The Brocard and Tucker Circles of a Cyclic Quadrilateral.

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1. The application of the geometrical properties of the Brocard and Tucker circles of a triangle to a quadrilateral appears never to have been adequately worked out, as far as the author can discover. Hence, the object of this paper.

Some of the problems involved have been published, under the author's name, as independent questions for solution, and where, in the author's opinion, solutions other than his own have seemed more satisfactory for the logical treatment of the subject, these solutions have been employed, with due acknowledgments to their authors.

2. *Condition for Brocard points.*

We shall first establish the condition necessary for the existence of Brocard points within a quadrilateral.

Let $ABCD$ (Fig. 1) be a quadrilateral in which a point X can be found such that the $\angle^s$ XAD, XBA, XCB, XDC are all equal; denote each of these angles by ω; the sides BC, CD, DA, AB by a, b, c, d; the diagonals BD, AC by e, f, and the area by Q; then

$$\angle AXB = \pi - \omega - (A - \omega) = \pi - A.$$

Similarly $\angle BXC = \pi - B$, $\angle CXD = \pi - C$, $\angle DXA = \pi - D$.

Now $AX : \sin\omega = AB : \sin AXB = d : \sin(\pi - A) = d : \sin A$

and $AX : \sin(D - \omega) = c : \sin(\pi - D) = c : \sin D$,

hence, eliminating AX by division

$$\sin(D - \omega) : \sin\omega = d \sin D : c \sin A,$$

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from which $\cot\omega = \frac{d}{c}\operatorname{cosec} A + \cot D.$

Similarly
$$\left.\begin{aligned}\cot\omega &= \frac{c}{b}\operatorname{cosec} D + \cot C,\\ &= \frac{b}{a}\operatorname{cosec} C + \cot B,\\ &= \frac{a}{d}\operatorname{cosec} B + \cot A,\end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

$$\therefore \quad \frac{d}{c}\operatorname{cosec} A + \cot D + \frac{b}{a}\operatorname{cosec} C + \cot B$$
$$= \frac{c}{b}\operatorname{cosec} D + \cot C + \frac{a}{d}\operatorname{cosec} B + \cot A = 2\cot\omega.$$

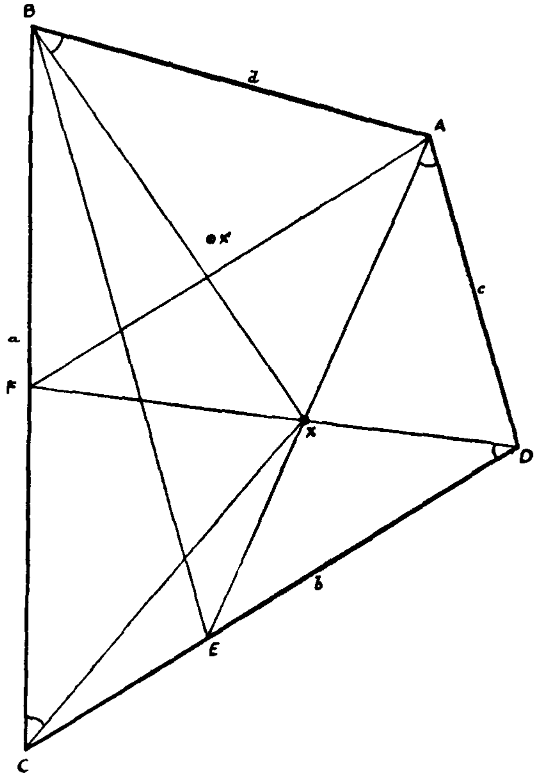

Fig. 1.

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Hence,

$$\frac{d}{c}\operatorname{cosec} A - \frac{a}{d}\operatorname{cosec} B + \frac{b}{a}\operatorname{cosec} C - \frac{c}{b}\operatorname{cosec} D = \cot A - \cot B + \cot C - \cot D \quad \ldots\ldots\ldots\ldots(2)$$

This is the general condition for all quadrilaterals in its most general form.*

3. *Reduction of Condition for any quadrilateral.*

Condition (2) involves the eight parts of the quadrilateral, but for given data, it will need reduction in order to express it in terms of the data. To effect this it is simpler to start with equations (1). Let us suppose, for example, that the four sides and the angle C are given, and the figure is subject to the limitation that it must be convex, then, since

$$2cd\cos A = c^2 + d^2 - e^2$$

and

$$e^2 = a^2 + b^2 - 2ab\cos C,$$

$\cos A$ and therefore $\angle A$ become known, there being only one value of $\cos A$ in the eliminant of e^2 since $A < \pi$.

Put $\operatorname{cosec} A = \alpha$, $\operatorname{cosec} C = \beta$, $\operatorname{cosec} B = x$, $\operatorname{cosec} D = y$ and $\cot\omega = k$, then α, β are known and x, y, k unknown constants. Equations (1) now become:—

$$k = \frac{d}{c}\cdot\alpha + \sqrt{y^2 - 1},$$

$$k = \frac{c}{b}\cdot y + \sqrt{\beta^2 - 1},$$

$$k = \frac{b}{a}\cdot\beta + \sqrt{x^2 - 1},$$

$$k = \frac{a}{d}\cdot x + \sqrt{\alpha^2 - 1},$$

The condition for Brocard points in terms of a, b, c, d and $\angle C$ is therefore the eliminant of x, y, k in the above equations.

* *Vide* the author's note in *Mathematical Gazette*, Vol. IX., pp. 83–85.

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The following method of elimination is due to Lt.-Col. Allan Cunningham, R.E.*

For shortness write $E = \sqrt{\beta^2 - 1}$, $F = \sqrt{\alpha^2 - 1}$, then the equations may be written

$$\text{(i)} \quad k - \frac{d}{c} \cdot \alpha = \sqrt{y^2 - 1}.$$

$$\text{(ii)} \quad (k - E) \cdot \frac{b}{c} = y,$$

$$\text{(iii)} \quad k - \frac{b}{a} \cdot \beta = \sqrt{x^2 - 1}.$$

$$\text{(iv)} \quad (k - F)\frac{d}{a} = x.$$

Squaring and taking the differences of (i) and (ii), and of (iii) and (iv), we get the two following equations independent of x and y :—

$$(k - E)^2 \cdot \frac{b^2}{c^2} - \left(k - \frac{d}{c}\alpha\right)^2 = 1\,; \quad (k - F)^2 \cdot \frac{d^2}{a^2} - \left(k - \frac{b}{a} \cdot \beta\right)^2 = 1,$$

which may be written as quadratics in k, thus

$$k^2(b^2 - c^2) + 2k(cd\alpha - b^2E) + (b^2E^2 - d^2\alpha^2 - c^2) = 0,$$

$$k^2(d^2 - a^2) + 2k(ab\beta - d^2F) + (d^2F^2 - b^2\beta^2 - a^2) = 0,$$

or

$$pk^2 + 2qk + r = 0,$$

$$p'k^2 + 2q'k + r' = 0,$$

the eliminant of which is

$$(pr' - p'r)^2 + 4(p'q - pq')(qr' - q'r) = 0, \quad \ldots\ldots\ldots\ldots (3)$$

which is therefore the required condition.

