

REFLECTION OF STATIONARY SETS AND THE TREE PROPERTY AT THE SUCCESSOR OF A SINGULAR CARDINAL

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Abstract. We show that from infinitely many supercompact cardinals one can force a model of ZFC where both the tree property and the stationary reflection hold at \aleph_{ω^2+1} .

§1. Introduction. One of the most fruitful research areas in set theory concerns the investigation of models of set theory where properties of large cardinals hold at small cardinals. The results presented in this paper focus on two properties of weakly compact cardinals that, under large cardinal assumptions, can be forced at small cardinals, the tree property and the reflection of stationary sets. To get a model of the tree property at the double successor of a regular cardinal it is enough to assume the consistency of a weakly compact cardinal and force with a classical poset due to Mitchell (see [6]). Forcing the tree property at the successor of a singular cardinal is harder and requires much stronger assumptions. The first model of the tree property at the successor of a singular cardinal, was defined by Magidor and Shelah in [4] who proved from large cardinals the consistency of the tree property at $\aleph_{\omega+1}$. The hypotheses used in such result have the consistency strength of a large cardinal between a huge cardinal and a 2-huge cardinal; this was later improved by Sinapova [10] who was able to force the tree property at $\aleph_{\omega+1}$ from weaker large cardinal assumptions, namely assuming the consistency of infinitely many supercompact cardinals. Another paper by Neeman [7] shows that, assuming the consistency of infinitely many supercompact cardinals, one can force a model where the tree property holds simultaneously at $\aleph_{\omega+1}$ and at every \aleph_n with $n \geq 2$. All these constructions can be adapted to force the tree property at the successor of any singular cardinal κ of countable cofinality, however in all these models the reflection of stationary sets fails at κ^+ . In the case of $\aleph_{\omega+1}$, for instance, these forcing constructions all add a bad scale. So it is natural to ask whether the tree property and the reflection of stationary sets are incompatible at the successor of a singular cardinal. In this paper we answer this question for a particular cardinal, namely \aleph_{ω^2+1} . We show that, assuming the consistency of infinitely many supercompact cardinals, one can force a model where both the tree property and the stationary

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set reflection hold at \aleph_{ω^2+1} . Whether the same can be proven for $\aleph_{\omega+1}$ remains an open problem.

We will use a forcing construction due to Magidor and Shelah [5] that was introduced to define a model where \aleph_{ω^2+1} satisfies a strong reflection principle, denoted $\Delta_{\aleph_{\omega^2}, \aleph_{\omega^2+1}}$. The principle is defined as follows.

DEFINITION 1.1. Given two cardinals $\kappa < \lambda$, $\Delta_{\kappa, \lambda}$ is the statement that for every cardinal $\mu < \kappa$, for every stationary set $S \subseteq E_{< \kappa}^\lambda := \{\alpha < \lambda; \text{cof}(\alpha) < \kappa\}$ and for every algebra A on λ with μ operations, there exists a subalgebra A' of order type a regular cardinal $\eta < \kappa$ such that $S \cap A'$ is stationary in $\text{sup}(A')$.

Let us denote by $\Delta_{< \lambda, \lambda}$ the principle $\forall \kappa < \lambda \Delta_{\kappa, \lambda}$. Under this principle it is possible to prove several ‘compactness’ results, namely theorems where, given a structure of size λ , properties of substructures of size $\leq \kappa$ imply a global property for the whole structure. For instance, assuming $\Delta_{< \lambda, \lambda}$ one can prove that every almost free Abelian group of size λ is free (where ‘almost free’ means that every subgroup of smaller size is free). Magidor and Shelah proved in [5] that assuming the consistency of infinitely many supercompact cardinals, there exists a model of ZFC + GCH where $\Delta_{\aleph_{\omega^2}, \aleph_{\omega^2+1}}$ holds. The results presented in [5] combined with other previous results (see [1], [8], [9]) showed also that \aleph_{ω^2+1} is the smallest regular cardinal λ that can consistently satisfy $\Delta_{< \lambda, \lambda}$. In this paper we prove that in the Magidor–Shelah’s model, \aleph_{ω^2+1} satisfies even the tree property.

THEOREM 1.2. *Assuming the consistency of infinitely many supercompact cardinals, there exists a model of ZFC where both $\Delta_{\aleph_{\omega^2}, \aleph_{\omega^2+1}}$ and the tree property at \aleph_{ω^2+1} hold.*

The principle $\Delta_{\kappa, \lambda}$ expresses a strong form of reflection, in particular $\Delta_{< \lambda, \lambda}$ implies the reflection of stationary subsets of λ . It follows from Theorem 1.2 that, assuming the consistency of infinitely many supercompact cardinals, \aleph_{ω^2+1} can consistently satisfy both the tree property and $\Delta_{\aleph_{\omega^2}, \aleph_{\omega^2+1}}$, hence the stationary set reflection.

In the second part of this paper we show that $\Delta_{\aleph_{\omega^2}, \aleph_{\omega^2+1}}$ does not imply the tree property at \aleph_{ω^2+1} . More precisely, we prove the following theorem.

THEOREM 1.3. *Assuming the consistency of infinitely many supercompact cardinals, we can force a model of ZFC where $\Delta_{\aleph_{\omega^2}, \aleph_{\omega^2+1}}$ holds while the tree property fails at \aleph_{ω^2+1} .*

The paper is organized as follows. In Section 2 we recall some classical results that will be used repeatedly in the proofs of Theorems 1.2 and 1.3. In Sections 3 and 4 we present Magidor–Shelah’s forcing construction for building a model of $\Delta_{\aleph_{\omega^2}, \aleph_{\omega^2+1}}$. In Section 5 we prove that in the Magidor–Shelah’s model the tree property holds at \aleph_{ω^2+1} , so we prove Theorem 1.2. Finally, Section 6 is devoted to the proof of Theorem 1.3.

§2. Preliminaries. In this section we list some classical results about forcing that preserve Aronszajn trees. We recall that a κ -Aronszajn tree is a κ -tree with no cofinal branches. To simplify the notation, we will always assume that our κ^+ -trees are subsets of $\kappa^+ \times \kappa$, and that for every $\alpha < \kappa^+$ the α -th level of the tree $\text{Lev}_\alpha(T)$ is a subset of $\{\alpha\} \times \kappa$.

A simple argument shows that if T is a κ -Aronszajn tree and \mathbb{P} is a κ -Knaster forcing notion, then T remains Aronszajn after forcing with \mathbb{P} : otherwise, if \dot{b} is a

name for a cofinal branch on T , then we can pick for every $\alpha < \kappa$ a condition p_α deciding the value of $\dot{b} \cap \alpha$. By the Knaster property there exists a cofinal subset $I \subseteq \kappa$ such that the conditions in the sequence $\langle p_\alpha; \alpha \in I \rangle$ are pairwise compatible. Say that $p_\alpha \Vdash \dot{b} \cap \alpha = t_\alpha$ for each α , then $\{t_\alpha; \alpha \in I\}$ is a cofinal branch for T , contradicting T is κ -Aronszajn. The following lemma by Unger show that for a forcing \mathbb{P} to preserve κ -Aronszajn trees it is enough that $\mathbb{P} \times \mathbb{P}$ has the κ -chain condition.

LEMMA 2.1 (Unger [11]). *Let \mathbb{P} be a forcing notion such that $\mathbb{P} \times \mathbb{P}$ is κ -c.c. Then \mathbb{P} has the κ -approximation property, i.e., given a set of ordinals A in a \mathbb{P} -generic extension $V[G]$, if $A \cap x \in V$ for every $x \in V$ of size $< \kappa$, then $A \in V$. In particular, if T is a κ -tree and $\mathbb{P} \times \mathbb{P}$ is κ -c.c., then forcing with \mathbb{P} does not add cofinal branches to T .*

PROOF. Suppose for a contradiction that for some ordinal τ there exists a \mathbb{P} -name \dot{A} such that

$$\Vdash_{\mathbb{P}} \dot{A} \subseteq \tau, \forall x \in [\tau]^{<\kappa} (\dot{A} \cap x \in V) \text{ and } \dot{A} \notin V.$$

We inductively define conditions $\langle p_i^0, p_i^1 \rangle_{i < \kappa}$ in $\mathbb{P} \times \mathbb{P}$, sets $\langle d_i^0, d_i^1 \rangle_{i < \kappa}$ in $[\tau]^{<\kappa}$ and a \subseteq -strictly increasing sequence $\langle x_i \rangle_{i < \kappa}$ in $[\tau]^{<\kappa}$ such that

- (1) for $\varepsilon \in \{0, 1\}$, $p_i^\varepsilon \Vdash \dot{A} \cap x_i = d_i^\varepsilon$;
- (2) $d_i^0 \neq d_i^1$ and $d_i^0 \cap (\bigcup_{j < i} x_j) = d_i^1 \cap (\bigcup_{j < i} x_j)$.

Suppose we have constructed $\langle p_i^0, p_i^1 \rangle_{i < j}$, $\langle d_i^0, d_i^1 \rangle_{i < j}$ and $\langle x_i \rangle_{i < j}$ successfully. Let $x := \bigcup_{i < j} x_i$ and let p be any condition in \mathbb{P} deciding the value of $\dot{A} \cap x$ to be $d \in [\tau]^{<\kappa}$. As \dot{A} does not belong to V , we can find $p_j^0, p_j^1 \leq p$, distinct d_j^0, d_j^1 and $x_j \supset x$ such that $p_j^\varepsilon \Vdash \dot{A} \cap x_j = d_j^\varepsilon$ for $\varepsilon \in \{0, 1\}$. Then $p_j^\varepsilon \Vdash d = \dot{A} \cap x = d_j^\varepsilon \cap x$ hence $d_j^0 \cap x = d = d_j^1 \cap x$.

Now we claim that $\langle p_i^0, p_i^1 \rangle_{i < \kappa}$ is an antichain, contradicting the κ -chain condition at $\mathbb{P} \times \mathbb{P}$. Suppose that for some $i < j$, the conditions (p_i^0, p_i^1) and (p_j^0, p_j^1) are compatible. Then $d_j^0 \cap x_i = d_i^0$ and $d_j^1 \cap x_i = d_i^1$. By construction $d_j^0 \cap (\bigcup_{l < j} x_l) = d_j^1 \cap (\bigcup_{l < j} x_l)$, in particular $d_j^0 \cap x_i = d_j^1 \cap x_i$, contradicting $d_i^0 \neq d_i^1$. \dashv

The following results by Magidor and Shelah shows that forcings with σ -closure preserve Aronszajn trees over successors of singular cardinals of countable cofinality.

LEMMA 2.2 (Magidor and Shelah [4, Theorem 5.2]). *Let λ be a singular cardinal of countable cofinality and let T be a λ^+ -tree. Assume that \mathbb{P} is a σ -closed poset, then forcing with \mathbb{P} does not add cofinal branches to T .*

PROOF. Let \dot{b} be a \mathbb{P} -name for a cofinal branch of T , and suppose for a contradiction that such a branch is not in V . We can inductively define for every $n < \omega$, conditions $\langle p_s; s \in {}^n \lambda \rangle$ and ordinals $\langle \alpha_s; s \in {}^n \lambda \rangle$ in λ^+ such that

- (1) $s \sqsubseteq t$ implies $p_t \leq p_s$,
- (2) $p_{s \smallfrown 0}$ and $p_{s \smallfrown 1}$ force contradictory information about $\dot{b} \cap \text{Lev}_{\alpha_s}(T)$.

