i} \int_{|z|=r} \overline{f}\, df = \frac{\pi|a|^2}{(1-p^2)^2} - \sum_{k=1}^{\infty} k|\alpha_k|^2 .$$

Since $\overline{A}(f)$ must be non-negative, we obtain the following inequality

(2)
$$\sum_{k=1}^{\infty} k|\alpha_k|^2 \le \frac{|a|^2}{(1-p^2)^2}$$

which we may call the Area Theorem for the Class $S_p(a)$. Equality holds in (2) if and only if $\overline{A}(f) = 0$. In [7], it was shown that there exists functions $f \in S_p(a)$ such that $\overline{A}(f) = 0$ for values of a in an open and dense subset D_p of Ω_p. Consequently, for $a \in D_p$, we may conclude that

$$\min_{f\in S_p(a)} \overline{A}(f) = 0$$

This minimal problem remains open if $a \in \Omega_p - D_p$.

In this paper, we are concerned with the corresponding maximal problem

$$\max_{f\in S_p(a)} \overline{A}(f) .$$

Expression (1) above suggests that the outer area would attain its greatest value only for a function of the form

(3)
$$F(z;p,a) = \frac{a}{z-p} + \frac{a}{p} + \left(1 + \frac{a}{p^2}\right) z .$$

In Section 2, we prove that this is indeed the case, provided that the residue a belongs to a certain disk $\Delta_p \subsetneq \Omega_p$. If $a \in \Omega_p - \Delta_p$, then this problem remains open. However, in Section 3, we introduce a variation for the class $S_p(a)$, (that is, a residue-preserving variation for the class S_p), which will be used to deduce information about the extremal functions in this case. Further results will depend upon either the successful invention of additional variational formulae for the class

$S_p(a)$, or the adaptation of variational formulae for the class S_p which already exist, such as the ones given in [5].

2. A Partial Solution.

In this section, we prove

THEOREM 1. *The solution to the extremal problem*

$$\max_{f \in S_p(a)} \overline{A}(f)$$

is given by (3) provided that the residue a *belongs to the disk* Δ_p *of values determined by the inequality*

$$\frac{|a|p^2}{|a + p^2|} \geq (1 + p)^2 \tag{4}$$

Proof. Case I: $a = -p^2$. In this case,

$$F(z;p,-p^2) = \frac{pz}{p - z} .$$

Since $\alpha_k = 0$ for $k \geq 1$, the formula for the outer area (1) gives $A(F) = \pi p^4/(1 - p^2)^2$. Moreover, since any other function in $S_p(-p^2)$ must have at least one non-zero coefficient α_k , any other function in this class must also have a strictly smaller outer area. Finally, since F is linear-fractional, it must map U onto the exterior of a certain disk. This disk is centered at $-2p/(1 - p^2)$ and has radius $p^2/(1 - p^2)$.

Case II: $a \neq -p^2$. Since (3) is meromorphic and satisfies the appropriate normalisations, it remains to show that it is univalent on U . Since F can be written in the form

$$F(z;p,a) = \frac{a}{z - p} + \left(\frac{2a}{p} + p\right) + \left(1 + \frac{a}{p^2}\right)(z - p)$$

It is sufficient to show that, under the assumed condition, the function $G(w) = A/w + Bw$ is univalent on the set $U_p = \{w : w = \zeta - p,\ \zeta \in U\}$,

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where $A = a$ and $B = 1 + a/p^2$. We note first that $|w_1w_2| < (1+p)^2$ for any $w_1, w_2 \in U_p$.