When the substitutions are made, however, the relation is so cumbersome that it is practically of very little value.

4. *Reduction of Condition for a cyclic quadrilateral.*

When the quadrilateral is inscribed in a circle, the condition for Brocard points becomes very simple, and many interesting analogies to the triangle are revealed.

Going back to (2), we have for a cyclic figure,

$$\operatorname{cosec} A = \operatorname{cosec} C, \quad \operatorname{cosec} B = \operatorname{cosec} D, \quad \cot A = -\cot C, \quad \cot B = -\cot D,$$

since

$$\angle A + \angle C = \angle B + \angle D = \pi,$$

* Mathematical Questions and Solutions (F. Hodgson, London), Vol. 2, p. 47.

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hence, (2) becomes

$$\left(\frac{d}{c}+\frac{b}{a}\right)\operatorname{cosec}A-\left(\frac{a}{d}+\frac{c}{b}\right)\operatorname{cosec}B=0$$

or $$bd(ad+bc)\sin B=ac(ab+cd)\sin A.$$

But $$2Q=(ab+cd)\sin A=(ad+bc)\sin B.$$

Hence, $$ac=bd,$$

and from Ptolemy's theorem, $ef=ac+bd$.

$$\therefore\quad ac=bd=\tfrac{1}{2}ef, \quad\text{.........................(4)}$$

the required condition.

We shall therefore confine the following investigation to a cyclic quadrilateral.

5. *Geometrical Proof of* (4).

This simple relation for Brocard points may also be established geometrically, and the following proof is based on one given by Mr W. F. Beard, M.A.*

Produce AX, DX (Fig. 1) to meet DC, BC in E, F respectively, join EB, FA, then

$$\angle BXA=\pi-A=\angle C$$

$$\angle XBA=\angle FDC.$$

$\therefore$ Δs BAX, FDC are similar,

$$\therefore\quad CD:DF=BX:AB$$

or $$AB\,.\,CD=DF\,.\,BX.$$

Again, $$\angle AXF=\text{supplement of }\angle AXD$$
$$=\text{supplement of }\pi-D=\angle D$$
$$=\text{supplement of }\angle B.$$

$\therefore$ A, B, F, X lie on a circle,

hence, $$\angle BFA=\angle BXA=\pi-A=\angle C.$$

$\therefore$ AF is parallel to DC.

Similarly BE is parallel to AD.

Now $$\angle DAX=\angle BCX$$
$$\angle BXC=\pi-B=\angle D.$$

$\therefore$ Δs ADE, BXC are similar.

* Mathematical Questions and Solutions (F. Hodgson, London), Vol. 3, pp. 2–3.

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and $AD : AE = XC : BC.$

$\therefore \quad AD \,.\, BC = AE \,.\, XC.$

Finally, $\angle BXC = \angle D$ and $\angle AXF = \angle D$, already proved.

$\therefore \quad \angle BXC = \angle AXF = \angle EXD$

and $\angle BCX = \angle EDX = \angle DFA, \quad \because \; AF \parallel DE.$

$\therefore \quad \triangle$s BXC, AFX, DXE are similar;

hence, $BX : XC = EX : XD = AX : XF = AE : DF.$

$\therefore \quad BX \,.\, DF = AE \,.\, XC;$

hence, $AB \,.\, CD = AD \,.\, BC.$

6. *Second Brocard Point.*

The point X has a corresponding one X' such that the angles $X'AB$, $X'BC$, $X'CD$, $X'DA$ are all equal, and it may be shown in a precisely similar manner as for X that the necessary condition for its existence is $ac = bd = \frac{1}{2}ef$.

Hence, when this condition is fulfilled, there are two Brocard points, just as in the case of a triangle.

Let $\omega' =$ each of the equal angles $X'AB$, etc., then, as in Art. 2, it may be shown that

$$\cot \omega' = \frac{c}{d} \,.\, \operatorname{cosec} A + \cot B.$$

But $\operatorname{cosec} A = \operatorname{cosec} C$, and from (4)

$$c : d = b : a.$$

$$\therefore \quad \cot \omega' = \frac{b}{a} \operatorname{cosec} C + \cot B = \cot \omega, \quad \text{from (1).}$$

$$\therefore \quad \omega' = \omega.$$

X and X' are therefore isogonal conjugates.

7. *Geometrical Construction for a Cyclic Quadrilateral having Brocard Points.*

A cyclic quadrilateral $ABCD$, such that $AB \,.\, CD = AD \,.\, BC$, may be constructed by the following general method.

Draw any straight line LL_1 (Fig. 2); take any point A in it and mark off $AL_2 = AL$. Through A draw any straight line QQ_1; join $QL\,QL_2$, and draw Q_1L_1 antiparallel to LQ with respect to $\angle Q_1AL_1$; hence, draw BD parallel to Q_1L_1 and equal to QL_2.

Describe the circumcircle to the $\triangle ABD$ and in it place a chord BC such that $\angle DBC = \angle AL_2Q$; join CD, then $ABCD$ is the required quadrilateral.

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Since A, B, C, D are concyclic points,

$$\therefore \quad \angle BCD = \text{supplement of } \angle BAD = \angle L_2AQ$$

$$\left.\begin{aligned} \text{and} \quad \angle DBC &= \angle AL_2Q \\ BD &= QL_2 \end{aligned}\right\} \text{By construction.}$$

$$\therefore \quad \triangle\text{s } BCD,\ QAL_2 \text{ are congruous.}$$

Hence, $BC = AL_2$ and $CD = AQ$.

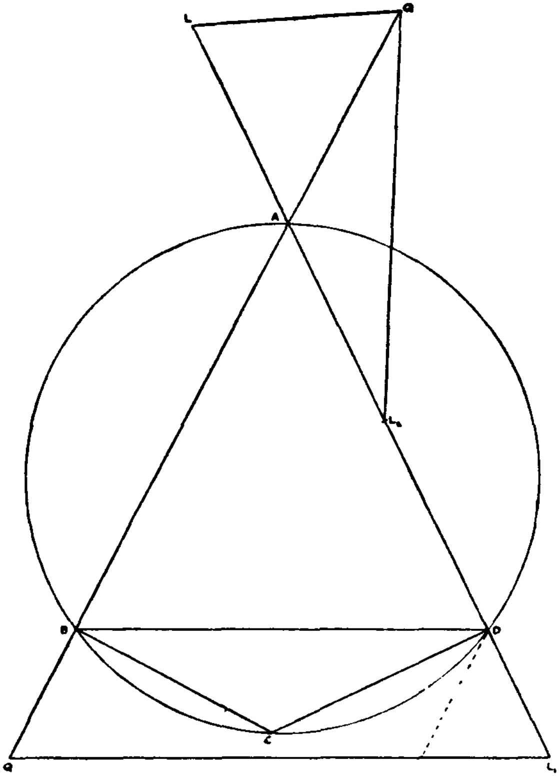

Fig. 2.