Then, we let α be the limit of all α_s and, for each $f \in {}^\omega \lambda$, we let p_f be a lower bound for the sequence $\langle p_{f \upharpoonright n}; n < \omega \rangle$ that decides the value of $\dot{b} \cap \text{Lev}_\alpha(T)$ as x_f . By construction the x_f 's are pairwise distinct. This implies that $\text{Lev}_\alpha(T)$ has size λ^ω , a contradiction. \dashv

When we work with a κ^+ -tree \dot{T} in a generic extension by some forcing notion \mathbb{P} , it is often useful to consider the following relations $<_p$ on $\kappa^+ \times \kappa$ where $p \in \mathbb{P}$: for $(\alpha, \zeta), (\beta, \eta) \in \kappa^+ \times \kappa$, we let $(\alpha, \zeta) <_p (\beta, \eta)$ when $p \Vdash (\alpha, \zeta) <_T (\beta, \eta)$. This lead us to the notion of *system* introduced by Magidor and Shelah in [5].

DEFINITION 2.3 (Magidor and Shelah [5]). Let D be a set of ordinals and τ a cardinal. A *system* over $D \times \tau$ is a collection of transitive, reflexive relations $\{R_i\}_{i \in I}$ on $D \times \tau$ such that:

- (1) if $(\alpha, \zeta) R_i (\beta, \eta)$ and $(\alpha, \zeta) \neq (\beta, \eta)$, then $\alpha < \beta$;
- (2) if (α_0, ζ_0) and (α_1, ζ_1) are both below (β, η) in R_i , then (α_0, ζ_0) and (α_1, ζ_1) are comparable in R_i (by condition (1) this implies that $(\alpha_0, \zeta_0) R_i (\alpha_1, \zeta_1)$ if $\alpha_0 < \alpha_1$, $(\alpha_1, \zeta_1) R_i (\alpha_0, \eta_0)$ if $\alpha_1 < \alpha_0$, and $\zeta_0 = \zeta_1$ if $\alpha_0 = \alpha_1$);
- (3) for every $\alpha < \beta$ both in D , there is $i \in I$ and $\zeta, \eta \in \tau$ such that $(\alpha, \zeta) R_i (\beta, \eta)$.

If $\mathcal{R} := \{R_i\}_{i \in I}$ is a system over $D \times \tau$, then every element of $D \times \tau$ is a *node* of \mathcal{R} . For every $\alpha \in D$, the α -th level of \mathcal{R} , denoted $\text{Lev}_\alpha(\mathcal{R})$, is the set $\{\alpha\} \times \kappa$.

DEFINITION 2.4 (Sinapova [10]). Let $\{R_i\}_{i \in I}$ be a system on $D \times \tau$. A *branch* through some R_i is a partial function from $b : D \rightarrow \tau$ such that for any $\beta \in \text{dom}(b)$ and any $\alpha < \beta$ in D , $\alpha \in \text{dom}(b)$ if and only if there exists ζ such that $(\alpha, \zeta) R_i (\beta, b(\beta))$ and $b(\alpha)$ is equal to the unique ζ witnessing this (ζ is unique by condition (2) of the definition of system). We say that b is a *cofinal* if $\text{dom}(b)$ is cofinal in D .

Sinapova proved a useful preserving theorem for systems.

THEOREM 2.5 (Sinapova [10]). *Suppose that v is a singular cardinal of countable cofinality and $\mathcal{R} = \{R_i\}_{i \in I}$ is a system on $D \times \tau$ with D cofinal in v^+ and $\max(|I|, \tau) < v$. Suppose that \mathbb{P} is a χ -closed forcing notion with $\chi > \max(\tau, |I|)^+$ and G is a generic filter for \mathbb{P} over V . Suppose that in $V[G]$ there are (not necessarily all cofinal) branches $\langle b_{i,\delta}; i \in I, \delta < \tau \rangle$ such that*

- (1) every $b_{i,\delta}$ is an R_i -branch, and for some (i, δ) , $b_{i,\delta}$ is cofinal;
- (2) for all $\alpha \in D$, there is (i, δ) such that $\text{Lev}_\alpha(\mathcal{R}) \cap b_{i,\delta}$ is non empty.

Then \mathcal{R} has a cofinal branch in V .

§3. The Main forcing. In this section we present Magidor–Shelah’s forcing construction for building models of $\Delta_{\aleph_{\omega^2}, \aleph_{\omega^2+1}}$. We assume that $\langle \kappa_n \rangle_{n < \omega}$ is an increasing sequence of supercompact cardinals which are indestructible by directed closed forcings (i.e., if \mathbb{P} is a κ_n -directed closed forcing notion, then κ_n is still supercompact in $V^\mathbb{P}$, see Laver [3]). We let $\lambda := \lim_{n < \omega} \kappa_n$ and we assume that, for every $n < \omega$, we have $2^{\kappa_n} = \kappa_n^+$. For every $n < \omega$, we let

$$\mathbb{S}_n := \prod_{m \geq n} \text{Coll}(\kappa_m^{+2}, < \kappa_{m+1}).$$

Since every \mathbb{S}_n is κ_n -directed closed, κ_n remains supercompact in $V^{\mathbb{S}_n}$ so we can fix, for every $n < \omega$, an \mathbb{S}_n -term \dot{F}_n for a normal ultrafilter on $\mathcal{P}_{\kappa_n}(\lambda^+)$ in $V^{\mathbb{S}_n}$. \dot{F}_n has a natural projection to a normal ultrafilter U_n on κ_n . The poset \mathbb{S}_n is actually κ_n^{+2} -closed and we assumed $2^{\kappa_n} = \kappa_n^+$, therefore forcing with \mathbb{S}_n does not introduce new sets which are hereditarily of size $\leq \kappa_n^+$, hence we have $U_n \in V$.

We let $\pi_n : V \rightarrow N_n$ be the elementary embedding corresponding to U_n . Consider $\text{Coll}^{N_n}(\kappa_n^{+\omega+2}, < \pi_n(\kappa_n))$, this forcing has the $\pi_n(\kappa_n)$ -chain condition in N_n and $\pi(\kappa_n)$ is inaccessible. Therefore, there are $\pi_n(\kappa_n)$ many dense subsets of this forcing which are in N_n . On the other hand $|\pi_n(\kappa_n)| = \kappa_n^+$ and the forcing is κ_n^+ -closed in N_n . Therefore, one can inductively define in V a generic filter K_n for $\text{Coll}^{N_n}(\kappa_n^{+\omega+2}, < \pi_n(\kappa_n))$ over N_n by meeting each dense set in N_n . We define the main forcing \mathbb{P} .

DEFINITION 3.1. Conditions of \mathbb{P} are sequences of the form

$$p = \langle \alpha_0, g_0, f_0, \dots, \alpha_{n-1}, g_{n-1}, f_{n-1}, A_n, g_n, F_n, \dots \rangle$$

such that:

- (1) every α_i is an inaccessible cardinal between κ_{i-1} and κ_i (with $\kappa_{-1} := \omega$);
- (2) $g_0 \in \text{Coll}(\omega, \alpha_0^{+\omega})$ and for $0 < i < n$, $g_i \in \text{Coll}(\kappa_{i-1}^{++}, < \alpha_i)$;
- (3) $f_i \in \text{Coll}(\alpha_i^{+\omega+2}, < \kappa_i)$;
- (4) $A_j \in U_j$ and every element of A_j is an inaccessible cardinal;
- (5) for $j \geq n$, $g_j \in \text{Coll}(\kappa_{j-1}^{++}, < \alpha)$ for the least α in A_j (hence for every $\alpha \in A_j$);
- (6) F_j is a function with domain A_j such that $F_j(\alpha) \in \text{Coll}(\alpha^{+\omega+2}, < \kappa_j)$ for every $\alpha \in A_j$, and such that $[F_j]_{U_j} \in K_j$ ($[F_j]_{U_j}$ is the equivalence class of F_j as a member of the ultrapower $\text{Ult}(V, U_j)$).

Given two conditions

$$p = \langle \alpha_0^p, g_0^p, f_0^p, \dots, \alpha_{n-1}^p, g_{n-1}^p, f_{n-1}^p, A_n^p, g_n^p, F_n^p, \dots \rangle$$

$$q = \langle \alpha_0^q, g_0^q, f_0^q, \dots, \alpha_{m-1}^q, g_{m-1}^q, f_{m-1}^q, A_m^q, g_m^q, F_m^q, \dots \rangle$$

we say that $p \leq q$ if and only if, the following hold

- (1) $m \leq n$ and for $i < m$, $\alpha_i^p = \alpha_i^q$ and $f_i^p \leq f_i^q$;
- (2) for every $i < \omega$, $g_i^p \leq g_i^q$;
- (3) for $m \leq j < n$, $\alpha_j^p \in A_j^q$ and $f_j^p \leq F_j^q(\alpha_j^p)$;
- (4) for $j \geq n$, $A_j^p \subseteq A_j^q$ and $F_j^p(\alpha) \leq F_j^q(\alpha)$ for all $\alpha \in A_j^p$.

The only difference with the forcing defined in [5] is in the definition of the g_0 coordinates. Magidor and Shelah showed that forcing with \mathbb{P} determines a model of $\Delta_{\aleph_{\omega^2}, \aleph_{\omega^2+1}}$, so in order to prove Theorem 1.2 we just need to show that there exists a generic extension by \mathbb{P} where the tree property holds at \aleph_{ω^2+1} .

Given a condition

$$p = \langle \alpha_0, g_0, f_0, \dots, \alpha_{n-1}, g_{n-1}, f_{n-1}, A_n, g_n, F_n, \dots \rangle$$

we say that

- (1) n is the *length* of p , and we denote it $lg(p)$;
- (2) the subsequence $\langle \alpha_0, g_0, f_0, \dots, \alpha_{n-1}, g_{n-1}, f_{n-1} \rangle$ is called the *lower part* of p or the *stem* of p , denoted $stem(p)$;
- (3) $\langle \alpha_0, \dots, \alpha_{n-1} \rangle$ is the α -*part* of p ;
- (4) $\langle g_0, \dots, g_{n-1} \rangle$ is the g -*part* of p ;
- (5) $\langle f_0, \dots, f_{n-1} \rangle$ is the f -*part* of p ;
- (6) $\langle A_j : j \geq n \rangle$ is the A -*part* of p ;
- (7) $\langle g_j : j \geq n \rangle$ is the S -*part* of p and for $k \geq n$, $\langle g_j : j \geq k \rangle$ is the S_k -*part* of p ;
- (8) $\langle F_j : j \geq n \rangle$ is the F -*part* of p ;

Given two conditions p and q , we write $p \leq_k q$ when $p \leq q$, $lg(p) = lg(q)$, $p \upharpoonright k = q \upharpoonright k$ and $g_k^p = g_k^q$.

§4. Basic properties of \mathbb{P} . We list some basic properties of \mathbb{P} .

PROPOSITION 4.1 (Magidor Shelah [5, Lemmas 2 and 3]). *The following hold for \mathbb{P} :*

- (1) every \leq_n -decreasing sequence $\langle p_\zeta : \zeta < \eta \rangle$ of less than κ_n -many conditions each of length n has a lower bound;
- (2) every \leq_k -decreasing sequence $\langle p_\zeta : \zeta < \eta \rangle$ of at most $\kappa_k^{p_0}$ -many conditions each of length $n > k$ has a lower bound;
- (3) \mathbb{P} satisfies the Prikry property in the following version: given a formula φ and a condition $p \in \mathbb{P}$ of length n and given $k \leq n$, there exists a condition q such that $q \leq_k p$ and q decides φ modulo k , that is if $r \leq q$ decides φ and q' is the condition obtained from q by replacing $q \upharpoonright k$ with $r \upharpoonright k$, then q' decides φ the same way r does.
- (4) \mathbb{P} preserves λ^+ ;
- (5) forcing with \mathbb{P} turns λ^+ into \aleph_{ω^2+1} .

