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Harold Blum , Yuchen Liu  and Lu Qi 

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ABSTRACT

We prove that the multiplicity of a filtration of a local ring satisfies various convexity properties. In particular, we show the multiplicity is convex along geodesics. As a consequence, we prove that the volume of a valuation is log convex on simplices of quasi-monomial valuations and give a new proof of a theorem of Xu and Zhuang on the uniqueness of normalized volume minimizers. In another direction, we generalize a theorem of Rees on multiplicities of ideals to filtrations and characterize when the Minkowski inequality for filtrations is an equality under mild assumptions.

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1. Introduction

Throughout this paper, (R, \mathfrak{m}) denotes an n -dimensional, analytically irreducible, Noetherian, local domain.

The Hilbert–Samuel multiplicity of an \mathfrak{m} -primary ideal $\mathfrak{a} \subset R$ is a fundamental invariant of the singularities of \mathfrak{a} and satisfies various convexity properties such as Teissier’s Minkowski inequality [Tei78]. In this paper, we consider \mathfrak{m} -filtrations of R , which generalize the filtrations of R given by the powers of a single ideal, and prove various convexity properties for such filtrations. The results have applications to the study of K-stability, volumes of valuations, and problems in commutative algebra.

An \mathfrak{m} -filtration is a collection $\mathfrak{a}_\bullet = (\mathfrak{a}_\lambda)_{\lambda \in \mathbb{R}_{>0}}$ of \mathfrak{m} -primary ideals of R that is decreasing, graded, and left continuous. The latter three conditions mean $\mathfrak{a}_\lambda \subset \mathfrak{a}_\mu$ when $\lambda > \mu$,

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$\mathfrak{a}_\lambda \cdot \mathfrak{a}_\mu \subset \mathfrak{a}_{\lambda+\mu}$, and $\mathfrak{a}_\lambda = \mathfrak{a}_{\lambda-\epsilon}$ when $0 < \epsilon \ll 1$, respectively. The key examples of \mathfrak{m} -filtrations are as follows.

- (1) A trivial example is given by taking powers $(\mathfrak{b}^{[\lambda]})_{\lambda \in \mathbb{R}_{>0}}$ of a fixed \mathfrak{m} -primary ideal $\mathfrak{b} \subset R$.
- (2) An important example in this paper is $\mathfrak{a}_\bullet(v) := (\mathfrak{a}_\lambda(v))_{\lambda \in \mathbb{R}_{>0}}$, where $v: \text{Frac}(R)^\times \rightarrow \mathbb{R}$ is a valuation centered at \mathfrak{m} and $\mathfrak{a}_\lambda(v) := \{f \in R \mid v(f) \geq \lambda\}$ (see § 2.2).
- (3) If $(\mathfrak{b}_\lambda)_{\lambda \in \mathbb{Z}_{>0}}$ is a decreasing, graded sequence of \mathfrak{m} -primary ideals,¹ then $(\mathfrak{b}_{\lceil \lambda \rceil})_{\lambda \in \mathbb{R}_{>0}}$ is an \mathfrak{m} -filtration. Such sequences have been well studied in the literature [Laz04, § 2.4.B].

Following work of Ein, Lazarsfeld, and Smith, the *multiplicity* of an \mathfrak{m} -filtration is

$$\text{vol}(\mathfrak{a}_\bullet) := \lim_{\mathbb{N} \ni m \rightarrow \infty} \frac{\ell(R/\mathfrak{a}_m)}{m^n/n!} = \lim_{\mathbb{N} \ni m \rightarrow \infty} \frac{e(\mathfrak{a}_m)}{m^n},$$

where the existence of the above limit and the equality were proven in increasing generality by [ELS03, Mus02, LM09, Cut13, Cut14]. This invariant is the local counterpart of the volume of a graded linear series of a line bundle and has been studied both in the context of commutative algebra [CSS19, Cut21] and recently in work of C. Li and others on the normalized volume of a valuation [Li18, LLX20].

In this paper, we prove two properties of the multiplicity of \mathfrak{m} -filtrations. The first is a convexity result, which has applications to volumes of valuations and C. Li’s normalized volume function. The second is a generalization of a classical theorem of Rees [Ree61] from the setting of ideals to filtrations and results in a characterization of when the Minkowski inequality for filtrations is an equality.

1.1 Multiplicity and geodesics

Given two \mathfrak{m} -filtrations $\mathfrak{a}_{\bullet,0}$ and $\mathfrak{a}_{\bullet,1}$, we define a segment of \mathfrak{m} -filtrations $(\mathfrak{a}_{\bullet,t})_{t \in [0,1]}$ interpolating between $\mathfrak{a}_{\bullet,0}$ and $\mathfrak{a}_{\bullet,1}$ by setting

$$\mathfrak{a}_{\lambda,t} := \sum_{\lambda=(1-t)\mu+t\nu} \mathfrak{a}_{\mu,0} \cap \mathfrak{a}_{\nu,1}.$$

We call $(\mathfrak{a}_{\bullet,t})_{t \in [0,1]}$ the *geodesic* between $\mathfrak{a}_{\bullet,0}$ and $\mathfrak{a}_{\bullet,1}$, since it is the local analogue of the geodesic between two filtrations of the section ring of a polarized variety [BLXZ23, Reb22]. This definition is also related to a construction in [XZ21].