Again, since BD, LQ are antiparallels with respect to $\angle BAD$,

$$\therefore \quad L,\ Q,\ D,\ B \text{ are concyclic points}$$

$$\therefore \quad LA \,.\, AD = QA \,.\, AB.$$

But $LA = AL_2 = BC$ and $QA = CD$, already proved.

$$\therefore \quad BC \,.\, AD = CD \,.\, AB.$$

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This general construction may, with a little modification, be adapted to most cases of particular given conditions.

8. *Location of Brocard Points.*

As in the case of the triangle, X, X' may be located by Milne's construction, *i.e.*, by describing circles on AB, BC, CD, DA touching the sides AD, AB, BC, CD respectively; X is then their common point of intersection. Similarly if the circles described on AB, BC, CD, DA touch BC, CD, DA, AB respectively, then X' is their common point of intersection.

The geometrical proof given in Art. 5 affords, however, a much simpler construction, for when AX, DX are produced to cut CD, BC in E, F respectively (Fig. 1), then BE, AF are parallel respectively to AD, CD, hence, the following construction:

Draw from two consecutive angular points, A, B, AF, BE parallel to CD, AD respectively to cut BC in F and CD in E; join EA, FD and their point of intersection will be X.

Similarly by drawing from C, D lines parallel to AD, AB respectively to cut AB in E' and BC in F', then X' is the point of intersection of $E'D$, $F'A$.

9. *Important Formulae.*

The expressions $ad+bc$, $ab+cd$, $ac+bd$ are of such frequent occurrence in connection with the cyclic quadrilateral that it will facilitate the discussion if we give the forms they may assume when condition (4) is fulfilled.

$$ad+bc = \frac{acd+bc^2}{c} = \frac{bd^2+bc^2}{c} = \frac{b}{c}\,(c^2+d^2),$$

also

$$ad+bc = \frac{abd+b^2c}{b} = \frac{a^2c+b^2c}{b} = \frac{c}{b}\,(a^2+b^2).$$

$$\left.\begin{aligned} \therefore\quad ad+bc &= \frac{b}{c}\,(c^2+d^2) = \frac{c}{b}\,(a^2+b^2) \\ \text{Similarly}\qquad ab+cd &= \frac{c}{d}\,(a^2+d^2) = \frac{d}{c}\,(b^2+c^2) \\ \text{and from (4)}\qquad ac+bd &= 2ac = 2bd = ef \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots(5)$$

The following formulae are now given for reference in their usual forms. Modification by (5) will be made as necessary.

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Let R = radius of circumcircle of quadrilateral, then

$$\left.\begin{aligned}\sin A = \sin C = \frac{e}{2R}\\ \sin B = \sin D = \frac{f}{2R}\end{aligned}\right\}, \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(6)$$

and since $\qquad 2Q = (ab + cd)\sin A = (ad + bc)\sin B,$

$$\therefore\quad 4RQ = e(ab + cd) = f(ad + bc). \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(7)$$

Also

$$\left.\begin{aligned}e^2(ab + cd) = (ac + bd)(ad + bc) = 2ac(ad + bc)\\ f^2(ad + bc) = (ac + bd)(ab + cd) = 2ac(ab + cd)\end{aligned}\right\}. \quad\ldots\ldots\ldots(8)$$

Again, from (1),

$$2\cot\omega = \frac{d}{c}\operatorname{cosec} A + \cot D + \frac{b}{a}\operatorname{cosec} C + \cot B$$

$$= \frac{ad + bc}{ac}\,.\operatorname{cosec} A$$

$$= \frac{ab + cd}{bd}\operatorname{cosec} B, \text{ similarly.}$$

$$\therefore\quad 2ac\cot\omega = (ad + bc)\operatorname{cosec} A = (ab + cd)\operatorname{cosec} B. \quad\ldots\ldots\ldots(9)$$

Finally, from (6) and (7),

$$2ac\cot\omega = \frac{4RQ}{f}\,.\,\frac{2R}{e} = \frac{8R^2Q}{2ac}, \text{ from (4)},$$

$$\therefore\quad \cot\omega = \frac{2R^2Q}{a^2c^2}. \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(10)$$

Other expressions for the functions of ω will be found in Art. 12.

10. *Distances from angular points and coordinates of X, X′.*

From the triangle AXB (Fig. 3),

$$XA : \sin\omega = AB : \sin AXB = d : \sin A = 2Rd : e, \text{ from (6).}$$

$$\left.\begin{aligned}
\therefore\quad XA &= \frac{2Rd}{e}\,.\sin\omega; \text{ similarly, } & X'A &= \frac{2Rc}{e}\,.\sin\omega,\\
\text{and}\quad XB &= \frac{2Ra}{f}\,.\sin\omega; \quad ,, & X'B &= \frac{2Rd}{f}\,.\sin\omega,\\
XC &= \frac{2Rb}{e}\,.\sin\omega; \quad ,, & X'C &= \frac{2Ra}{e}\,.\sin\omega,\\
XD &= \frac{2Rc}{f}\,.\sin\omega; \quad ,, & X'D &= \frac{2Rb}{f}\,.\sin\omega
\end{aligned}\right\}\quad(11)$$

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From these values the following important geometrical results are readily deduced :

$$\left.\begin{aligned} &(a)\ XA \,.\, XB \,.\, XC \,.\, XD = X'A \,.\, X'B \,.\, X'C \,.\, X'D = 4R^4 \sin^4 \omega \\ &(b)\ XA \,.\, X'D = XB \,.\, X'A = XC \,.\, X'B = XD \,.\, X'C = 2R^2 \sin^2 \omega \\ &(c)\ \frac{XA \,.\, XC}{XB \,.\, XD} = \frac{X'A \,.\, X'C}{X'B \cdot X'D} = \frac{f^2}{e^2} = \frac{AC^2}{BD^2} = \frac{(ab+cd)^2}{(ad+bc)^2} \end{aligned}\right\} \quad (12)$$