We fix a generic filter G for \mathbb{S}_0 over V . In $V[G]$, we define

$$\mathbb{P}^* := \{p \in \mathbb{P}; \text{ the } S\text{-part of } p \text{ is in } G\}$$

ordered as a subset of \mathbb{P} . We should point out that if the length of $p \in \mathbb{P}$ is $k > 1$, then its S -part formally does not belong to \mathbb{S}_0 but to \mathbb{S}_{k+1} ; however \mathbb{S}_{k+1} naturally embeds into \mathbb{S}_0 in such a way that G naturally induces an \mathbb{S}_{k+1} -generic filter.

LEMMA 4.2 (Magidor and Shelah [5, Lemma 6]). $V^{\mathbb{P}^*} \subseteq V^{\mathbb{S}_0 * \mathbb{P}^*}$.

We will perform the proof of Theorem 1.2 in $V^{\mathbb{S}_0}$ where we will work with conditions of \mathbb{P}^* . The nice feature of $V^{\mathbb{S}_0}$ is that in this model κ_0 is still supercompact and, for every $n < \omega$, we can easily get a generic supercompact embedding with critical point κ_n . Moreover, working with conditions of \mathbb{P}^* allow us to use the following nice property.

REMARK 4.3. Every two conditions of \mathbb{P}^* with the same stem are compatible.

It is convenient to introduce a notation for the S -part of a condition p in \mathbb{P}^* , say $Spart(p)$. For a condition $p \in \mathbb{P}^*$ and for $j \geq lg(p)$ we denote by $Spart(p)(j)$ the j -th coordinate of the sequence $Spart(p)$.

A stem h and a condition $r \in \mathbb{S}_0$ determine a unique condition $p \in \mathbb{P}$ that we call the *closure of h with r* and we denote it $cl(h, r)$. This is the condition $p = \langle \alpha_0, g_0, f_0, \dots, \alpha_{n-1}, g_{n-1}, f_{n-1}, A_n, g_n, F_n, \dots \rangle$ whose stem is h and such that for all j above n , $g_j = r(j)$, $A_j^p = \kappa_j$ and F_j^p is the function that associate to every $\alpha \in A_j^p = \kappa_j$ the maximal condition of $\text{Coll}(\alpha^{+\omega+2}, < \kappa_j)$.

Assume $\langle g_n \rangle_{n < \omega}$ and $\langle h_n \rangle_{n < \omega}$ are two conditions in \mathbb{S}_0 , we let

$$\langle g_n \rangle_{n < \omega} \sim \langle h_n \rangle_{n < \omega} : \iff \text{ for large enough } i, g_i = h_i.$$

For $g \in \mathbb{S}_0$, we denote by $[g]$ its equivalence class. Define $\mathbb{S}_0/\sim := \{[g]; g \in \mathbb{S}_0\}$, with the ordering $[g] \leq [h]$ if and only if for large enough i , $g_i \leq h_i$. Let $G^* := \{[g]; g \in G\}$, then $V[G]$ is a generic extension of $V[G^*]$ via the forcing \mathbb{S}_0/G^* .

LEMMA 4.4 (Magidor and Shelah [5]). *If H is a generic filter for \mathbb{P}^* over $V[G^*]$, then $V[G^*][H] = V[H]$.*

PROOF. It is enough to show that G^* can be defined in $V[H]$. We prove that

$$G^* = \{[r] \in \mathbb{S}_0/\sim; \exists p \in H ([Spart(p)] \leq [r])\}.$$

If $p \in H$, then by definition of \mathbb{P}^* we have $Spart(p) \in G$, hence $[Spart(p)] \in G^*$. Conversely, if $[r] \in G^*$, then an easy density argument shows that there exists $p \in H$ such that $[Spart(p)] \leq [r]$. Indeed, if $q \in \mathbb{P}^*$, then $[Spart(q)]$ is in G^* and it is therefore compatible with $[r]$; then, one can easily extend q to a condition $p \in \mathbb{P}^*$ such that $[Spart(p)] \leq [r]$. \dashv

We now define in $V[G^*]$ the poset

$$\mathbb{P}^{**} := \{p \in \mathbb{P}; [Spart(p)] \in G^*\}.$$

PROPOSITION 4.5. *In $V[G^*]$, the product $\mathbb{P}^{**} \times \mathbb{S}_0/G^* \times \mathbb{S}_0/G^*$ has the λ^+ -chain condition.*

PROOF. Assume that for some $r \in \mathbb{S}_0$,

$$[r] \Vdash \dot{A} \subseteq \mathbb{P}^{**} \times \mathbb{S}_0/G^* \times \mathbb{S}_0/G^* \text{ is a maximal antichain.}$$

For every $m < \omega$, we define two sets \mathbb{Q}_m and H_m as follows:

- $\mathbb{Q}_m := \prod_{i < m} \text{Coll}(\kappa_i^{+2}, < \kappa_{i+1})$.
- H_m is the set of all sequences of the form

$$(\alpha_0, g_0, f_0, \dots, \alpha_{k-1}, g_{k-1}, f_{k-1}, g_k, \dots, g_{m-1}),$$

where $k \leq m$ and there is $p \in \mathbb{P}$ of length k with stem

$$(\alpha_0, g_0, f_0, \dots, \alpha_{k-1}, g_{k-1}, f_{k-1})$$

such that $g_i = g_i^p$ for $k \leq i < m$.

For a condition $p \in \mathbb{P}$ of length $k \leq m$, we denote by $h_m(p)$ the sequence

$$(\alpha_0^p, g_0^p, f_0^p, \dots, \alpha_{k-1}^p, g_{k-1}^p, f_{k-1}^p, g_k^p, \dots, g_{m-1}^p).$$

Note that $h_m(p)$ belongs to H_m .

As every $H_m \times \mathbb{Q}_m \times \mathbb{Q}_m$ has size κ_m , the union $\bigcup_{m < \omega} H_m \times \mathbb{Q}_m \times \mathbb{Q}_m$ has λ -many elements and we can enumerate this poset as $\{(h_\beta, s_\beta^0, s_\beta^1); \beta < \lambda\}$, where for $\kappa_i < \beta < \kappa_{i+1}$, $(h_\beta, s_\beta^0, s_\beta^1)$ belongs to $H_{i+1} \times \mathbb{Q}_{i+1} \times \mathbb{Q}_{i+1}$. For $q \in \mathbb{S}_0$ and $p \in \mathbb{Q}_m$, we denote by $q * p$ the unique condition $u \in \mathbb{S}_0$ such that $u \upharpoonright m = p$ and $u(i) = q(i)$, for every $i \geq m$. We inductively define a decreasing sequence $\langle r_\beta; \beta < \lambda \rangle$ of conditions in \mathbb{S}_0 , such that for all β, γ between κ_i and κ_{i+1} , we have $r_\beta \upharpoonright i + 1 = r_\gamma \upharpoonright i + 1$.

We let $r_0 := r$. For β limit, r_β is defined by $r_\beta(i) := \bigcup_{\gamma < \beta} r_\gamma(i)$, for every i (the inductive hypothesis and the closure of $\text{Coll}(\kappa_i^{+2}, < \kappa_{i+1})$, for $\kappa_i > \beta$, guarantee that r_β is a condition in \mathbb{S}_0).

Suppose that r_β has been defined, we want to define $r_{\beta+1}$. Let m be the least such that $\beta < \kappa_m$, and let φ_β be the following statement:

“There exists $(p, q^0, q^1) \in \dot{A}$ such that

- (1) $h_m(p) = h_\beta$ (hence p is a condition of length m),
- (2) $q^0 \upharpoonright m = s_\beta^0$ and $q^1 \upharpoonright m = s_\beta^1$,
- (3) for $j \geq m$ the conditions $q^0(j), q^1(j), r_\beta(j)$ and g_j^p (the j -th coordinate of the S -part of p) are pairwise compatible”.

There exists a condition $r_{\beta+1}$ such that $[r_{\beta+1}] \leq [r_\beta]$ and $[r_{\beta+1}]$ decides the statement φ_β . We can assume without loss of generality that $r_{\beta+1} \upharpoonright m+1 = r_\beta \upharpoonright m+1$ or we replace $r_{\beta+1}$ by an equivalent condition. So the inductive hypothesis is satisfied.

If $[r_{\beta+1}] \Vdash \varphi_\beta$, then let $(p_\beta, q_\beta^0, q_\beta^1)$ witness it. The condition $[r_{\beta+1}]$ forces that $[q_\beta^0]$, $[q_\beta^1]$, $[r_\beta]$ and the class of the S -part of p_β are in G^* , hence $[r_{\beta+1}]$ must be stronger than all of them. It follows that for all j above some k , we have

$$r_{\beta+1}(j) \leq q_\beta^0(j), q_\beta^1(j), r_\beta(j), g_j^{p_\beta}.$$

If $m < k$, then for j between m and k the conditions $q_\beta^0(j)$, $q_\beta^1(j)$, $r_\beta(j)$, and $g_j^{p_\beta}$ are compatible by item 3, hence $q_\beta^0(j) \cup q_\beta^1(j) \cup r_\beta(j) \cup g_j^{p_\beta}$ is a condition in $\text{Coll}(\kappa_i^{++}, < \kappa_{i+1})$. Since we can replace $r_{\beta+1}$ by an equivalent condition, we can assume without loss of generality that for all j between m and k , we have $r_{\beta+1}(j) = q_\beta^0(j) \cup q_\beta^1(j) \cup r_\beta(j) \cup g_j^{p_\beta}$. It follows that

- (1) $r_{\beta+1} * s_\beta^0 \leq q_\beta^0$,
- (2) $r_{\beta+1} * s_\beta^1 \leq q_\beta^1$,
- (3) for all $j \geq m$, $r_{\beta+1}(j) \leq g_j^{p_\beta}$.

Now, we let $r_\infty \in \mathbb{S}_0$ be defined by $r_\infty(i) := \bigcup_{\beta < \lambda} r_\beta(i)$. We define

$$E := \{(p_\beta, q_\beta^0, q_\beta^1); \beta < \lambda, [r_{\beta+1}] \Vdash \varphi_\beta\}.$$

We show that $[r_\infty]$ forces that every element of \dot{A} is compatible with an element of E . Since E has size λ , this will prove that the size of the antichain is at most λ .

Assume that for some $s \in \mathbb{S}_0$ and for some (p, q^0, q^1) we have $[s] \leq [r_\infty]$ and $[s]$ forces that $(p, q^0, q^1) \in \dot{A}$. Without loss of generality $s \leq r_\infty$. Also $[s]$ forces that $[q^0]$, $[q^1]$ and the class of the S -part of p are in G^* , hence $[s]$ is stronger than all those conditions. For some m , we have $s(i) \supseteq q^0(i) \cup q^1(i) \cup g_i^p$ for every $i \geq m$. The triple $(h_m(p), q^0 \upharpoonright m, q^1 \upharpoonright m)$ appears in our enumeration as $(h_\beta, s_\beta^0, s_\beta^1)$ for some $\beta < \kappa_m$. Clearly (p, q^0, q^1) witnesses the truth of φ_β thus $[s] \Vdash \varphi_\beta$ because $[s] \leq [r_{\beta+1}]$. So $[s]$ forces that both (p, q^0, q^1) and $(p_\beta, q_\beta^0, q_\beta^1)$ are in the antichain. We prove that they are compatible, hence they are equal. The conditions p and p_β have the same stem, let i be their common length. We claim that g_i^p and $g_i^{p_\beta}$ are compatible. For $i < m$ this is true because $s(i)$ extends both g_i^p and $g_i^{p_\beta}$. Similarly q^0 and q^1 are compatible with q_β^0 and q_β^1 respectively. This completes the proof. \dashv

COROLLARY 4.6. $\Vdash_{\mathbb{P}^{**}} \mathbb{S}_0/G^* \times \mathbb{S}_0/G^*$ is λ^+ -c.c.

