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ON APPROXIMATION BY TRIGONOMETRIC LAGRANGE INTERPOLATING POLYNOMIALS

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It is well-known that the approximation to $f(x) \in C_{2\pi}$ by nth trigonometric Lagrange interpolating polynomials with equally spaced nodes in $C_{2\pi}$ has an upper bound $\ln(n)E_n(f)$, where $E_n(f)$ is the nth best approximation of $f(x)$. For various natural reasons, one can ask what might happen in L^p space? The present paper indicates that the result about the trigonometric Lagrange interoplating approximation in L^p space for $1 < p < \infty$ may be "bad" to an arbitrary degree.

Let $L^p_{2\pi}$ be the class of integrable functions of power p and of period 2π, $C_{2\pi}$ be the class of continuous 2π-periodic functions and T_n be the trigonometric polynomials of degree at most n.

For $f \in L^1_{2\pi}$, $S_n(f,x)$ is the nth partial sum of the Fourier series of $f(x)$; for $f \in L^p_{2\pi}$, $E_n(f)_p$, is the nth best approximation of $f(x)$ in L^p space; for $f \in C_{2\pi}$ $L_n(f,x)$ is the nth trigonometric Lagrange interpolating polynomial of $f(x)$ with equally spaced nodes; that is

$$L_n(f,x) = \sum_{k=0}^{2n} f(x_k) l_k(x),$$

where

$$l_k(x) = \frac{1}{2n+1} \frac{\sin(n+1/2)(x-x_k)}{\sin 1/2(x-x_k)},$$

$$x_k = \frac{2k\pi}{2n+1}, \qquad k = 0, 1, \ldots, 2n.$$

The norm of $f \in L^p_{2\pi}$ is defined as follows.

$$\|f\|_{L^p} = \left(\int_0^{2\pi} |f(x)|^p dx\right)^{1/p}, \qquad 1 \leqslant p < \infty,$$

$$\|f\| = \|f\|_{L^\infty} = \max_{0 \leqslant x \leqslant 2\pi} |f(x)|, \qquad p = \infty.$$

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It is well-known that

$$\|S_n\| = \sup\{\|S_n f\|: \ \|f\| = 1\} \sim \ln(n+1), \tag{1}$$

which means that there exists a positive constant M independent of n such that

$$M^{-1}\ln(n+1) \leqslant \|S_n\| \leqslant M\ln(n+1),$$

so the factor $\ln(n+1)$ in the following inequality

$$\|f - S_n(f)\| = 0(\ln(n+1)E_n(f)) \text{ for } f \in C_{2\pi}$$

cannot be omitted. However, in L^p space for $1 < p < \infty$, by the Riesz theorem (see [2]), a beautiful result is obtained for $f \in L^p_{2\pi}$, namely:

$$\|f - S_n(f)\|_{L^p} \leqslant c_p E_n(f)_p, \tag{2}$$

where c_p is a positive constant depending only upon p. Below for convenience the symbol c_i will denote a positive constant depending only upon at most p.

On the other hand, we can also see that (see [1])

$$\|L_n\| \sim \ln(n+1). \tag{3}$$

From (1) and (3), together with (2), it seems reasonable to guess that for $1 < p < \infty$,

$$\|f - L_n(f)\|_{L^p} \leqslant c_p E_n(f)_p, \quad f \in C_{2\pi}.$$

Unfortunately, this is not true, as the following example shows.

THEOREM. *Let $1 < p < \infty$ and let $\{\lambda_n\}$ be a positive decreasing sequence of real numbers such that $n^s\lambda_n \to 0$ for any $s > 0$. Then there exists an infinitely differentiable function $f \in C_{2\pi}$ such that*

$$\overline{\lim_{n\to\infty}} \frac{\|f - L_n(f)\|_{L^p}}{\lambda_n^{-1}\|f - S_n(f)\|_{L^p}} > 0.$$

LEMMA 1. *Let $1 < p \leqslant 2$. Then there exists a trigonometric polynomial $g_n(x)$ such that*

$$\|g_n - L_n(g_n)\|_{L^p} \geqslant c\lambda_n^{-3}n^{1/q}, \quad 1/p + 1/q = 1, \tag{4}$$

$$\|g_n - S_n(g_n)\|_{L^p} = 0\left(\lambda_n^{-3/2}n^{1/2}\right). \tag{5}$$

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PROOF: Set

$$g_n(x) = \sum_{m=1}^{[\lambda_n^{-3}]} \sum_{k=m(2n+1)}^{m(2n+1)+n-1} \cos kx.$$