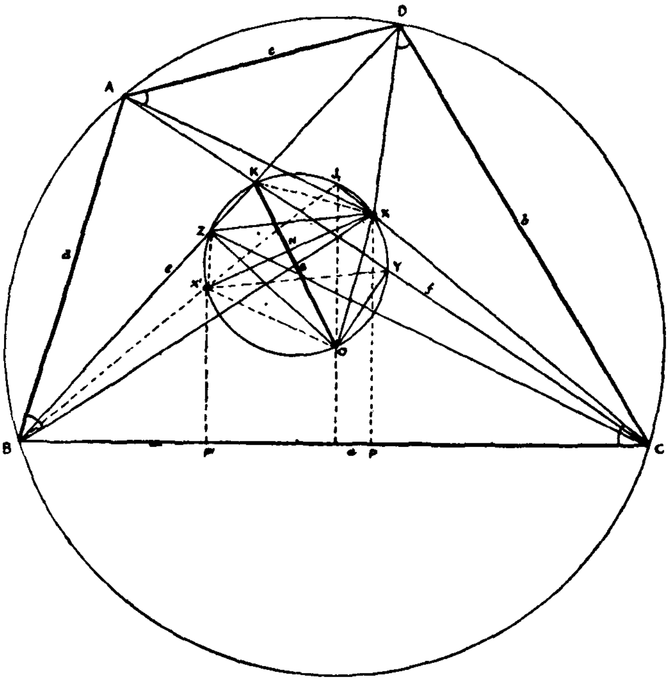

Fig. 3.

To determine the Cartesian co-ordinates of X, X' with reference to any of the sides, say BC, let P, P' be the feet of the perpendiculars from X, X' to BC, then

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$$CP = CX \cos\omega = \frac{2Rb}{e} \cdot \sin\omega \cdot \cos\omega, \quad \text{from (11)},$$

$$\left.\begin{aligned} &= \frac{Rb}{e} \sin 2\omega \\ PX = CX \sin\omega &= \frac{2Rb}{e} \sin^2\omega \\ \text{Similarly,} \qquad BP' &= \frac{Rd}{f} \cdot \sin 2\omega \\ P'X' &= \frac{2Rd}{f} \cdot \sin^2\omega \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots (13)$$

Like expressions may similarly be found with reference to each of the other sides.

From (13) and (4) we have

$$BP' \cdot CP = \tfrac{1}{2}R^2 \sin^2 2\omega \quad \text{and} \quad PX \cdot P'X' = 2R^2 \sin^4\omega, ;$$

these rectangles, being thus constant, are therefore the same for every side.

To find the distance between X and X', we have

$$\begin{aligned} XX'^2 &= (BP - BP')^2 + (PX - P'X')^2 \\ &= BX^2 + BX'^2 - 2(a - CP) \cdot BP' - 2PX \cdot P'X' \\ &= 4R^2 \sin^2\omega \left[\frac{a^2 + d^2}{f^2} - \frac{ad}{fR} \cot\omega + \cos^2\omega - \sin^2\omega\right] \quad \text{from (11) and (13)} \\ &= 4R^2 \sin^2\omega \cdot \cos 2\omega, \quad \text{from (6) and (9).} \end{aligned}$$

$$\therefore \; XX' = 2R \sin\omega \cdot \cos^{\frac{1}{2}} 2\omega. \quad \ldots\ldots\ldots\ldots\ldots\ldots (14)$$

11. *The Brocard Circle.*

Let O be the circumcentre, K the intersection of the diagonals AC, BD, and Y, Z their respective mid-points (Fig. 3). Join

$$OK, OY, OZ, OX, OX', XY, XZ, ZK, X'Y, X'Z.$$

Then, since $\angle KYO = \angle OZK = \frac{1}{2}\pi$,

$\therefore$ O, Y, K, Z, are concyclic.

Again, $\angle ZBC = \angle CAD$, and from (4),

$$a : \tfrac{1}{2}e = f : c,$$

or $BC : BZ = AC : AD$.

$\therefore$ $\triangle$s ZBC, ACD are similar;

hence, $\angle BZC = \angle ADC = \angle D = \angle BXC$.

$\therefore$ Z, X, C, B, are concyclic points.

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Similarly, it may be shown that $XYCD$, $X'YBC$, $X'ZDC$ are each cyclic.

Now, because $ZXCB$ is cyclic,

$$\therefore \quad \angle KZX = \text{supplement of } \angle BZX = \angle BCX = \omega.$$

Because $XYCD$ is cyclic,

$$\angle KYX = \text{supplement of } \angle XYC = \angle XDC = \omega.$$

$$\therefore \quad \angle KZX = \angle KYX;$$

hence, X, K, Z, Y are concyclic.

Similarly $\angle KYX'$ and $\angle KZX' = \pi - \omega$,

therefore X', K, Z, Y are concyclic,

hence, the six points O, Y, X, K, Z, X' lie on a circle whose diameter is OK; this is the Brocard Circle of the quadrilateral.

Let CX, BX' intersect at J_1, then because

$$\angle J_1CB = \angle J_1BC = \omega,$$

$$\therefore \quad J_1B = J_1C,$$

and J_1 lies on the perpendicular to BC at its mid-point,

$\therefore$ J_1O produced intersects BC at right angles at its mid-point.

$$\therefore \quad \angle OJ_1X = \angle OJ_1X' = \tfrac{1}{2}\pi - \omega,$$

$$\therefore \quad OJ_1 \text{ bisects } \angle X'J_1X,$$

and because $\angle XYO = \angle OYK + \angle KYX = \frac{1}{2}\pi + \omega$,

$$\therefore \quad \angle\text{s } OJ_1X,\ XYO \text{ are supplementary,}$$

$$\therefore \quad J_1 \text{ lies on the Brocard circle.}$$

Similarly if J_2 be the intersection of DX, CX', J_3 that of AX, DX', and J_4 that of BX, AX', it may be shown that J_2, J_3, J_4 lie on the Brocard circle.

Hence the Brocard circle passes through the following ten points:
- the two Brocard points (X, X'),
- the mid-points of the diagonals (Y, Z),
- the intersection of the diagonals (K),
- the circumcentre (O),
- the intersections of the joins of the Brocard points and the angular points of the quadrilateral, *i.e.*, the apices of the isosceles triangles having the sides as their respective bases and ω as the equal angles.

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12. *Functions of* ω *and its maximum value.*

Before proceeding to find the Brocard radius it is necessary to evaluate some of the functions of ω.

Let u_a, u_b, u_c, u_d be the lengths of the perpendiculars from K to the sides BC, CD, DA, AB respectively; then, since OK is the diameter of the Brocard circle,

$$\therefore \quad \angle KJ_1 O = \tfrac{1}{2}\pi.$$

$$\therefore \quad KJ_1 \text{ is parallel to } BC.$$

$$\therefore \quad u_a = \text{perpendicular from } J_1 \text{ to } BC = \tfrac{1}{2} a \,.\, \tan\omega.$$

Similarly, $u_b = \tfrac{1}{2} b \,.\, \tan\omega$, $u_c = \tfrac{1}{2} c \,.\, \tan\omega$, $u_d = \tfrac{1}{2} d \,.\, \tan\omega$.