PROOF. Suppose for a contradiction that for some $p \in \mathbb{P}^{**}$, we have

$$p \Vdash \dot{A} \text{ is an antichain of size } \lambda^+.$$

For every $\alpha < \lambda^+$, we fix a condition $p_\alpha \leq p$ that decides the value of the α -th element of \dot{A} as a pair $(s_\alpha^0, s_\alpha^1) \in \mathbb{S}_0/G^* \times \mathbb{S}_0/G^*$. For $\alpha < \beta < \lambda^+$, if $(q, t^0, t^1) \leq (p_\alpha, s_\alpha^0, s_\alpha^1), (p_\beta, s_\beta^0, s_\beta^1)$, then

$$q \Vdash (s_\alpha^0, s_\alpha^1), (s_\beta^0, s_\beta^1) \in \dot{A} \text{ and } (t_0, t^1) \leq (s_\alpha^0, s_\alpha^1), (s_\beta^0, s_\beta^1)$$

contradicting the fact that \dot{A} is an antichain. It follows that $\{(p_\alpha, s_\alpha^0, s_\alpha^1); \alpha < \lambda^+\}$ is an antichain of $\mathbb{P}^* \times \mathbb{S}_0/G^* \times \mathbb{S}_0/G^*$. By Proposition 4.5 this product has the λ^+ -chain condition, so we have a contradiction. \dashv

§5. Forcing the tree property at \aleph_{ω^2+1} with \mathbb{P} . In this section, we prove that there exists a \mathbb{P} -generic extension of V in which \aleph_{ω^2+1} has the tree property, thus we demonstrate Theorem 1.2. In the proof we will use a technique described in Sinapova’s paper [10] for getting the tree property at the successor of a singular from a diagonal Prikry-type forcing. However, Sinapova’s approach is based on a typical property of diagonal Prikry-type forcings, namely that any two conditions with the same stem are compatible; this is not true for the forcing \mathbb{P} . The forcing \mathbb{P}^* on the contrary does satisfy this property, so we will work with \mathbb{P}^* over $V[G]$ and make the relevant changes to apply Sinapova’s technique to our case.

Suppose for a contradiction that no \mathbb{P} -generic extension of V forces the tree property at λ^+ , then we can find a \mathbb{P} -name \dot{T} such that

$$\emptyset \Vdash_{\mathbb{P}} \dot{T} \text{ is a } \lambda^+\text{-Aronszajn tree.}$$

We can assume that \dot{T} is a name for a subset of $\lambda^+ \times \lambda$.

We are going to prove that in $V[G]$ there exists a sequence of pairwise compatible conditions $\langle p_\beta; \beta \in J \rangle$ in \mathbb{P}^* and a sequence of elements $\langle u_\beta; \beta \in J \rangle$ in $\lambda^+ \times \lambda$, where J is a cofinal subset of λ^+ , such that for all $\beta < \beta'$ in J the weakest common extension of p_β and $p_{\beta'}$ forces $u_\beta <_{\dot{T}} u_{\beta'}$. Once those sequences are defined, we get a contradiction with the following argument. We claim that there exists a generic filter H for \mathbb{P}^* over $V[G]$ such that $\{\beta; p_\beta \in H\}$ is cofinal in λ^+ . Indeed, if no such filter exists, then we would have $\emptyset \Vdash_{\mathbb{P}^*} \{\beta; p_\beta \in \dot{G}\}$ is bounded. As \mathbb{P}^* is λ^+ -c.c. there would be $\delta < \lambda^+$ such that $\emptyset \Vdash_{\mathbb{P}^*} \{\beta; p_\beta \in \dot{G}\} \subseteq \delta$. Let $\delta' \in J$ above δ , then $p_{\delta'} \Vdash p_{\delta'} \notin \dot{G}$, a contradiction. It follows that $B := \{u_\beta; p_\beta \in H\}$ is a cofinal branch for the tree in $V[G][H]$. We recall that H was \mathbb{P}^* -generic over $V[G]$, it is also \mathbb{P}^{**} -generic over $V[G^*]$. We have $V[G][H] = V[G^*][H][G/G^*] = V[H][G/G^*]$, namely $V[G][H]$ can be seen as a generic extension of $V[H]$ via the forcing \mathbb{S}_0/G^* . By Corollary 4.6, we know that $\mathbb{S}_0/G^* \times \mathbb{S}_0/G^*$ is λ^+ -c.c. in $V[H]$. Since B is approximated, we can apply Lemma 2.1, thus B exists in $V[H]$. So we found a cofinal branch for \dot{T} in a \mathbb{P} -generic extension of V , contradicting the assumption $\emptyset \Vdash_{\mathbb{P}} \dot{T}$ is Aronszajn.

The first step is to prove the following.

LEMMA 5.1. *In $V[G]$ there are $n, m < \omega$ and a cofinal set $I \subseteq \lambda^+$ such that for all $\alpha < \beta$ in I , one can find $\zeta, \eta < \kappa_m$ and a condition $q \in \mathbb{P}^*$ of length n such that $q \Vdash (\alpha, \zeta) <_{\dot{T}} (\beta, \eta)$.*

PROOF. We let $j : V[G] \rightarrow M$ be the elementary embedding corresponding to \dot{F}_0^G . So the critical point of j is κ_0 , $j(\kappa_0) > \lambda^+$ and M is closed by sequences of length λ^+ . We fix \bar{H} a generic filter for $j(\mathbb{P}^*)$ over $V[G]$ such that the first element of the α -sequence added by \bar{H} is κ_0 . This implies that λ^+ is not collapsed, because $\lambda^+ = \kappa_0^{+\omega+1}$ and, by definition of our forcing, \bar{H} does not collapse any cardinal between $\kappa_0^{+\omega}$ and $\kappa_0^{+\omega+2}$. Fix any ordinal γ between $\sup j''\lambda^+$ and $j(\lambda^+)$. Let $T^* := j(\dot{T})^{\bar{H}}$. T^* is a $j(\lambda^+)$ -tree, so we can fix a node u of T^* on level γ . For every $\beta < \lambda^+$, there are $m_\beta < \omega$, $\zeta_\beta < j(\kappa_{m_\beta})$ and a condition $p_\beta \in \bar{H}$ such that

$$p_\beta \Vdash_{j(\mathbb{P}^*)} (j(\beta), \zeta_\beta) <_{j(\dot{T})} u.$$

For some $n, m < \omega$ and for a cofinal $I^* \subseteq \lambda^+$, we have $lg(p_\beta) = n$ and $m_\beta = m$, for every $\beta \in I^*$. We can write each condition p_β with $\beta \in I^*$ as

$$p_\beta = \langle \kappa_0, g_0^\beta, f_0^\beta, \dots, \alpha_{n-1}^\beta, g_{n-1}^\beta, f_{n-1}^\beta, A_n^\beta, g_n^\beta, F_n^\beta, \dots \rangle.$$

As the p_β 's are pairwise compatible conditions, they satisfy the following properties:

- (1) the p_β 's have the same α -part $\langle \alpha_0, \dots, \alpha_{n-1} \rangle$;
- (2) we can assume that for some $g_0 \in \text{Coll}(\omega, \kappa_0^{+\omega})$, we have $g_0^\beta = g_0$ for all β^* (we can shrink I^* if necessary);
- (3) for every $0 < i < n$, the sequence $\langle g_i^\beta \rangle_{\beta \in I}$ has a lower bound g_i (indeed $\text{Coll}(j(\kappa_{i-1}), < \alpha_i)$ has closure $> \lambda^+$);
- (4) for $i < n$, the sequence $\langle f_i^\beta \rangle_{\beta \in I}$ has a lower bound f_i (these are conditions in $\text{Coll}(\alpha_i^{+\omega+2}, j(\kappa_i))$ which has closure $> \kappa_0^{+\omega+1} = \lambda^+$).

So we defined a stem

$$s := \langle \kappa_0, g_0, f_0, \dots, \alpha_{n-1}, g_{n-1}, f_{n-1} \rangle$$

such that for all $\beta \in I^*$, there exists a condition $p \in j(\mathbb{P}^*)$ with stem s forcing $(j(\beta), \zeta_\beta) <_{T^*} u$. By letting

$$I := \{ \beta < \lambda^+; \exists p \in j(\mathbb{P}^*)(\text{stem}(p) = s \wedge \exists \zeta < j(\kappa_m)(p \Vdash (j(\beta), \zeta) < u)) \}$$

we get a cofinal subset of λ^+ which is in $V[G]$ and has the desired property. Indeed, if $\beta < \beta'$ are in I , then there are $p, p' \in j(\mathbb{P}^*)$ with stem s and $\zeta, \eta < j(\kappa_m)$ such that

- (1) $p \Vdash (j(\beta), \zeta) <_{j(\dot{T})} u$,
- (2) $p' \Vdash (j(\beta'), \eta) <_{j(\dot{T})} u$.

As they have the same stem, the two conditions are compatible, so there exists a condition $q^* \leq p, p'$ forcing $(j(\beta), \zeta) <_{j(\dot{T})} u$ and $(j(\beta'), \eta) <_{j(\dot{T})} u$.

It follows that $q^* \Vdash (j(\beta), \zeta) <_{j(\dot{T})} (j(\beta'), \eta)$. By elementarily, we can find a condition $q \in \mathbb{P}^*$ of length n and two ordinals $\zeta, \eta < \kappa_m$ such that $q \Vdash_{\mathbb{P}^*}^{V[G]} (\beta, \zeta) <_{\dot{T}} (\beta', \eta)$. That completes the proof of the lemma. \dashv

We fix n, m , and I as in the conclusion of the above lemma, without loss of generality $n \leq m$. In $V[G]$ we say that a stem s ‘forces’ a statement φ and we write $s \Vdash \varphi$, when there is a condition $p \in \mathbb{P}^*$ with stem s such that $p \Vdash \varphi$. We prove the following.

LEMMA 5.2. *In $V[G]$, there exists a cofinal $J \subseteq I$, a stem h of \mathbb{P}^* length n and a sequence $\langle u_\beta; \beta \in J \rangle$ with $u_\beta \in \{\beta\} \times \kappa_m$ for $\beta \in J$, such that for every $\beta < \beta'$ in J , we have $h \Vdash u_\beta <_{\dot{T}} u_{\beta'}$.*

PROOF. Let $l = m + 2$. If V_l is the \mathcal{S}_l -generic extension determined by G , then κ_l is supercompact in V_l and there is a λ^+ -supercompact elementary embedding $j : V_l \rightarrow M_l$ with critical point κ_l . Let G_C be the generic filter for $\text{Coll}(\kappa_0^{+2}, < \kappa_1) \times \dots \times \text{Coll}(\kappa_{l-1}^{+2}, < \kappa_l)$ determined by G , then by forcing with $\mathbb{C}_{\text{Tail}} := \text{Coll}(\kappa_{l-1}^{+2}, < j(\kappa_l) \setminus \kappa_l)$ over $V[G]$ we get a generic object H^* such that $j[G] \subseteq G_C * H^*$, hence we can lift j to an embedding $j^* : V[G] \rightarrow M_l[G_C][H^*]$ that we rename j .