Since $\cos(m(2n+1)+j)x_k = \cos jx_k$ for $0 \leqslant j \leqslant n-1$ and $0 \leqslant k \leqslant 2n$, and $L_n(f,x) \in T_n$, $L_n(g_n, x) = \sum_{j=0}^{n-1} [\lambda_n^{-3}] \cos jx$. Applying the Hausdorff-Young inequality (see [**2**]) we have

$$\|g_n - L_n(g_n)\|_{L^p} \geqslant c_3 \left(\sum_{j=1}^{n-1} [\lambda_n^{-3}]^q \right)^{1/q} \geqslant c_4 \lambda_n^{-3} n^{1/q}.$$

On the other hand $\|g_n - S_n(g_n)\|_{L^p} = \|g_n\|_{L^p} \leqslant c_5 \|g_n\|_{L^2}$, so, from the Parseval equality, we have $\|g_n - S_n(g_n)\|_{L^p} = 0\left(\lambda_n^{-3/2} n^{1/2}\right)$, and we have proved (4) and (5). ▯

Similarly, with a slight change to $g_n(x)$, applying the Hölder inequality to $\|g_n - L_n(g_n)\|_{L^p}$, and the Hausdorff-Young inequality to $\|g_n - S_n(g_n)\|_{L^p}$, we can obtain the following lemma in the case of $2 < p < \infty$.

LEMMA 2. *Let $2 < p < \infty$. Then there exists a trigonometric polynomial $g_n^*(x)$ such that*

$$\|g_n^* - L_n(g_n^*)\|_{L^p} \geqslant c_2 \lambda_n^{-t} n^{1/2}, \tag{6}$$

$$\|g_n^* - S_n(g_n^*)\|_{L^p} = 0\left(\lambda_n^{-t/q} n^{1/q}\right), \qquad 1/p + 1/q = 1. \tag{7}$$

PROOF OF THE THOEREM: First suppose that $1 < p \leqslant 2$. Let $n_i = 8$, select n_{j+1} such that

$$\lambda_{n_{j+1}} \leqslant \lambda_{n_j}^2 \text{ and } n_{j+1} \geqslant \lambda_{n_j}^{-3}.$$

Define $f(x)$ by

$$f(x) = \sum_{j=1}^{\infty} \lambda_{n_j}^{n_j} g_{n_j}(x).$$

It is clear that $f \in C_{2\pi}$ is infinitely differentiable. Minkowski's inequality implies that

$$\|f - L_{n_k}(f)\|_{L^p} \geqslant \lambda_{n_k}^{n_k} \|g_{n_k} - L_{n_k}(g_{n_k})\|_{L^p} - \sum_{j=k+1}^{\infty} \lambda_{n_j}^{n_j} \sum_{m=1}^{[\lambda_n^{-3}]} \sum_{l=m(2n_j+1)}^{m(2n_j+1)+n_j-1} \|\cos lx - L_{n_k}(\cos lt, x)\|_{L^p},$$

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so, by (3) and (4),

$$(8)\quad \|f - L_{n_k}(f)_{L^p}\| \geqslant c_6 \lambda_{n_k}^{n_k - 3} n_k^{1/q} - 0\left(\sum_{j=k+1}^{\infty} \lambda_{n_j}^{n_j - 4}\right) \geqslant c_6 \lambda_{n_k}^{n_k - 3} n_k^{1/q} - 0\left(\lambda_{n_k}^{n_k}\right).$$

At the same time, from (5),

$$(9)\quad \|f - S_{n_k}(f)\|_{L^p} \leqslant \sum_{j=k}^{\infty} \lambda_{n_j}^{n_j} \|g_{n_j} - S_{n_k}\left(g_{n_j}\right)\|_{L^p} = 0\left(\lambda_{n_k}^{n_k - 3/2} n_k^{1/2}\right) = 0\left(\lambda_{n_k}^{n_k - 2}\right).$$

Combining (8) and (9) we get, for sufficiently large k,

$$\frac{\|f - L_{n_k}(f)\|_{L^p}}{\|f - S_{n_k}(f)\|_{L^p}} \geqslant c_7 \lambda_{n_k}^{-1} n_k^{1/q};$$

thus in this case the theorem is proved. For the case $2 < p < \infty$, taking $t = 3p/2$ and starting from (6) and (7), we can construct the function required in a similar way. The proof of the theorem is completed. ∎

In L^1 space there is also such a "bad result"; we will discuss it in another paper using a different method of construction.

References

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