Now the area of the quadrilateral

$$\begin{aligned} Q &= \triangle AKB + \triangle BKC + \triangle CKD + \triangle DKA \\ &= \tfrac{1}{2} a u_a + \tfrac{1}{2} b u_b + \tfrac{1}{2} c u_c + \tfrac{1}{2} d u_d \\ &= \tfrac{1}{4}(a^2 + b^2 + c^2 + d^2) \,.\, \tan\omega. \end{aligned}$$

$$\therefore \quad \cot\omega = \frac{\Sigma a^2}{4Q} \,. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (15a)$$

But $\quad \operatorname{cosec}^2\omega = 1 + \cot^2\omega = \dfrac{16Q^2 + (\Sigma a^2)^2}{16Q^2}$

$$\left.\begin{aligned} &= \frac{\Sigma a^2 b^2 + 2abcd}{4Q^2} \\ &= \frac{(ab + cd)^2 + (ad + bc)^2}{4Q^2} \quad \text{by (4)} \\ &= \operatorname{cosec}^2 A + \operatorname{cosec}^2 B \end{aligned}\right\} \ldots (15b)$$

Hence $\quad \cot^2\omega = 1 + \cot^2 A + \cot^2 B. \quad \ldots\ldots\ldots\ldots\ldots\ldots (15c)$

Again, $\quad \cos^2\omega = \cot^2\omega \,.\, \sin^2\omega = \dfrac{1}{4} \,.\, \dfrac{(\Sigma a^2)^2}{\Sigma a^2 b^2 + 2a^2 c^2} \quad \ldots\ldots\ldots\ldots (15d)$

since by (4), $abcd = a^2 c^2 = b^2 d^2$

$$\sin^2 2\omega = 4\sin^2\omega \,.\, \cos^2\omega = \frac{4(\Sigma a^2)^2 \,.\, Q^2}{(\Sigma a^2 b^2 + 2a^2 c^2)^2}$$

$$\therefore \quad \sin 2\omega = \frac{2 \,.\, \Sigma a^2 \,.\, Q}{\Sigma a^2 b^2 + 2a^2 c^2} \quad \ldots\ldots\ldots\ldots\ldots (15e)$$

$$\cos 2\omega = 2\cos^2\omega - 1$$

$$\left.\begin{aligned} &= \frac{\Sigma a^4 - 4a^2 c^2}{2(\Sigma a^2 b^2 + 2a^2 c^2)} \\ &= \frac{1}{2} \,.\, \frac{(a^2 - c^2)^2 + (b^2 - d^2)^2}{(ab + cd)^2 + (ad + bc)^2} \end{aligned}\right\} . \quad \ldots\ldots (15f)$$

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From (15*b*) we have

$$\operatorname{cosec}\omega = (\operatorname{cosec}^2 A + \operatorname{cosec}^2 B)^{\frac{1}{2}},$$

and since the right-hand side is the sum of two squares, it is always positive, hence cosec ω, and therefore ω, is always real.

Now the minimum value of either cosec A or cosec B is 1, hence cosec ω is not less than $\sqrt{2}$, *i.e.*, ω cannot exceed $\frac{\pi}{4}$ or 45°.

13. *Radius of the Brocard Circle.*

From Art. 11 we have

$$\angle KZX = \angle KYX = \angle KYX' = \omega \quad \text{(Fig. 3)}.$$

$$\therefore \quad KY \text{ bisects } \angle XYX'.$$

But $\angle KYX = \angle KOX$ in the same segment,

and $\angle KYX' = \angle KOX'$ „ „ „

$$\angle KOX = \angle KOX' = \omega,$$

and OK bisects $\angle XOX'$;

hence, since OK is a diameter,

$$\therefore \quad OX = OX', \quad KX = KX'$$

and XX' is perpendicular to OK.

Let XX' intersect OK at N, then

$$XN = NX' = \tfrac{1}{2}XX' = R\sin\omega \,.\, \cos^{\frac{1}{2}} 2\omega, \quad \text{from (14)}.$$

But from above, $\angle XON = \angle XOK = \omega$,

$$\angle ONX = \angle OXK = \tfrac{1}{2}\pi.$$

$$\therefore \quad \angle KXN = \angle XON = \omega.$$

Let β = radius of Brocard circle, then

$$\begin{aligned} 2\beta = OK &= ON + NK \\ &= XN(\cot\omega + \tan\omega) \\ &= 2R\sin\omega \,.\, \cos^{\frac{1}{2}} 2\omega \,.\, \operatorname{cosec} 2\omega, \quad \text{from above,} \\ &= R\sec\omega \,.\, \cos^{\frac{1}{2}} 2\omega \end{aligned}$$

$$\left.\begin{aligned} \therefore \quad \beta &= \tfrac{1}{2}R\sec\omega \,.\, \cos^{\frac{1}{2}} 2\omega \\ &= \frac{R(\Sigma a^4 - 4a^2c^2)^{\frac{1}{2}}}{\sqrt{2}\,.\,\Sigma a^2} \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots (16)$$

from (15*d*) and (15*f*).

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The distance from each of the Brocard points to the circumcentre may now easily be found, for

$$OX = NX \operatorname{cosec} XON$$
$$= R \sin\omega \,.\, \cos^{\frac{1}{2}} 2\omega \,.\, \operatorname{cosec}\omega$$
$$= R \cos^{\frac{1}{2}} 2\omega.$$

$$\left.\begin{aligned} \therefore \quad & OX = OX' = R\cos^{\frac{1}{2}} 2\omega. \\ \text{Similarly} \quad & XK = KX' = R\tan\omega \,.\, \cos^{\frac{1}{2}} 2\omega. \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots (17)$$

14. *Tucker Circles.*

Take any point A' (Fig. 4) in KA and through it draw $HA'H'$ parallel to AB to intersect BD in B', and $GA'G'$ parallel to AD to intersect BD in D'. Through D' draw $FD'F'$ parallel to CD to intersect AC in C'; join $B'C'$ and produce it in both directions to meet AB, CD in E, E' respectively, then

$$KB' : B'B = KA' : A'A = KD' : D'D = KC' : C'C.$$
$$\therefore \quad B'C' \text{ is parallel to } BC.$$

Join HG', EH', FE', GF', then

$$A'H : A'A = \sin A'AH : \sin A'HA = \sin CAD : \sin A'HD$$
$$= b : 2R\sin A,$$

since $2R\sin CAD = b$.