Let $\gamma \in j(I)$ be above $\sup j''\lambda^+$. For every $\beta \in I$, we fix $\zeta_\beta, \eta_\beta < \kappa_m$ and a condition $p_\beta \in j(\mathbb{P}^*)$ of length n with stem h_β such that

$$p_\beta \Vdash (j(\beta), \zeta_\beta) <_{j(\dot{T})} (\gamma, \eta_\beta).$$

In $V[G]$ we define a system $\{R_h; h \text{ is a stem}\}$ over $\lambda^+ \times \kappa_m$ by letting

$$R_h := \{\langle u, v \rangle; \exists p \in \mathbb{P}, \text{stem}(p) = h, p \Vdash u <_{\dot{T}} v\}.$$

Every R_h is transitive, because in $V[G]$ two conditions of \mathbb{P} with the same stem are compatible. We also define for every stem h and for every $\eta < \kappa_m$ a set

$$b_{h,\eta} := \{(\beta, \zeta) \in \lambda^+ \times \kappa_m; \exists p \in j(\mathbb{P})(\text{stem}(p) = h \wedge p \Vdash (j(\beta), \zeta) <_{j(\dot{T})} (\gamma, \eta))\}.$$

Note that every $b_{h,\delta}$ is an R_h -branch. Moreover, in $V[G]$ λ^+ is regular and the stems of $j(\mathbb{P})$ of length n are $< \kappa_n$. The forcing \mathbb{C}_{Tail} is κ_{l-1}^{+2} -closed, hence it doesn't add $< \kappa_n$ -sequences. So we can find in $V[G][H^*]$ a cofinal $J \subseteq I$, a stem h^* and two ordinals ζ^*, η^* such that for $\beta \in J$, we have $h_\beta = h^*, \zeta_\beta = \zeta^*$ and $\eta_\beta = \eta^*$. Thus b_{h^*,η^*} is a cofinal R_{h^*} -branch and we can apply Theorem 2.5. We get that the system has a cofinal branch in $V[G]$, i.e., for some stem h , there exists a cofinal $J \subseteq I$, and a sequence $\langle \sigma_\beta; \beta \in J \rangle$ such that for $\beta < \beta'$ in J , $h \Vdash (\beta, \sigma_\beta) <_{\dot{T}} (\beta', \sigma_{\beta'})$. Set $u_\beta := (\beta, \sigma_\beta)$ for $\beta \in J$, then J and $\langle u_\beta; \beta \in J \rangle$ are as required. \dashv

Let h, J , and $\alpha \mapsto u_\alpha$ be as in the conclusion of the above lemma. By shrinking J , we may assume that for some $\zeta < \kappa_m$, we have $u_\alpha = (\alpha, \zeta)$ for each $\alpha \in J$.

LEMMA 5.3. *Suppose that s is a stem of length k , $L \subseteq \lambda^+$ is unbounded and for all $\alpha < \beta$ with $\alpha, \beta \in L$, $s \Vdash u_\alpha <_{\dot{T}} u_\beta$. Then, there are $\rho < \lambda^+$ and sets $\langle A^\alpha, g^\alpha, F^\alpha \rangle_{\alpha \in L \setminus \rho}$ in $V[G]$ such that:*

- (1) A^α is in U_k , g^α is in the generic filter for $\text{Coll}(\kappa_{k-1}^{++}, < \min(A^\alpha))$ induced by G , F^α is a function of domain A^α such that for every $x \in A^\alpha$, $F^\alpha(x) \in \text{Coll}(\kappa_x^{+\omega+2}, < \kappa_k)$ and $[F^\alpha]_{U_k} \in K_k$;
- (2) for every $\alpha < \beta$ in $L \setminus \rho$, for all $x \in A^\alpha \cap A^\beta$,

$$s \frown \langle x, g^\alpha \cup g^\beta, F^\alpha(x) \cup F^\beta(x) \rangle \Vdash u_\alpha <_{\dot{T}} u_\beta.$$

PROOF. Let l be $k + 3$. If V_l is the \mathbb{S}_l -generic extension determined by G , then κ_l is supercompact in V_l and there is a λ^+ -supercompact elementary embedding $j : V_l \rightarrow M_l$ with critical point κ_l . Let G_C be the generic filter for $\text{Coll}(\kappa_0^{+2}, < \kappa_1) \times \dots \times \text{Coll}(\kappa_{l-1}^{+2}, < \kappa_l)$ determined by G , then forcing with $\mathbb{C}_{\text{tail}} := \text{Coll}(\kappa_{l-1}^{+2}, < j(\kappa_l) \setminus \kappa_l)$ over $V[G]$ we get a generic object H^* such that $j[G] \subseteq G_C * H^*$, hence we can lift j to an embedding $j^* : V[G] \rightarrow M_l[G_C][H^*]$ that we rename j . Note that the forcing \mathbb{C}_{tail} is κ_{l-1} -closed, that is κ_{k+2} -closed. Choose $\gamma \in j(L)$ above $j[\lambda^+]$. By elementarily, we can pick for all $\alpha \in L$ a condition $p_\alpha \in j(\mathbb{P}^*)$ with stem s such that

$$p_\alpha \Vdash (j(\alpha), \zeta) <_{j(\dot{T})} (\gamma, \zeta).$$

Every p_α is of the form $s \frown (A_k^\alpha, g_k^\alpha, F_k^\alpha, \dots)$ where $A_k^\alpha \in j(U_k) = U_k \subseteq \mathcal{P}(\kappa_k)$, g_k^α is in the generic filter for $\text{Coll}(\kappa_{k-1}^{++}, < \min(A_k^\alpha))$ induced by G , and F_k^α is a function with domain A_k^α such that for every $\beta \in A_k^\alpha$, $F_k^\alpha(\beta)$ is a condition in $\text{Coll}(\beta^{+\omega+2}, < \kappa_k)$ and $[F_k^\alpha]_{U_k} \in K_k$. There are $|\mathcal{P}(\kappa_k)| = \kappa_{k+1}$ -many possible triples (A, g, F) in the range of the function $\alpha \mapsto (A_k^\alpha, g_k^\alpha, F_k^\alpha)$. Since λ^+ is regular

in $V[G][H^*]$ and \mathbb{C}_{tail} adds no sequences of length less than κ_{k+2} , the function $\alpha \mapsto (A_k^\alpha, g_k^\alpha, F_k^\alpha)$ must be constant on an unbounded subset L' of L . So there is (A^*, g^*, F^*) such that $(A^*, g^*, F^*) = (A_k^\alpha, g_k^\alpha, F_k^\alpha)$ for every $\alpha \in L'$.

Now we step back to $V[G]$ where we define E as the set of all triples (A, g, F) such that $A \in U_k, g$ is in the generic filter for $\text{Coll}(\kappa_{k-1}^{++}, < \min(A))$ derived by G , and F is a function with domain A such that $[F]_{U_k} \in K_k$ and $F(x) \in \text{Coll}(x^{+\omega+2}, < \kappa_k)$ for all $x \in A$. For every $(A, g, F) \in E$, we define a relation $R_{A,g,F}$ on $L \times \{\zeta\}$, by letting $a R_{A,g,F} b$ when there exists a condition p such that $p \upharpoonright k+1 = s \cap (A, g, F)$ and $p \Vdash a <_{\dot{T}} b$. Then $\{R_{A,g,F}\}_{(A,g,F) \in E}$ is a system, because for every two conditions $p, p' \in \mathbb{P}^*$ extending $s \cap (A, g, F)$ and forcing a statement φ , we can find a third condition $q \leq p, p'$ extending $s \cap (A, g, F)$ and forcing the same statement φ .

Now in $V[G][H^*]$ we can define a system of branches for $\{R_{A,g,F}\}_{(A,g,F) \in E}$ as follows. We let $b_{A,g,F}$ be the set of all pairs (α, ζ) such that there is a condition p in $j(\mathbb{P})$ such that $p \upharpoonright k+1 = s \cap (A, g, F)$ and $p \Vdash (j(\alpha), \zeta) <_{j(\dot{T})} (\gamma, \zeta)$. The triple (A^*, g^*, F^*) defined above determines a cofinal branch b_{A^*,g^*,F^*} . By Theorem 2.5 (applied to \mathbb{C}_{tail} which is κ_{k+2} -closed), a cofinal branch for the system exists also in $V[G]$. So there exists $L^* \subseteq L$ and (A, g, F) such that for all $\alpha < \beta$ in L^* , there exists a condition $p \in \mathbb{P}^*$ extending $s \cap (A, g, F)$ that forces $u_\alpha <_{\dot{T}} u_\beta$.

Let ρ be the least element of L^* and, for every $\alpha \in (L \setminus \rho) \setminus L^*$, let α^* be the least element of L^* above α . For $\alpha \in (L \setminus \rho) \setminus L^*$ there exists $(\bar{A}^\alpha, \bar{g}^\alpha, \bar{F}^\alpha)$ such that some condition extending $s \cap (\bar{A}^\alpha, \bar{g}^\alpha, \bar{F}^\alpha)$ forces $u_\alpha <_{\dot{T}} u_{\alpha^*}$. So given $\alpha \in L \setminus \rho$, we define $(A^\alpha, g^\alpha, F^\alpha)$ as follows. If $\alpha \in L^*$, then we let $(A^\alpha, g^\alpha, F^\alpha)$ be (A, g, F) . If $\alpha \notin L^*$, then we let A^α be a subset of $A \cap \bar{A}^\alpha$ such that for every $x \in A^\alpha$, $\bar{F}^\alpha(x)$ and $F(x)$ are compatible; we let g^α be a condition in the generic filter for $\text{Coll}(\kappa_{k-1}^{++}, < \min(A \cap \bar{A}^\alpha))$ induced by G , and we let $F^\alpha(x) = \bar{F}^\alpha(x) \cup F(x)$. The sequence $\langle A^\alpha, g^\alpha, F^\alpha \rangle_{\alpha \in L \setminus \rho}$ is as required. \dashv

We are now ready for the final step.

LEMMA 5.4. *In $V[G]$ there exists a sequence of pairwise compatible conditions $\langle p_\beta; \beta \in J \setminus \rho \rangle$ in \mathbb{P}^* with stem h where $\rho < \lambda^+$ and for all $\beta < \beta'$ in J the weakest common extension of p_β and $p_{\beta'}$ forces in \mathbb{P}^* that $u_\beta <_{\dot{T}} u_{\beta'}$.*

PROOF. By induction on $k < \omega$, we define $\langle \rho_k; k \geq n \rangle$, and $\langle A_k^\alpha, g_k^\alpha, F_k^\alpha; k \geq n, \alpha \in J \setminus \rho_k \rangle$ such that:

- (1) for all $\alpha \in J \setminus \rho_k$, we have $A_k^\alpha \in U_k, g_k$ is in the generic filter for $\text{Coll}(\kappa_{k-1}^{++}, < \min(A_k^\alpha))$ induced by G , and F_k^α is a function with domain A_k^α with $[F_k^\alpha]_{U_k} \in K_k$ such that $F_k^\alpha(\beta) \in \text{Coll}(\beta^{+\omega+2}, < \kappa_k)$ for every $\beta \in A_k^\alpha$;
- (2) for all $\beta < \beta'$ in $J \setminus \rho_k$, if s is a stem of the form

$$h \cap \langle \alpha_n, g_n, f_n, \dots, \alpha_k, g_k, f_k \rangle$$

such that for $n \leq i \leq k, \alpha_i \in A_i^\beta \cap A_i^{\beta'}, g_i$ is in the generic filter for $\text{Coll}(\kappa_{i-1}^{++}, \alpha_i)$ derived from G , and $f_i \leq F_i^\beta(\alpha_i) \cup F_i^{\beta'}(\alpha_i)$, there is a condition of \mathbb{P}^* with stem s that forces $u_\beta <_{\dot{T}} u_{\beta'}$

ρ_n and $\langle A_n^\alpha, g_n^\alpha, F_n^\alpha; \alpha \in J \setminus \rho_n \rangle$ are given by the above lemma applied to h . Assume that we have defined ρ_k , and $\langle A_k^\alpha, g_k^\alpha, F_k^\alpha; \alpha \in J \setminus \rho_k \rangle$, we want to define ρ_{k+1} , and $\langle A_{k+1}^\alpha, g_{k+1}^\alpha, F_{k+1}^\alpha; \alpha \in J \setminus \rho_{k+1} \rangle$. For a stem $s = \langle \alpha_0, g_0, f_0, \dots, \alpha_k, g_k, f_k \rangle$

extending h we let

$$J^s := \{\alpha \in J \setminus \rho_k; \forall i \leq k, \alpha_i \in A_i^\alpha, g_i \leq g_i^\alpha \text{ and } f_i \leq F_i^\alpha(\alpha_i)\}.$$