Suppose now that G is not univalent; that is, that there exists $w_1, w_2 \in U_p$, $w_1 \neq w_2$, such that

$$G(w_1) = \frac{A}{w_1} + Bw_1 = \frac{A}{w_2} + Bw_2 = G(w_2) .$$

Then a little arithmetic shows that

$$(1+p)^2 > |w_1w_2| = \left|\frac{A}{B}\right| = \frac{|a|p^2}{|a + p^2|} ,$$

contrary to assumption. Hence, (4) implies that $F(z;p,a)$ is univalent on U and thereby belongs to $S_p(a)$. Since (4) may be rewritten in the form

$$\left|\frac{1}{a} + \frac{1}{p^2}\right| \le \frac{1}{(1+p)^2}$$

it is clear that the reciprocals $\{1/a\}$ described by this condition belong to a disk; consequently, the set of residues $\{a\}$ do as well. In explicit form, we may write

$$\Delta_p = \{s : |s + (1+p)^2\delta_p| \le p^2\delta_p\} ,$$

where

$$\delta_p = \frac{p^2(1+p)^2}{(1+2p)(1+2p+2p^2)} .$$

This concludes the proof of the theorem.

For the sake of completeness, we now briefly discuss the mapping properties of $F(z;p,a)$. Since F can be rewritten in the form

$$F(z;p,a) = \frac{2a}{p} + p + 2\left(\frac{a}{p^2} + 1\right)\left(\frac{z-p}{2} + \frac{1}{2}\left(\frac{ap^2}{a+p^2}\right)\left(\frac{1}{z-p}\right)\right) ,$$

it is easily seen to be the successive composition of three maps:

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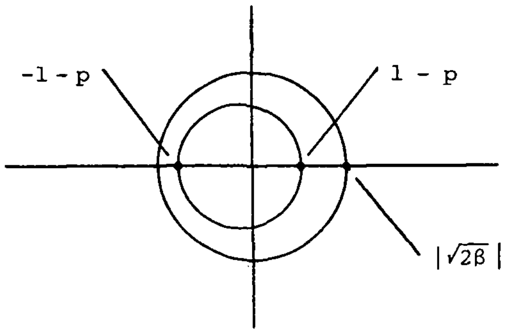
(1) $\zeta = C(z) = z - p$,

(2) $\xi = D(\zeta) = \zeta/2 + \beta/\zeta$, $\qquad \beta = \dfrac{1}{2}\left(\dfrac{ap^2}{a + p^2}\right)$,

and (3) $w = E(\xi) = (2a/p + p) + 2(1 + a/p^2)\xi$.

Maps C and E are linear. Map D is discussed in [1, p. 64-67]; it univalently takes either the interior or the exterior of the circle $\{|\zeta| = |\sqrt{\beta}|\}$ onto the plane, cut from $-\sqrt{2\beta}$ to $+\sqrt{2\beta}$.



Since $|\sqrt{2\beta}| = \left|\dfrac{ap^2}{a + p^2}\right|^{1/2} \geq (1 + p)$, the set $C(U)$ is contained entirely in the interior of the circle $\{|\zeta| = |\sqrt{2\beta}|\}$. The set $D(C(U))$ is the plane, except for a compact set of positive area, which contains the interval $[-\sqrt{2\beta}, +\sqrt{2\beta}]$. Map E stretches and translates the set $D(C(U))$. If strict inequality holds in (4), then the set $F(U;p,a)$ will have an analytic boundary. If equality holds in (4), then $F(U;p,a)$ will have an analytic boundary except for one point; the tear-shaped exterior in this case is sometimes called a Joukowski aerofoil and is obtained whenever $a \in \partial\Delta_p$.

3. Additional Information.

The results to follow may be easily discussed in terms of the framework introduced here.

Let L_p denote the linear subspace of all functions of the form

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$$f(z) = \sum_{-\infty}^{+\infty} a_n z^n$$

which are analytic on the annulus $A_p = \{z: p < |z| < 1\}$, and satisfy the condition

$$\sum_{-\infty}^{+\infty} |n||a_n|^2 < +\infty .$$

For each $f(z) = \sum_{-\infty}^{+\infty} a_n z^n$ and $h(z) = \sum_{-\infty}^{+\infty} b_n z^n$ in L_p , the Hermitian Product

$$\langle f,h\rangle = -\sum_{-\infty}^{+\infty} n a_n \overline{b}_n$$

exists due to the Cauchy-Schwarz Inequality.