Similarly $\quad A'G' : A'A = a : 2R\sin A.$

Hence, by division $\quad A'H : A'G' = b : a$

$$= c : d \quad \text{from (4)}$$
$$= AD : AB$$
$$= A'D' : A'B'.$$
$$\therefore \quad A'H \,.\, A'B' = A'G' \,.\, A'D'.$$
$$\therefore \quad B', G', H, D' \text{ are concyclic;}$$

hence, $G'H$ is antiparallel to BD with respect to $\angle A$.

Similarly FE', $H'E$, $F'G$ are antiparallel to BD and AC with respect to $\angle$s C, B, D respectively.

Let KA, KB, KC, KD intersect HG', EH' FE' GF' in T_1, T_2, T_3, T_4 respectively, then, since $AG'A'H$ is a parallelogram,

$$\therefore \quad T_1 \text{ is the mid-point of } HG'.$$

$\therefore \quad KA$ is the symmedian of the $\triangle ABD$ with respect to $\angle A$.

Similarly KB, KC, KD are the symmedians of $\triangle$s ABC, CBD, ADC with respect to $\angle$s B, C, D respectively.

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Again, because HG' is an antiparallel to BD and O is the circumcentre of the $\triangle ABD$,

$$\therefore \quad OA \text{ is perpendicular to } HG'.$$

Draw $T_1 U$ parallel to OA to meet OK in U, then

$$UT_1 \text{ also is perpendicular to } HG'.$$

$$\therefore \quad UG'^2 = UT_1^2 + T_1 G'^2 = UT_1^2 + T_1 H^2 = UH^2.$$

$$\therefore \quad UG' = UH.$$

Similarly, $\quad UF' = UG,\ UE' = UF,\ UE = UH'.$

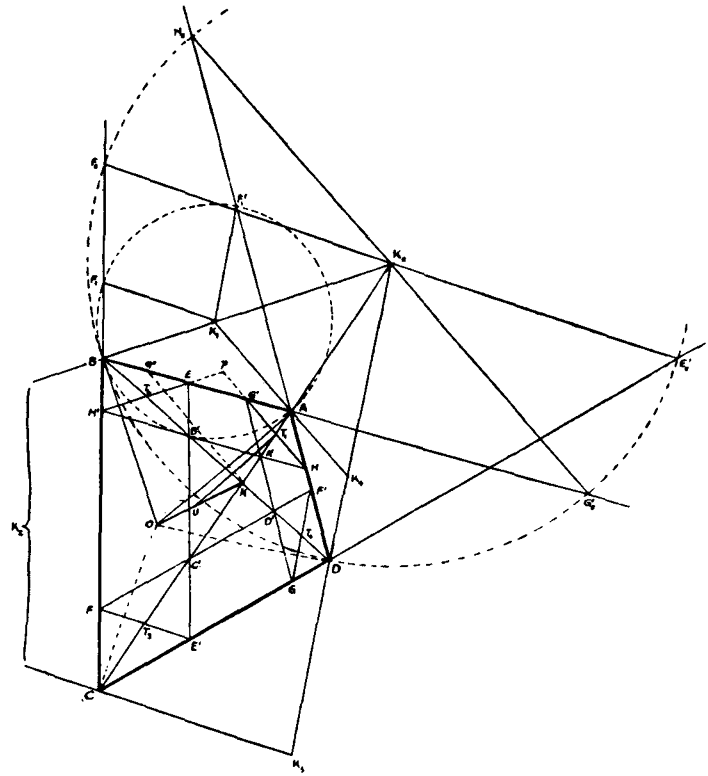

Fig. 4.

Let $H'E$, HG' produced intersect in p, then

$$\angle pHH' = \angle HG'A = \angle BDA = \angle BCA = \angle BEH' = \angle pH'H.$$

$$\therefore \quad pH' = pH,$$

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and since EG' is parallel to HH',

$$\therefore \quad pE = pG'.$$

Hence, $$EH' = G'H.$$

Similarly, $$EH' = E'F, \ E'F = FG, \ FG = HG'.$$

$$\therefore \quad HG' = EH' = FE' = GF'.$$

Further, $$UT_1 : OA = UK : OK = UT_2 : OB.$$

$$\therefore \quad UT_1 = UT_2.$$

Similarly, $$UT_2 = UT_3 = UT_4.$$

Hence T_1, T_2, T_3, T_4 lie on a circle whose centre is U, to which $E'F$, $H'E$, $G'H$, $F'G$ are tangents.

Since, however, these tangents have already been proved equal, $\therefore$ the eight points E, E', F, F', G, G', H, H' also lie on a circle whose centre is U. This is a Tucker Circle of the quadrilateral.

15. *Radius of a Tucker Circle.*

It is clear that after having once established the geometrical properties of the figure, as in the preceding article, we may take U to be any point in OK and then proceed to determine the radius of the Tucker Circle having U as its centre, by finding the points where it intersects the sides of the quadrilateral. OK is thus the locus of the centres of all the Tucker Circles. To find a general expression for their radii, let

$$OU : OK = \lambda : 1,$$

and let ρ = radius of the circle whose centre is K; then ρ is the semi-length of the antiparallels through K.

Now $$UT_1 : OA = UK : OK$$

or $$UT_1 : \ R = 1 - \lambda : 1.$$

$$\therefore \quad UT_1 = R\,(1 - \lambda).$$

This is the radius of the circle which passes through T_1, T_2, T_3, T_4.

Also $$G'T_1 : \rho = AT_1 : AK$$

$$= OU : OK$$

$$= \lambda : 1.$$

$$\therefore \quad G'T_1 = \lambda\rho.$$

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If therefore τ = radius of a Tucker Circle,

$$\tau^2 = UG''^2 = UT_1^2 + T_1 G''^2$$
$$= R^2 (1-\lambda)^2 + \lambda^2 \rho^2.$$

To completely define τ, however, it is necessary to find a value for the unknown ρ.

Now, if the parallel to HG' through K meets AB in G'', then the perpendicular from K to $AB = KG'' \sin AG''K = \rho \sin ADB = \frac{1}{2}\rho \cdot \frac{d}{R}$.

$$\therefore \quad 2Ru_d = \rho\, d$$

or
$$\rho = R \tan \omega. \quad \text{.........................(18)}$$

Putting this value in the above expression for τ, we get

$$\tau^2 = R^2 (1-\lambda)^2 + \lambda^2 R^2 \tan^2 \omega,$$
$$= R^2 (1 - 2\lambda + \lambda^2 \sec^2 \omega). \quad \text{....................(19)}$$

16. *Particular cases of Tucker Circles.*

When $\lambda = 1$, U is coincident with K, and from (19),

$$\tau = R \tan \omega = \rho.$$

This is therefore the radius of the Tucker Circle, which in the triangle is known as the Cosine Circle. It may, however, be appropriately called the Cosine Circle of the quadrilateral, since the intercepts made by it on the sides BC, CD, DA, AB are $2\rho \cos BAC$, $2\rho \cos CBD$, $2\rho \cos ACD$, $2\rho \cos ADB$ respectively.