We define ρ^s as follows: if J^s is bounded in λ^+ , then ρ^s is a bound; otherwise we let ρ^s and $\langle A^{s,\alpha}, g^{s,\alpha}, F^{s,\alpha}; \alpha \in J^s \setminus \rho^s \rangle$ be given by the above lemma applied to s and J^s . Let

$$\rho_{k+1} := \sup\{\rho^s; s \text{ is a stem of length } k + 1 \text{ extending } h\}.$$

For each $\alpha \in J \setminus \rho_{n+1}$, let

$$H_\alpha(k + 1) := \{s; s \text{ is a stem of length } k + 1 \text{ such that } h \sqsubseteq s \text{ and } \alpha \in J^s\}.$$

Note that the set $A_{k+1}^\alpha := \bigcap_{s \in H_\alpha(k+1)} A^{s,\alpha}$ is in U_{k+1} and $[F^\alpha]_{U_{k+1}} := \bigcup\{[F^{s,\alpha}]_{U_{k+1}}; s \in H_\alpha(k + 1)\}$ is in K_{k+1} . By shrinking A_{k+1}^α we can assume that for all $x \in A_{k+1}^\alpha$, $F^\alpha(x) = \bigcup\{F^{s,\alpha}(x); x > \alpha_k\}$. We let g^α be the condition $\bigcup_{s \in H_\alpha(k+1)} g^{s,\alpha}$. We check that (1) and (2) hold for $k + 1$. Condition (1) holds by construction. For condition (2), consider $\alpha < \beta$ both in $J \setminus \rho_{k+1}$ and suppose that $s = h \cap \langle A_n, g_n, F_n, \dots, A_{k+1}, g_{k+1}, F_{k+1} \rangle$ is a stem of length $k + 2$ such that $\alpha, \beta \in J^s$. As both α and β are in J^s and $\rho_{k+1} < \alpha, \beta$, we have that J^s is unbounded and ρ^s was obtained by applying the previous lemma. Therefore $s \in H_\alpha(k + 1) \cap H_\beta(k + 1)$, $\alpha_{k+1} \in A^{s,\alpha} \cap A^{s,\beta}$, $g_{k+1} \leq g^{s,\alpha}, g^{s,\beta}$ and $f_{k+1} \leq F^{s,\alpha}(\alpha_{k+1}) \cup F^{s,\beta}(\alpha_{k+1})$. So by construction of $A^{s,\alpha}, A^{s,\beta}, F^{s,\alpha}, F^{s,\beta}$ we have $s \Vdash u_\alpha <_{\dot{J}} u_\beta$.

This completes the definition. Finally, we let $\rho := \sup_{n < \omega} \rho_n$ and for every $\alpha \in J \setminus \rho$, we let p_α be the condition $h \cap \langle A_n^\alpha, g_n^\alpha, F_n^\alpha, \dots, A_j^\alpha, g_j^\alpha, F_j^\alpha, \dots \rangle$. Then $\langle p_\alpha; \alpha \in J \setminus \rho \rangle$ is as desired. Indeed, if $\alpha < \beta$ are in $J \setminus \rho$, then p_α and p_β are compatible in \mathbb{P}^* and if $q \leq p_\alpha, p_\beta$ is in \mathbb{P}^* , then by construction $\text{stem}(q) \Vdash u_\alpha <_{\dot{J}} u_\beta$, i.e., there is a condition of \mathbb{P}^* with the same stem as q that forces $u_\alpha <_{\dot{J}} u_\beta$. Two conditions of \mathbb{P}^* with the same stem are compatible, hence $q \not\Vdash u_\alpha <_{\dot{J}} u_\beta$. It follows that the weakest common extension of p_α and p_β forces $u_\alpha <_{\dot{J}} u_\beta$. \dashv

That completes the proof of Theorem 1.2.

§6. Stationary set reflection and the failure of the tree property at \aleph_{ω^2+1} . In this section we show that the stationary set reflection at \aleph_{ω^2+1} does not imply the tree property. We force from large cardinals a model of $\Delta_{\aleph_{\omega^2}, \aleph_{\omega^2+1}}$ where the tree property at \aleph_{ω^2+1} fails, thus we demonstrate Theorem 1.3. The proof combines Magidor and Shelah’s technique for getting $\Delta_{\aleph_{\omega^2}, \aleph_{\omega^2+1}}$ with some ideas from Kunen’s paper (see [2]).

We start by assuming the existence of infinitely many supercompact cardinals $\langle \kappa_n \rangle_{n < \omega}$; as usual we assume that the sequence is increasing and that the supercompactness of each κ_n is indestructible by κ_n -directed closed forcings. We let $\lambda := \lim_{n < \omega} \kappa_n$ and we assume that $2^{\kappa_n} = \kappa_n^+$ for every $n < \omega$. We consider the following forcing notion \mathbb{R} : a condition t of \mathbb{R} is either the one-point tree $\{0\}$ or an homogeneous tree $t \subseteq \lambda^+ 2$ of successor height such that $\langle 0 \rangle$ and $\langle 1 \rangle$ are in t . \mathbb{R} is partially ordered by end-extension. It is proven in Kunen’s paper that forcing with \mathbb{R} adds a Suslin tree, let T be such a tree. Kunen observed that the iterated forcing $\mathbb{R} * T$ is equivalent to a λ^+ -closed forcing

$$\mathbb{Q} := \{(r, t); r \in \mathbb{R}, t \in r, \text{ht}(r) = \text{dom}(t) + 1, \text{ and } r \Vdash t \in \dot{T}\}.$$

We observe that \mathbb{Q} is even λ^+ -directed closed: if $\langle r_\alpha, t_\alpha \rangle_{\alpha < \lambda}$ is a directed sequence of conditions in \mathbb{Q} , then $r := \bigcup_{\alpha < \lambda} r_\alpha$ is an homogeneous tree of height $\gamma := \sup_{\alpha < \lambda} \text{ht}(r_\alpha)$ and $t := \bigcup_{\lambda} t_\alpha$ is a path through r . We can extend r to an homogeneous tree of height $\gamma + 1$ by adding at level γ all sequences $s \frown (t \setminus \alpha)$ such that $s \in r$, where $t \setminus \alpha$ denotes the unique s' such that $t \upharpoonright \alpha \frown s' = t$. This provides a common extension for the sequence $\langle r_\alpha, t_\alpha \rangle_{\alpha < \lambda}$.

Since we assumed that the supercompactness of each κ_n is indestructible by directed closed forcing, κ_n remains supercompact in $V^{\mathbb{R} * T}$ as well as in the model $V^{(\mathbb{R} * T) \times \mathbb{S}_n}$ (\mathbb{S}_n denotes the same poset as in the previous sections). We use this fact to define in V a variation of the forcing \mathbb{P} discussed in the previous sections: each \dot{F}_n is replaced by an $(\mathbb{R} * T) \times \mathbb{S}_n$ -name for a normal ultrafilter on $\mathcal{P}_{\kappa_n}(\lambda^+)$, and U_n denotes now the projection of such ultrafilter to a normal ultrafilter on κ_n . Since $2^{\kappa_n} = \kappa_n^+$ and forcing with $(\mathbb{R} * T) \times \mathbb{S}_n$ does not add sets which are hereditarily of size $\leq \kappa_n^+$, once again we have $U_n \in V$. The generic filters K_i are defined as before.

We want to prove that forcing with $\mathbb{R} * \mathbb{P}$ we obtain the desired model. Let be $\mathbb{S} := \text{Coll}(\omega, < \kappa_0) \times \mathbb{S}_0$, and let $G_{\mathbb{S}_0}$ be a generic filter for \mathbb{S}_0 over V . As before, we denote by \mathbb{P}^* the poset whose conditions are the conditions p of \mathbb{P} such that the S -part of p is in $G_{\mathbb{S}_0}$.

We will need the following lemma.

LEMMA 6.1. *T is λ^+ -strategically closed in $V^{\mathbb{R}}$.*

PROOF. T is λ^+ -strategically closed if Even has a winning strategy in the game $G_{\lambda^+}(T)$ where two players Odd and Even take turns to play conditions t_β for λ^+ many moves with Odd playing at odd stages and Even playing at even and limit stages. Even must play the maximal condition of T at move zero and, at move β , the condition t_β must be stronger than any condition played until then. Even wins the game if he can respond at any move. We describe the strategy as follows. At each move β , Even chooses a condition $r_\beta \in \mathbb{R}$ in addition to $t_\beta \in T$ in such a way that $t_\beta \in r_\beta$ and $\text{ht}(r_\beta) = \text{dom}(t_\beta) + 1$. It follows that for every β even or limit, the pair (r_β, t_β) belongs to \mathbb{Q} . The closure of \mathbb{Q} ensures that Even can chose (r_β, t_β) at each stage. \dashv

THEOREM 6.2. *The tree property at \aleph_{ω^2+1} fails in $V^{\mathbb{R} * \mathbb{P}}$.*

PROOF. T is λ^+ -Suslin in $V^{\mathbb{R}}$, we prove that forcing with $\mathbb{S} * \mathbb{P}^*$ over $V^{\mathbb{R}}$ does not add a cofinal branch to T ; in particular T remains Aronszajn in $V^{\mathbb{R} * \mathbb{P}}$ which is a submodel of $V^{\mathbb{R} * \mathbb{S} * \mathbb{P}^*}$. The forcing \mathbb{S} is a product of a forcing of size κ_0 , namely $\text{Coll}(\omega, < \kappa_0)$, with a σ -closed forcing, namely \mathbb{S}_0 . \mathbb{S}_0 is σ -closed, hence by Lemma 2.2 it cannot add cofinal branches to T over $V^{\mathbb{R}}$. The poset $\text{Coll}(\omega, < \kappa_0)$ is λ^+ -Knaster in $V^{\mathbb{R} * \mathbb{S}_0}$, therefore it cannot add cofinal branches to T over $V^{\mathbb{R} * \mathbb{S}_0}$. It follows that T remains Aronszajn in $V^{\mathbb{R} * \mathbb{S}}$.

CLAIM 6.3. *\mathbb{P}^* is λ^+ -Knaster in $V^{\mathbb{R} * \mathbb{S}}$.*

PROOF. Given a sequence $\langle p_\beta; \beta < \lambda^+ \rangle$ of conditions in \mathbb{P}^* , there exists $n < \omega$ and a stationary set $S^* \subseteq \lambda^+$ such that p_β has length n for every $\beta \in S^*$. The possible stems of \mathbb{P}^* length n are $\leq \kappa_{n+1} < \lambda^+$ hence there exists s and a stationary subset $S \subseteq S^*$ such that $\text{stem}(p_\beta) = s$ for every $\beta \in S$. Two conditions of \mathbb{P}^* with the same stem are compatible, hence the conditions in the subsequence $\langle p_\beta; \beta \in S \rangle$ are pairwise compatible. \dashv

It follows that \mathbb{P}^* cannot add cofinal branches to T , hence T remains Aronszajn in $V^{\mathbb{R} * \mathbb{S} * \mathbb{P}^*}$. \dashv

THEOREM 6.4. $V^{\mathbb{R}*\mathbb{P}} \models \Delta_{\aleph_{\omega^2}, \aleph_{\omega^2+1}}$.

To prove the theorem we want to use the old argument from Magidor and Shelah’s paper, the main difficulty is to deal with the presence of the generic Suslin tree T . For the argument to work, we need T to be λ^+ -c.c. in $V^{\mathbb{R}*\mathbb{S}_n}$; this motivates the following lemma.