Suppose now that $f \in L_p$ and that G is analytic on the range of f , with $G \circ f \in L_p$. We wish to show that the Hermitian product $\langle f, G \circ f\rangle$ has an alternate formulation in terms of a contour integral involving f and G . If we write

$$f(z) = \sum_{-\infty}^{+\infty} a_n z^n \quad \text{and} \quad (G \circ f)(z) = \sum_{-\infty}^{+\infty} c_n z^n ,$$

Then after a short computation, we obtain the relation

$$\sum_{-\infty}^{+\infty} n a_n \overline{c}_n r^{2n} = \frac{1}{2\pi i} \int_{f(|z|=r)} \overline{G(w)}\, dw .$$

Consequently, it is now clear that

$$\langle f, G \circ f\rangle = \lim_{r\to 1} \left(\frac{-1}{2\pi i} \int_{f(|z|=r)} \overline{G(w)}\, dw \right) ,$$

which is the formulation that we desired. Observe that, as a special case, if we let $G(w) = w$, we obtain, for $f \in S_p$, the relation $\langle f,f\rangle = \overline{A}(f)/\pi$. It is now apparent that $\langle f,f\rangle \geq 0$, and that equality holds if and only if $\overline{A}(f) = 0$.

Now let D be any open connected set in the complex plane. By $\Lambda(D)$ we shall denote the set of functions $h(w)$ which are holomorphic in D, and satisfy the Lipschitz condition

$$|h(w_1) - h(w_2)| \le K_h |w_1 - w_2|$$

for all $w_1, w_2 \in D$, and some constant K_h which depends only upon h.

We are now ready to state and prove a Variational Lemma for the Class $S_p(a)$.

LEMMA. *Suppose that* $g(z) \in S_p(a)$ *and that* $ext\{g(U)\} \neq \phi$. *Let* $L_c(w) = cw/(c + w)$ *and* $f_c(z) = L_c(g(z))$, *where* $-c \in ext\{g(U)\}$. *Then*

$$g_t(z) = g(z) + tQ(g(z)) + O(t^2) \in S_p(a)$$

for all complex t, $|t|$ *sufficiently small, and* $O(t^2)$ *is uniform on compact subsets of* U. *Here,*

$$Q(w) = \left(\frac{c + w}{c}\right)^2 h\left(\frac{cw}{c + w}\right)$$

where $h \in \Lambda(f_c(U))$ *and* $h(0) = h'(0) = h(c) = h'(c) = 0$.

Proof. Let S denote the class of functions $f(z)$ which are analytic and univalent in the unit disk U and are normalized so that $f(0) = 0$ and $f'(0) = 1$. A short argument shows that the function

$$f_c(z) = (L_c \circ g)(z) = \frac{cg(z)}{c + g(z)}$$

belongs to the class S and that $\text{Res}_{z=p}\, g(z) = -f_c^2(p)/f_c'(p)$. Now let $h(z) \in \Lambda(f_c(U))$ and assume that $h(z)$ and its first derivative vanish at $z = 0$ and $z = c = f_c(p)$. Let t be any complex number and consider the function

$$f_{ct}(z) = f_c(z) + th(f_c(z)).$$

For all t having sufficiently small absolute value, $f_{ct}(z)$ is univalent in U. Indeed, let z_1 and z_2 be distinct points of U.

Then

$$f_{ct}(z_2) - f_{ct}(z_1) = f_c(z_2) - f_c(z_1)$$
$$+ t[h(f_c(z_2)) - h(f_c(z_1))]$$

and so

$$|f_{ct}(z_2) - f_{ct}(z_1)| \geq |f_c(z_2) - f_c(z_1)| \cdot (1 - K_h|t|) .$$

Thus, if $|t| < 1/K_h$, then $f_{ct}(z_1) \neq f_{ct}(z_2)$.

Due to the additional restrictions placed on $h(z)$, we must also have $f_{ct}(0) = 0$ and $f'_{ct}(0) = 1$. Therefore, $f_{ct} \in S$ for all t sufficiently small in absolute value.