When $\lambda = \frac{1}{2}$, U is the mid-point of OK and therefore coincident with the Brocard centre. The Tucker Circle having this point as its centre corresponds to the Lemoine or Triplicate Ratio Circle of the triangle. If τ' be its radius, then putting $\lambda = \frac{1}{2}$ in (19), we have
$$\tau' = \tfrac{1}{2} R \sec \omega. \quad \text{.........................(20)}$$

The intercepts made on the sides are not, however, directly proportional to the simple cubes of the sides, for

$$H'F = u_a (\cot B + \cot C) = \tfrac{1}{2} a (\cot B + \cot C) \tan \omega,$$

or
$$H'F = a - EE' = a - EK - E'K = a - u_d \operatorname{cosec} B - u_b \operatorname{cosec} C$$
$$= a - \tfrac{1}{2} (d \operatorname{cosec} B + b \operatorname{cosec} C) \tan \omega$$
$$= a \left(1 - \frac{cd}{ab+cd} - \frac{bc}{ad+bc}\right), \quad \text{from (9)}$$
$$= \frac{a\,(a^2 - c^2)}{\Sigma a^2} \quad \text{from (4).}$$

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Hence, $H'F = \frac{1}{2}a\,(\cot B + \cot C)\,.\tan\omega = \dfrac{a\,(a^2 - c^2)}{\Sigma a^2}$,

Similarly,

$$E'G = \tfrac{1}{2}b\,(\cot C + \cot D)\,.\tan\omega = \frac{b\,(d^2 - b^2)}{\Sigma a^2}\,,$$

$$HF' = \tfrac{1}{2}c\,(\cot B + \cot C)\,.\tan\omega = \frac{c\,(a^2 - c^2)}{\Sigma a^2}\,,$$

$$EG' = \tfrac{1}{2}d\,(\cot C + \cot D)\,.\tan\omega = \frac{d\,(d^2 - b^2)}{\Sigma a^2}\,,$$

$$\therefore\quad H'F : E'G : HF' : EG' = a^3 - ac^2 : bd^2 - b^3 : a^2c - c^3 : d^3 - b^2d$$
$$= a^3 - bcd : acd - b^3 : abd - c^3 : d^3 - abc.$$

When $\lambda = 0$, U is coincident with O and $\tau = R$; the Tucker Circle thus becomes the circumcircle.

17. *Ex-Cosine Circles.*

Let the tangents to the circumcircle drawn at A, B, C, D intersect at K_1, K_2, K_3, K_4 respectively.

Through K_1 draw K_1F_1 parallel to $E'F$ to meet CB produced in F_1, and K_1F_1' parallel to GF' to meet DA produced in F_1', then

$$\angle K_1F_1B = \angle E'FC = \angle BDC = \angle BAC = \angle BH'E = \angle K_1BF,$$

since $K_1B \parallel EH'$, both being perpendicular to OB.

$$\therefore\quad K_1F_1 = K_1B.$$

Similarly, $$K_1F_1' = K_1A.$$

But $\angle K_1AO = \angle K_1BO = \frac{1}{2}\pi$, and $OA = OB$,

$$\therefore\quad K_1A = K_1B,$$

hence, $$K_1A = K_1B = K_1F_1 = K_1F_1'.$$

$\therefore$ A, B, F_1, F_1' lie on a circle whose centre is K_1.

Let ρ_1 = radius of this circle, then

$$d = AB = 2\rho_1\cos K_1AB = 2\rho_1\cos AG'H = 2\rho_1\cos ADB.$$

This circle is thus an ex-cosine circle of the quadrilateral.

Again, $$2\rho_1\cos ADB = d = 2R\sin ADB,$$

$$\therefore\quad \rho_1 = R\tan ADB = R\tan ACB$$

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Similarly, it may be shown that there are three other ex-cosine circles having K_2, K_3, K_4 as their centres. Let ρ_2, ρ_3, ρ_4 be their radii, then

$$\left.\begin{aligned}\rho_1^2 &= R^2 \tan^2 ACB = \frac{R^2 d^2}{4R^2 - d^2}\\ \rho_2^2 &= R^2 \tan^2 BDC = \frac{R^2 a^2}{4R^2 - a^2}\\ \rho_3^2 &= R^2 \tan^2 CAD = \frac{R^2 b^2}{4R^2 - b^2}\\ \rho_4^2 &= R^2 \tan^2 ABD = \frac{R^2 c^2}{4R^2 - c^2}\end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots(21)$$

There are, however, two other ex-cosine circles having the diagonals as their respective chords. Let the tangents at B, D intersect at K_e; through K_e draw $H_e G_e'$ parallel to HG' to meet DA, BA produced in H_e, G_e', and draw $E_e'F_e$ parallel to $E'F$ to meet CD, CB produced in E_e', F_e. Then it may readily be shown that $K_e B = K_e F_e = K_e H_e = K_e E_e' = K_e G_e' = K_e D$. Hence, B, F_e, H_e, E_e', G_e', D lie on a circle whose centre is K_e.

Let ρ_e be its radius, then

$$e = BD = 2\rho_e \cos K_e BD = 2\rho_e \cos C.$$

But $\quad e = 2R \sin C.$

$$\therefore \quad \rho_e = R \tan C.$$

Similarly it may be shown that there is another circle whose centre K_f is the point of intersection of the tangents at A and C: if ρ_f is its radius, then

$$\rho_e = R \tan C, \quad \rho_f = R \tan B. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(22)$$

There are thus six ex-cosine circles to the quadrilateral.

The points K_e, K_f bear also another significance, since $F_e E_e$ is parallel to FE' and is bisected at K_e.

$$\therefore \quad CA \text{ and } AK_e \text{ are collinear.}$$

Now $\quad AT_1 : AK = OU : OK = CT_3 : CK.$

$$\therefore \quad AT_1 : CT_3 = AK : CK.$$

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Again, $T_1 H : K_e G_e' = AT_1 : AK_e$

and $T_3 E' : K_e E_e' = CT_3 : CK_e$.

But $T_1 H = T_3 E'$ (Art. 14), and $K_e G_e' = K_e E_e'$.

$\therefore$ By division $AT_1 : AK_e = CT_3 : CK_e$.

or $AK_e : CK_e = AT_1 : CT_3$

$= AK : CK$, from above,

neglecting signs.

Hence C, K, A, K_e form a harmonic range, and therefore K_e lies on the diagonal joining the intersection of BA, CD produced with that of CB, DA produced. Similarly, K_f is the intersecting point of the diagonal through K_e and BDprodúced; thus the $\triangle KK_eK_f$ is the diagonal triangle of the complete quadrilateral formed by the four straight lines AB, BC, CD, DA.