LEMMA 6.5. *For every $n < \omega$, the tree T remains λ^+ -Suslin in $V^{\mathbb{R}*\mathbb{S}_n}$.*

PROOF. We work in $V^{\mathbb{R}}$. Let be $s \in \mathbb{S}_n$ and \dot{A} such that

$$s \Vdash \dot{A} \text{ is a maximal antichain in } T.$$

We let θ be some regular cardinal much larger than any cardinal under discussion and let \mathcal{H}_θ be some expansion of $\langle H_\theta, \in \rangle$ by at most countably many constants, functions and relations including s , T , \mathbb{S}_n and everything relevant to this proof. We fix a λ^+ -approximating sequence, namely a continuous increasing sequence $\langle M_\alpha \rangle_{\alpha \in \lambda^+}$ of elementary substructures of \mathcal{H}_θ of size $< \lambda^+$ such that for all α , $\lambda^+ \in M_\alpha$, $\langle M_\beta; \beta \leq \alpha \rangle \in M_{\alpha+1}$ and $M_\alpha \cap \lambda^+$ is an ordinal of λ^+ .

By Lemma 6.1 T is λ^+ -strategically closed, we assume that every M_α contains the corresponding strategy τ .

CLAIM 6.6. *Given a model M in the approximating sequence, for every $k \geq n$, $q \in \mathbb{S}_n$ and $x \in T \cap M$, there exists $t^* > x$ in $T \cap M$ and a condition $q^* \leq q$ with $q^* \upharpoonright k = q \upharpoonright k$ such that $q^* \Vdash t^*$ is above some element of \dot{A} .*

PROOF. Let $\{p_\gamma; \gamma < \kappa_k\}$ enumerate all the sequences $\bar{q} \upharpoonright k$ for $\bar{q} \leq q$. We inductively define a decreasing sequence of conditions $\langle q_\alpha \rangle_{\alpha \in \kappa_k}$ in $\mathbb{S}_n \cap M$ and an increasing sequence of nodes $\langle t_\alpha \rangle_{\alpha \in \kappa_k}$ in $T \cap M$ such that

- (1) $q_0 := q$ and $t_0 := x$,
- (2) $q_\alpha \upharpoonright k = q \upharpoonright k$ for every α

moreover, we make sure that the nodes are chosen according to the strategy τ . For α limit ordinal, we let q_α be the union of all q_β where $\beta < \alpha$, and t_α is the node given by the strategy τ applied to $\langle t_\beta \rangle_{\beta < \alpha}$. Suppose that q_α and t_α are defined, we define $q_{\alpha+1}$ and $t_{\alpha+1}$ as follows. We denote by $q_\alpha * p_\alpha$ the unique condition $r \in \mathbb{S}_n$ such that $r \upharpoonright k = p_\alpha$ and $r(i) = q_\alpha(i)$ for all $i \geq k$. Let φ_α be the statement:

“There is a condition \bar{q} compatible with $q_\alpha * p_\alpha$ and there is a node $t > t_\alpha$ such that $\bar{q} \Vdash t$ is above some element of \dot{A} .”

If the statement is true, then we let $t_{\alpha+1} > t$ be the node given by the strategy τ and we let $q_{\alpha+1}$ be given by

$$q_{\alpha+1}(i) := \begin{cases} q_\alpha(i) & \text{if } i < k, \\ q_\alpha(i) \cup \bar{q}(i) & \text{otherwise.} \end{cases}$$

If the statement is false, we let $q_{\alpha+1}$ be q_α and we let $t_{\alpha+1}$ be t_α .

Using the closure of \mathbb{S}_k , let q^* be a lower bound for $\langle q_\alpha; \alpha \in \kappa_k \rangle$ (such a lower bound exists because the conditions in the sequence have the same k -lower part). We also let t^* be the node given by the strategy τ applied to $\langle t_\alpha; \alpha \in \kappa_k \rangle$. By elementarity of M we can assume that both q^* and t^* belong to M . We show that

$$q^* \Vdash t^* \text{ is above some element of } \dot{A}.$$

Suppose otherwise, we will reach a contradiction. q^* forces that \dot{A} is a maximal antichain, so we have

$$q^* \Vdash t^* \text{ is compatible with some element of } \dot{A}.$$

Let be t^{**} and q^{**} be such that t^{**} is compatible with t^* , $q^{**} \leq q^*$ and q^{**} forces that t^{**} is in \dot{A} . The sequence $q^{**} \upharpoonright k$ appears in the enumeration as p_α for some α , so by construction q^{**} is compatible with $q_\alpha * p_\alpha$ and t^{**} is compatible with t_α . It follows that the statement φ_α is true, hence $t_{\alpha+1}$ and $q_{\alpha+1}$ were defined so that $t_{\alpha+1} > t_\alpha$ and

$$q_{\alpha+1} \Vdash t_{\alpha+1} \text{ is above some element of } \dot{A}.$$

This prove that q^{**} and t^{**} are as required. ⊢

We resume the proof of the lemma. Using the claim, we can inductively define a decreasing sequence $\langle s_\alpha \rangle_{\alpha < \lambda^+}$ such that

- (a) $s_0 := s$ and $s_\alpha \upharpoonright k = s \upharpoonright k$ for every α ,
- (b) for every $x \in T \cap M_\alpha$ there exists $y > x$ in $T \cap M_\alpha$ such that

$$s_\alpha \Vdash y \text{ is above some element of } \dot{A}.$$

Finally let be $s_\infty := \bigcup_{\alpha < \lambda^+} s_\alpha$ and $M_\infty := \bigcup_{\alpha < \lambda^+} M_\alpha$. We show that

$$s_\infty \Vdash \text{every element of } \dot{A} \text{ can be extended to an element of } T \cap M_\infty;$$

this will complete the proof as $T \cap M_\infty$ belongs to $V^{\mathbb{R}}$ where T is λ^+ -Suslin. Assume $s_\infty \Vdash \dot{x} \in \dot{A}$, we take $s \leq s_\infty$ and $x \in T$ such that $s \Vdash \dot{x} = x$. By elementarity we can assume without loss of generality that $x \in M_\infty$. Let α be the least ordinal such that $x \in M_\alpha$. By (b) there exists $y > x$ in $T \cap M_\alpha$ such that

$$s_\alpha \Vdash y \text{ is above some element of } \dot{A}.$$

In particular, s_∞ forces the same. By maximality of \dot{A} we have $s_\infty \Vdash y$ is above \dot{x} that completes the proof of the lemma. ⊢

We are now ready to prove Theorem 6.4. From now on, we will essentially follow the arguments of [5] with minor adjustments.

Let $G_{\mathbb{R}}$ be an \mathbb{R} -generic filter over V , let G_T be a T -generic filter over $V[G_{\mathbb{R}}]$ and let G_n be an \mathbb{S}_n -generic filter over V . In $V[G_{\mathbb{R}}]$ we let $p \in \mathbb{P}$, \dot{A} , \dot{S} and $\mu < \lambda$ be such that the S -part of p is in G_n and

$$p \Vdash \dot{A} \text{ is an algebra on } \lambda^+ \text{ with } \mu \text{ operations and } \dot{S} \subseteq \lambda^+ \text{ is a stationary set.}$$

Let $l < \omega$ be such that $\mu < \kappa_l$, we can assume without loss of generality that the length of p is $n > l$ and p forces that every ordinal in \dot{S} has cofinality $< \kappa_l$. In any \mathbb{P} -generic extension $V[G_{\mathbb{R}}][G_{\mathbb{P}}]$ we can observe that for every β in \dot{S} there is a condition $q_\beta \in G_{\mathbb{P}}$ such that $q_\beta \Vdash \beta \in \dot{S}$. Since there are less than λ^+ many possible stems of q_β , there is a stationary subset \dot{E} of \dot{S} such that for every $\beta \in \dot{E}$ the stem of q_β is fixed. We can assume without loss of generality that the stem of p extends this fixed stem, hence p forces that

$$\dot{E} = \{\beta < \lambda^+; \exists q \leq p \text{ in } G_{\mathbb{P}} \text{ such that } stem(q) = stem(p) \text{ and } q \Vdash \beta \in \dot{S}\}$$

is stationary in λ^+ .

In the rest of the proof we will work in $V[G_{\mathbb{R}} * G_T][G_n]$, recall that κ_n is still supercompact in this model. We define \mathbb{P}_n^* to be the set of all conditions $q \in \mathbb{P}$ of

length $\geq n$ such that the S_n -part of q belongs to G_n . We let

$$E^* := \{\alpha < \lambda^+; \exists q \leq p \text{ in } \mathbb{P}_n^* \text{ with the same stem of } p \text{ such that } q \Vdash \alpha \in \dot{E}\}.$$

LEMMA 6.7. $V[G_{\mathbb{R}}][G_T][G_n] \models E^*$ is a stationary subset of λ^+ .

PROOF. Note that $V[G_{\mathbb{R}}][G_T][G_n] = V[G_{\mathbb{R}}][G_n][G_T]$. Assume for a contradiction that we can find in $V[G_{\mathbb{R}}][G_n][G_T]$ a club $C \subseteq \lambda^+$ which is disjoint from E^* . By Lemma 6.5, T is λ^+ -c.c., so there exists a club $D \subseteq C$ that lies in $V[G_{\mathbb{R}}][G_n]$, thus we can assume without loss of generality that C is in $V[G_{\mathbb{R}}][G_n]$. We work in $V[G_{\mathbb{R}}]$. As we did for S_0 in the previous section, we define an equivalence relation \sim on S_n by

$$\langle g_i \rangle_{n-1 < i} \sim \langle h_i \rangle_{n-1 < i} \iff \text{for large enough } i, g_i = h_i.$$

For a condition $r \in S_n$, we denote by $[r]$ its equivalence class. Let $H_n := \{[r]; r \in G_n\}$, then we have

$$V[G_{\mathbb{R}}] \subseteq V[G_{\mathbb{R}}][H_n] \subseteq V[G_{\mathbb{R}}][G_n].$$

The same arguments for the proof of Proposition 4.5 show that S_n/\sim is λ^+ -c.c.

Therefore, we can assume that C lies in $V[G_{\mathbb{R}}][H_n]$. We fix a generic $G_{\mathbb{P}}$ for \mathbb{P} over $V[G_{\mathbb{R}}]$ and we let E be the interpretation of \dot{E} in this model. We observe that $V[G_{\mathbb{R}}][G_{\mathbb{P}}]$ is obtained by forcing with \mathbb{P}_n^* over $V[G_{\mathbb{R}}][G_T][G_n]$. By the definition of E^* we have $E \subseteq E^*$. By Lemma 4.4 we have $V[G_{\mathbb{R}}][H_n] \subseteq V[G_{\mathbb{R}}][G_{\mathbb{P}}]$, in particular C belongs to $V[G_{\mathbb{R}}][G_{\mathbb{P}}]$, but C is disjoint from E^* , hence from E . That contradicts the fact that E is stationary in $V[G_{\mathbb{R}}][G_{\mathbb{P}}]$. \dashv

Now we define in $V[G_{\mathbb{R}}][G_T][G_n]$ a forcing notion C_n whose conditions are sequences $\langle \alpha_0, g_0, f_0, \dots, \alpha_{n-1}, g_{n-1}, f_{n-1}, s \rangle$ such that

- (1) $\langle \alpha_0, \dots, \alpha_{n-1} \rangle$ is the α -part of p ;
- (2) $\langle f_0, \dots, f_{n-1} \rangle$ belongs to $\prod_{i < n} \text{Coll}(\alpha_i^{+\omega+2}, \kappa_i)$;
- (3) $\langle g_0, \dots, g_{n-1} \rangle$ belongs to $\prod_{i < n} \text{Coll}(\kappa_i^{+\omega+2}, < \alpha_i)$;
- (4) $s \in \text{Coll}(\kappa_n^{++}, < \kappa_n)$.