Now consider the function

$$g_t(z) = (L^{-1}_{f_{ct}(p)} \circ f_{ct})(z) = \frac{f_{ct}(p)\, f_{ct}(z)}{f_{ct}(p) - f_{ct}(z)} .$$

A short argument shows that $g_t(z) \in S_p$ for all t, $|t| < 1/K_h$. Also, $\text{Res}_{z=p} g_t(z) = -f^2_{ct}(p)/f'_{ct}(p) = -f^2_c(p)/f'_c(p) = \text{Res}_{z=p} g(z) = a$, since $f_{ct}(p) = f_c(p)$ and $f'_{ct}(p) = f'_c(p)$. Hence, $g_t(z) \in S_p(a)$ for all t, $|t| < 1/K_h$. It remains to show that $g_t(z)$ has the prescribed form. Note first that, for any constant d, we have

$$L^{-1}_d(x + ty) = L^{-1}_d(x)\left[1 + ty\left(\frac{1}{x} + \frac{1}{d - x}\right) + O(t^2)\right] .$$

Using this relation, we get

$$\begin{aligned} g_t(z) &= L^{-1}_{f_{ct}(p)}(f_{ct}(z)) \\ &= L^{-1}_c(f_c(z) + th(f_c(z))) \\ &= L^{-1}_c(f_c(z))\left[1 + th(f_c(z)) \left(\frac{1}{f_c(z)} + \frac{1}{c - f_c(z)}\right) + O(t^2)\right] \\ &= g(z) + tg(z)(h \circ L_c \circ g)(z)\left[\frac{1}{L_c \circ g(z)} + \frac{1}{c - L_c \circ g(z)}\right] + O(t^2) \\ &= g(z) + tQ(g(z)) + O(t^2) , \end{aligned}$$

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where

$$Q(w) = w[(h \circ L_c)(w)]\left(\frac{1}{L_c(w)} + \frac{1}{c - L_c(w)}\right)$$

$$= \left(\frac{c + w}{c}\right)^2 h\left(\frac{cw}{c + w}\right) .$$

A major consequence of this Variational Lemma is the following necessary orthogonality condition.

THEOREM 2. *Let* $g(z)$ *be an extremal function for the problem*

$$\max_{f \in S_p(a)} \overline{A}(f)$$

and assume that $ext\{g(U)\} \neq \phi$. *Then*

$$\langle g, Q \circ g\rangle = \lim_{r \to 1} \left(\frac{-1}{2\pi i} \int_{g(|z|=r} \overline{Q(w)}\, dw\right) = 0$$

for every Q *satisfying the conditions of the Variational Lemma.*

Proof. If $g(z)$ has an expansion of the form

$$g(z) = \frac{a}{z - p} + \sum_0^\infty \alpha_k z^k$$

and $Q(g(z))$ has an expansion of the form

$$Q(g(z)) = \sum_2^\infty q_k z^k$$

then $g_t(z)$ will have an expansion of the form

$$g_t(z) = \frac{a}{z - p} + \alpha_0 + \alpha_1 z + \sum_2^\infty (\alpha_k + tq_k)z^k + O(t^2) .$$

Since $g(z)$ is an extremal function for the maximal problem, we must have $\overline{A}(g) \geq \overline{A}(g_t)$ for every complex t with $|t| < 1/K_h$. It follows from (1) that

$$\sum_2^\infty k|\alpha_k + tq_k|^2 + O(|t|^2) \geq \sum_2^\infty k|\alpha_k|^2$$

Equivalently, we have

$$\sum_{2}^{\infty} 2k \operatorname{Re}\left(\frac{\bar{t}}{|t|} \alpha_k \bar{q}_k\right) + O(|t|^2) \geq 0$$

for all t , $|t| < 1/K_h$; but this last inequality can hold only if

$$\sum_{2}^{\infty} k\alpha_k \bar{q}_k = 0 .$$

The proof is now complete, since this last equality is equivalent to the result in the statement of the theorem, as we have shown earlier in this section.

As a specific example, we now show that the extremal function $g(z)$ and the functions $f_c^N(z)$, $N \geq 2$, defined within the Variational Lemma above, are orthogonal in the space L_p with respect to the Hermitian product $<\,,\,>$.