18. *Some relations between radii.*

From (20), $\tau'^2 = \frac{1}{4}R^2 \sec^2 \omega$

$= \frac{1}{4}R^2 (1 + \tan^2 \omega)$

$= \frac{1}{4}(R^2 + \rho^2)$, from (18). ....(23)

From (16), $\beta^2 = \frac{1}{4}R^2 \sec^2 \omega \,.\, \cos 2\omega$

$= \frac{1}{4}R^2 (2 - \sec^2 \omega)$

$= \frac{1}{4}R^2 (1 - \tan^2 \omega)$

$= \frac{1}{4}(R^2 - \rho^2)$, from (18). ....(24)

Subtracting (24) from (23), we have

$\tau'^2 = \frac{1}{2}\rho^2 + \beta^2$. ....(25)

From (15a), $\cot^2 \omega = \dfrac{(\Sigma a^2)^2}{16Q^2}$,

and from (18),

$\rho^2 (\Sigma a^2)^2 = 16R^2 Q^2$

$= e^2 (ab + cd)^2$, from (7),

$= 2ac(ad + bc)(ab + cd)$, from (8),

$= 2a^2 c^2 \,.\, \Sigma a^2$, from (4)

$\therefore \quad \rho^2 \,.\, \Sigma a^2 = 2a^2 c^2 = 2b^2 d^2 = 2abcd$. ....(26)

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From Fig. 3 we have

$$a = u_a(\cot DBC + \cot ACB).$$

$$\therefore \quad \cot\omega = \tfrac{1}{2}(\cot DBC + \cot ACB).$$

But from (21), $R\left(\dfrac{1}{\rho_1} + \dfrac{1}{\rho_3}\right) = \cot ACB + \cot CAD$

$$= \cot ACB + \cot DBC$$

$$= 2\cot\omega,$$

$$= \frac{2R}{\rho}, \quad \text{from (18)},$$

$$\therefore \quad \frac{1}{\rho_1} + \frac{1}{\rho_3} = \frac{2}{\rho}.$$

Similarly, $\dfrac{1}{\rho_2} + \dfrac{1}{\rho_4} = \dfrac{2}{\rho}.$

$$\therefore \quad \frac{1}{\rho_1} + \frac{1}{\rho_3} = \frac{1}{\rho_2} + \frac{1}{\rho_4} = \frac{2}{\rho}. \quad \text{......................(27)}$$

Again, from (22),

$$R^2\left(\frac{1}{\rho_e^2} + \frac{1}{\rho_f^2}\right) = \cot^2 C + \cot^2 B$$

$$= \cot^2 A + \cot^2 B, \text{ for } \cot C = -\cot A$$

$$= \cot^2\omega - 1, \quad \text{from (15}c\text{)},$$

$$= \frac{R^2}{\rho^2} - 1$$

$$= \frac{R^2 - \rho^2}{\rho^2} = \frac{4\beta^2}{\rho^2}, \quad \text{from (24)},$$

$$\therefore \quad \frac{1}{\rho_e^2} + \frac{1}{\rho_f^2} = \frac{1}{\rho^2} - \frac{1}{R^2} = \frac{4\beta^2}{R^2\rho^2}. \quad \text{..................(28)}$$

19. *Comparison of Formulae.*

To summarise the analogy, the principal formulae for both the triangle and the cyclic quadrilateral are here collected and placed side by side.

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TRIANGLE.

$$XX' = 2R\sin\omega\,.\,(2\cos 2\omega - 1)^{\frac{1}{2}}$$

$$\omega < 30^\circ$$

$$\cot\omega = \Sigma\cot A = \frac{\Sigma a^2}{4\triangle}$$

$$\operatorname{cosec}^2\omega = \frac{\Sigma a^2 b^2}{4\triangle^2} = \Sigma\operatorname{cosec}^2 A$$

$$\cos^2\omega = \frac{(\Sigma a^2)^2}{4\,\Sigma a^2 b^2}$$

$$\sin 2\omega = \frac{2\triangle\,.\,\Sigma a^2}{\Sigma a^2 b^2}$$

$$\cos 2\omega = \frac{\Sigma a^4}{2\,\Sigma a^2 b^2}$$

$$\beta = \tfrac{1}{2}R\sec\omega\,.\,(2\cos 2\omega - 1)^{\frac{1}{2}}$$

$$\rho = R\tan\omega$$

$$\tau^2 = R^2(1 - 2\lambda + \lambda^2\sec^2\omega)$$

$$\rho_a = R\tan A$$

$$\tau'^2 = \tfrac{1}{4}(R^2 + \rho^2) = \tfrac{1}{4}R^2\sec^2\omega$$

$$\beta^2 = \tfrac{1}{4}(R^2 - 3\rho^2) = \tau'^2 - \rho^2$$

$$\Sigma\frac{1}{\rho_a} = \frac{1}{\rho}.$$

CYCLIC QUADRILATERAL.

$$ac = bd.$$

$$XX' = 2R\sin\omega\,.\,\cos^{\frac{1}{2}} 2\omega$$

$$\omega < 45^\circ$$

$$\cot^2\omega = 1 + \cot^2 A + \cot^2 B$$

$$\cot\omega = \frac{\Sigma a^2}{4Q}$$

$$\operatorname{cosec}^2\omega = \frac{\Sigma a^2 b^2 + 2a^2 c^2}{4\,Q^2} = \operatorname{cosec}^2 A + \operatorname{cosec}^2 B$$

$$\cos^2\omega = \frac{(\Sigma a^2)^2}{4\,(\Sigma a^2 b^2 + 2a^2 c^2)}$$

$$\sin 2\omega = \frac{2\,Q\,.\,\Sigma a^2}{\Sigma a^2 b^2 + 2a^2 c^2}$$

$$\cos 2\omega = \frac{\Sigma a^4 - 4a^2 c^2}{2\,(\Sigma a^2 b^2 + 2a^2 c^2)}$$

$$\beta = \tfrac{1}{2}R\sec\omega\,.\,\cos^{\frac{1}{2}} 2\omega$$

$$\rho = R\tan\omega$$

$$\tau^2 = R^2(1 - 2\lambda + \lambda^2\sec^2\omega)$$

$$\rho_c = R\tan C$$

$$\tau'^2 = \tfrac{1}{4}(R^2 + \rho^2) = \tfrac{1}{4}R^2\sec^2\omega$$

$$\beta^2 = \tfrac{1}{4}(R^2 - \rho^2) = \tau'^2 - \tfrac{1}{2}\rho^2$$

$$\Sigma\frac{1}{\rho_1} = \frac{4}{\rho}.$$

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