The ordering is:

$$\langle \alpha_0, g_0, f_0, \dots, \alpha_{n-1}, g_{n-1}, f_{n-1}, s \rangle \leq \langle \alpha_0, g'_0, f'_0, \dots, \alpha_{n-1}, g'_{n-1}, f'_{n-1}, s' \rangle$$

if and only if for all i , $f_i \leq f'_i$, $g_i \leq g'_i$ and $s \leq s'$. Let G_{C_n} be any generic filter for C_n that contains the stem of p . In $V[G_{\mathbb{R}}][G_T][G_n][G_{C_n}]$ we define $\mathbb{P}_n^{**} \subseteq \mathbb{P}_n^*$ as the set of all conditions q of length n such that the stem of q is in the generic G_{C_n} for C_n . Every two elements of \mathbb{P}_n^{**} are compatible. Moreover, by the closure of $S_n \times C_n$ we have that \mathbb{P}_n^{**} is σ -closed. Both \mathbb{P}_n^* and \mathbb{P}_n^{**} satisfy the property that for every statement φ in the forcing language of \mathbb{P} there exists $r \in \mathbb{P}_n^*$, respectively $r \in \mathbb{P}_n^{**}$, such that $r \leq p$ and r decides φ (see [5, Lemmas 6 and 8]).

We are going to define an algebra A^* in $V[G_{\mathbb{R}}][G_T][G_n][G_{C_n}]$ that will represent a version of the algebra \dot{A} . Without loss of generality we can assume that the order type of \dot{A} , namely the sequence of the cardinals specifying for each n how many n -ary operations are in \dot{A} , is in $V[\mathbb{R}]$. The algebra A^* will be generated by finite sequences of ordinals less than λ^+ . We must specify for each two terms, whether they denote the same element of the algebra, thus the elements of A^* are actually the equivalence classes of terms. Suppose that \dot{A} can be written as $\langle \lambda^+, \langle \dot{o}_i \rangle_{i < \mu} \rangle$. We can assume without loss of generality that one of the operations of \dot{A} is the

identity on the terms of the algebra. Given $\rho, \eta < \mu$ and given two sequences of ordinals $\vec{\beta}$ and $\vec{\gamma}$ in λ^+ , we let $\rho(\vec{\beta}) =^* \eta(\vec{\gamma})$ if some condition q in \mathbb{P}_n^{**} forces that $\dot{o}_\rho(\vec{\beta}) = \dot{o}_\eta(\vec{\gamma})$ (in the sense of the forcing language for \mathbb{P}). The elements of A^* are the equivalence classes under the relation $=^*$. Note that A^* is well defined because any two conditions of \mathbb{P}_n^{**} are compatible.

We define an ordering on A^* by letting $\rho(\vec{\beta}) <^* \eta(\vec{\gamma})$ if some condition of \mathbb{P}_n^{**} forces that $\dot{o}_\rho(\vec{\beta}) < \dot{o}_\eta(\vec{\gamma})$ as ordinals.

LEMMA 6.8 ([5, Lemmas 9, 10, 11]). *A^* is well ordered by $<^*$ in order type λ^+ .*

Now we work in $V[G_{\mathbb{R}}][G_T][G_n]$ where we let \dot{A}^* be a \mathbb{C}_n -term for the algebra A^* . Let U^* be the interpretation of \dot{F}_n in this model (recall that \dot{F}_n was an $(\mathbb{R} * T) \times \mathbb{S}_n$ -name for a normal ultrafilter on $\mathcal{P}_{\kappa_n}(\lambda^+)$ and U_n is the projection of such an ultrafilter). We let j be the supercompact elementary embedding corresponding to U^* . We consider θ a regular cardinal much larger than any cardinal under discussion and we let $\mathcal{H}_\theta := \langle H_\theta, \lambda^+, E^*, \mathbb{P}_n^*, p, \mathbb{C}_n, \dot{A}^* \rangle$. We define

$$B := \{M \cap \lambda^+; M \prec \mathcal{H}_\theta, |M| < \kappa_n, M \cap \lambda^+ \in \bigcap_{W \in U^* \cap M} W\}.$$

Since $j[\lambda^+] \in j(B)$, we have $B \in U^*$. We let

$$B^* := \{X \in \mathcal{P}_{\kappa_n}(\lambda^+); X \cap \kappa_n \text{ is inaccessible, o.t.}(X) = (X \cap \kappa_n)^{+\omega+1}, E^* \cap X \text{ is stationary in } \text{sup}(X)\},$$

then B^* also belongs to U^* .

LEMMA 6.9. *Let $X \in B \cap B^*$ such that $X \cap \kappa_n \in A_n^p$ and let $M \prec \mathcal{H}_\theta$ be witnessing the fact that $X \in B$. There exists a condition $q \in \mathbb{P}_n^*$ of length $n + 1$ such that $\alpha_n^q = X \cap \kappa_n$ and q extends every extension of p in $\mathbb{P}_n^* \cap M$ with the same stem as p .*

PROOF. The proof is just as in [5, Lemma 13] we include it for the sake of completeness. Let $\alpha_n := X \cap \kappa_n$. We use the closure of \mathbb{S}_n to define q as

$$\langle \alpha_0, g_0, f_0, \dots, \alpha_{n-1}, g_{n-1}, f_{n-1}, \alpha_n, g_n, f_n^*, A_{n+1}^*, g_{n+1}^*, F_{n+1}^*, \dots \rangle$$

where

- (1) $\alpha_0, g_0, f_0, \dots, \alpha_{n-1}, g_{n-1}, f_{n-1}, \alpha_n, g_n$ are like in p ;
- (2) for $j > n$, $A_j^* := \bigcap_{W \in U_j \cap M} W$;
- (3) for $j > n$, g_j^* is the union of the j -th components of $G_n \cap M$;
- (4) $f_n^* := \bigcup \{F(\alpha_n); F \in M, [F]_{U_n} \in K_n\}$;
- (5) for $j > n$, $F_j^*(\beta) := \bigcup \{F(\beta); F \in M, [F]_{U_j} \in K_j\}$.

Every A_j^* belongs to U_j because the cardinality of M is less than $\kappa_n < \kappa_j$ (moreover, $U^* \cap M$ belongs to V by the closure of $(\mathbb{R} * T) \times \mathbb{S}_n$, so we are intersecting a family of sets in V).

We show that f_n^* is in $\text{Coll}(\alpha_n^{+\omega+2}, < \kappa_n)$: we assumed that X belongs to every $W \in U_n^* \cap M$, hence, since U_n is the projection of U_n^* , we have $X \cap \kappa_n = \alpha_n \in W$ for $W \in U_n \cap M$. If $F, F' \in M$ and $[F]_{U_n}, [F']_{U_n} \in K_n$, then the set $P := \{\beta < \kappa_n; F(\beta) \text{ and } F^*(\beta) \text{ are compatible}\}$ belongs to U_n and we have $P \in U_n \cap M$, so $\alpha_n \in P$. It follows that f_n^* is the union of $|M|$ many mutually compatible conditions of $\text{Coll}(\alpha_n^{+\omega+2}, < \kappa_n)$ and by the closure of \mathbb{S}_n this union is in $V[G_{\mathbb{R}}]$. We have

$|M| = |X| = \alpha_n^{+\omega+1}$, hence f_n^* is in $\text{Coll}(\alpha_n^{+\omega+2}, < \kappa_n)$. The same argument shows that the F_j^* are well defined. \dashv

We resume the proof of the theorem. Take $X \in B \cap B^*$ and let M witness the fact that X is in B . Let $\alpha = X \cap \kappa_n = M \cap \kappa_n$ and let q be as in Lemma 6.9.

CLAIM 6.10. *q forces that the subalgebra of \dot{A} generated by X has the same order type as X and X is cofinal in it.*

PROOF. We sketch the proof, for more details see [5, p. 804]. Let B^* be the subalgebra of A^* generated by X . It is not difficult to see that B^* has order type $|X|$ and X is cofinal in it. B^* is defined in $V[G_{\mathbb{R}}][G_T][G_n][G_{\mathbb{C}_n}]$, nevertheless by the κ_n -c.c. of \mathbb{C}_n we can see that B^* exists in the smaller model $V[G_{\mathbb{R}}][G_T][G_n][J]$, where J is the generic object added by G_n for the set of all conditions in \mathbb{C}_n that have as last coordinate a condition in $\text{Coll}(\kappa_{n-1}, < \alpha_n^q)$ (instead of $\text{Coll}(\kappa_{n-1}, < \kappa_n)$). By the closure of $(\mathbb{R} * T) \times \mathbb{S}_n$, we have $B^* \in V[G_{\mathbb{R}}][J]$. Now, let $G_{\mathbb{P}} \subseteq \mathbb{P}$ be a generic filter for \mathbb{P} containing the condition q . Since the stem of q is in J , we have $V[G_{\mathbb{R}}][J] \subseteq V[G_{\mathbb{R}}][G_{\mathbb{P}}]$. Thus B^* is in $V[G_{\mathbb{R}}][G_{\mathbb{P}}]$. Let $\mathcal{A} := (\lambda^+, \langle o_\rho \rangle_{\rho < \mu})$ be the interpretation of \dot{A} by $G_{\mathbb{P}}$ and let \mathcal{B} be the subalgebra of \mathcal{A} generated by X . To prove the claim, it suffices to show that \mathcal{B} is isomorphic to B^* via an isomorphism that is the identity on X . The isomorphism associates to $o_\rho(\vec{\beta})$ the equivalence class of $\rho(\vec{\beta})$, denoted $[\rho(\vec{\beta})]$. To show that this is an isomorphism, consider $[\rho(\vec{\beta})]$ and $[\eta(\vec{\gamma})]$ two terms of B^* and suppose that $[\rho(\vec{\beta})] = [\eta(\vec{\gamma})]$ (resp. $[\rho(\vec{\beta})] < [\eta(\vec{\gamma})]$) in the sense of B^* . This means that there exists $r \leq p$ with the same stem as p and there exists $t \in J$ such that, if r^* is a condition like r except that the stem is t , then r^* forces that $\dot{o}_\rho(\vec{\beta}) = \dot{o}_\eta(\vec{\gamma})$ (resp. $\dot{o}_\rho(\vec{\beta}) < \dot{o}_\eta(\vec{\gamma})$). By elementarity of M , we can assume that $r \in M$, hence $q \leq r$, so $r \in G_{\mathbb{P}}$. By definition of J , we have $r^* \in G_{\mathbb{P}}$, therefore $o_\rho(\vec{\beta}) = o_\eta(\vec{\gamma})$ (resp. $o_\rho(\vec{\beta}) < o_\eta(\vec{\gamma})$). That completes the proof of the claim. \dashv

In conclusion, q forces that the order type of X is a regular cardinal. Since $\alpha_n^q = X \cap \kappa_n$, the order type of X is $(X \cap \kappa_n)^{+\omega+1} = (\alpha_n^q)^{+\omega+1}$ and no cardinal are collapsed between α_n and $\alpha_n^{+\omega+2}$. We also note that for $\beta \in E^* \cap X$, some extension of p in \mathbb{P}_n^* with the same stem forces $\beta \in E$, but by elementarity such an extension is in M , so q extends it, hence q forces $\beta \in E$. Since $X \in B^*$, we have $X \cap E^*$ is stationary in the $\text{sup}(X)$, so we get that q forces that the subalgebra generated by X is a witness to $\Delta_{\aleph_{\omega^2}, \aleph_{\omega^2+1}}$. The other direction is analogous, so that completes the proof of Theorem 6.4 and consequently of Theorem 1.3.

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