COROLLARY 2.1. *Suppose that*

$$g(z) = \frac{a}{z-p} + \sum_{0}^{\infty} \alpha_k z_k$$

is an extremal function for the problem

$$\max_{f \in S_p(a)} \bar{A}(f)$$

and that $ext\{g(U)\} \neq \phi$. *If we set*

$$f_c(z) = \frac{cg(z)}{(c + g(z))} = z + \sum_{2}^{\infty} a_k(c) z^k$$

and

$$f_c^N(z) = z^N + \sum_{N+1}^{\infty} a_k^{(N)}(c) z^k ,$$

then

$$<g, f_c^N> = \sum_{N}^{\infty} k\alpha_k \overline{a_k^{(N)}(c)} = 0$$

for every $N \geq 2$ *and every* c *such that* $-c \in ext\{g(U)\}$.

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Proof. Choose

$$Q(w) = c^2\left(\frac{cw}{c+w}\right)^N . \qquad (N \geq 2)$$

Introducing the change of variable $w = ct/(c-t)$, we have

$$\frac{1}{2\pi i}\int_{g(|z|=r}\overline{Q(w)}\,dw = \frac{\bar{c}^2}{2\pi i}\int_{g(|z|=r)}\overline{\left(\frac{cw}{c+w}\right)^N}\,dw$$

$$= \frac{|c|^4}{2\pi i}\int_{f_c(|z|=r)}\bar{t}^N\frac{dt}{(c-t)^2}$$

$$= \frac{\bar{c}^2}{2\pi i}\int_{|z|=r}\overline{f_c^N(z)}\,g'(z)dz$$

$$= \bar{c}^2\sum_N k\alpha_k\overline{a_k^{(N)}(c)}r^{2k} .$$

Now observe that since $f_c(z)$ is bounded, $f_c^N(z) \in L_p$. The corollary now follows from Theorem 2 by letting r tend to one.

Another Corollary to Theorem 2 concerning the coefficients of extremal functions is of interest.

COROLLARY 2.2. *Suppose that*

$$g(z) = \frac{a}{z-p} + \sum_0^\infty \alpha_k z^k$$

is an extremal function for the problem

$$\max_{f\in S_p(a)} \bar{A}(f)$$

and that $ext\{g(U)\} \neq \phi$. *Then either* $\alpha_k = 0$ *for all* $k \geq 2$ *or* $\alpha_k \neq 0$ *for infinitely many* $k \geq 2$.

Proof. Suppose that there exists a K such that $\alpha_K \neq 0$ and

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$\alpha_k = 0$ for all $k > K$. Applying the equality in Corollary 2.1, with $N = K$, we get

$$0 = \sum_{K}^{\infty} k\alpha_k \overline{a_k^{(K)}(c)} = K\alpha_K \overline{a_K^{(K)}(c)}$$

But $a_K^{(K)}(c) = 1$. Thus we have obtained a contradiction.

References

[1] H. Kober, *Dictionary of Conformal Representations*, Dover Publications, New York (1957).

[2] Y. Komatu, "Note on the theory of conformal representation by meromorphic functions I and II", *Proc. Japan Acad.* 21(1945), 269-277; 278-284.

[3] K. Ladegast, "Beiträge zur Theorie der schlichten Funktionen", *Math. Z.*, 58 (1953), 115-159.

[4] Z. Lewandowski and E. Złotkiewicz, "Variational formulae for functions meromorphic and univalent in the unit disc", *Ann. Univ. Mariae Curie - Skłodowska* 17 (1963), 47-53.

[5] Z. Lewandowski and E. Złotkiewicz, "Variational formulae for functions meromorphic and univalent in the unit disc", *Bull. Acad. Polon. Sci., Math.*, XII (1964), No.5, 253-254.

[6] Z. Lewandowski, R. J. Libera and E. Złotkiewicz, "Mapping properites of a class of univalent functions with pre-assigned zero and pole", *Ann. Polon. Math.*, XL (1983), 283-289.

[7] S. Zemyan, "A minimal outer area problem in conformal mapping", *J. Analyse Math.*, 39 (1981), 11-23.

[8] S. Zemyan, "The range of the residue functional for the class S_p", *Michigan Math. J.*, 31 (1984), 73-77.

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