In [BLXZ23], it was shown that several non-Archimedean functionals from the theory of K-stability are strictly convex along geodesics in the global setting. In a similar spirit, we prove a convexity result in the local setting for the multiplicity along geodesics.

THEOREM 1.1. *Assume R contains a field. If $\mathfrak{a}_{\bullet,0}$ and $\mathfrak{a}_{\bullet,1}$ are \mathfrak{m} -filtrations with positive multiplicity, then the function $E(t): [0,1] \rightarrow \mathbb{R}$ defined by $E(t) := e(\mathfrak{a}_{\bullet,t})$ satisfies the following properties:*

- (1) $E(t)$ is smooth;
- (2) $E(t)^{-1/n}$ is concave, meaning

$$E(t)^{-1/n} \geq (1-t)E(0)^{-1/n} + tE(1)^{-1/n} \quad \text{for all } t \in [0,1];$$

- (3) $E(t)^{-1/n}$ is linear if and only if $\tilde{\mathfrak{a}}_{\bullet,0} = \tilde{\mathfrak{a}}_{\bullet,1}$ for some $c \in \mathbb{R}_{>0}$.

The term ‘smooth’ in Theorem 1.1(1) means $E(t)$ extends to a C^∞ function on $(-\epsilon, 1 + \epsilon)$ for some $\epsilon > 0$. The symbol $\tilde{\cdot}$ in Theorem 1.1(3) denotes the saturation of an \mathfrak{m} -filtration, which

¹ In [Cut21], an \mathfrak{m} -filtration is indexed by $\mathbb{Z}_{>0}$ and, hence, coincides with a decreasing graded sequence of \mathfrak{m} -primary ideals.

is defined in § 3.1. This notion is an analogue of the integral closure of an ideal in the setting of filtrations and discussed further in § 1.2 below.

The proof of Theorem 1.1 is inspired by a related argument in the global setting [BLXZ23] and relies on constructing a measure on \mathbb{R}^2 that encodes the multiplicities of the filtrations along the geodesic. In the special case when $\mathfrak{a}_{\bullet,0}$ is the \mathfrak{m} -filtration of a valuation minimizing the normalized volume function over a Kawamata log terminal (klt) singularity, the proof of [XZ21] can be used to show Theorem 1.1(2). Theorem 1.1 removes these strong restrictions and is proven without the theory of K-stability for valuations introduced in [XZ21, § 3.1].

1.1.1 *Applications to volume.* As first defined by Ein, Lazarsfeld, Smith, the *volume* of a valuation $v: \text{Frac}(R)^\times \rightarrow \mathbb{R}$ centered at \mathfrak{m} is

$$\text{vol}(v) := e(\mathfrak{a}_{\bullet}(v)) = \lim_{m \rightarrow \infty} \frac{\ell(R/\mathfrak{a}_m(v))}{m^n/n!}.$$

This invariant is a local analogue of the volume of a line bundle and also plays a role in the study of K-stability of Fano varieties and Fano cone singularities.

Theorem 1.1 can be applied to show that the volume of a valuation is strictly log convex on simplices of quasi-monomial valuations in the valuation space of (R, \mathfrak{m}) . This gives an affirmative answer to [LLX20, Question 6.23]. We note that the volume was previously shown to be Lipschitz continuous on such a simplex [BFJ14, Corollary D].

COROLLARY 1.2 (Convexity of volume). *Assume R contains a field. Let $\eta \in (Y, D_1 + \dots + D_r)$ be a log smooth birational model of (R, \mathfrak{m}) . For any $\alpha, \beta \in \mathbb{R}_{>0}^r$ and $t \in (0, 1)$,*

$$\text{vol}(v_{(1-t)\alpha+t\beta})^{-1/n} \geq (1-t) \text{vol}(v_\alpha)^{-1/n} + t \text{vol}(v_\beta)^{-1/n}$$

and equality holds if and only if $\alpha = c\beta$ for some $c \in \mathbb{R}_{>0}$.

In the above theorem, v_α denotes the quasi-monomial valuation of $\text{Frac}(R)$ with weights α on D_1, \dots, D_r . See § 2.2.3 for a detailed definition.

1.1.2 *Applications to normalized volume.* In [Li18], Chi Li defined the normalized volume of a valuation over a klt singularity and proposed the problem of studying its minimizer. The notion plays an important role in the study of K-stability of Fano varieties and in the study of klt singularities. The invariant has been extensively studied in the recent years; see [LLX20] and [Zhu23] for surveys on this topic.

The fundamental problem in the study of the normalized volume function is the stable degeneration conjecture proposed by Li [Li18, Conjecture 7.1] and Li and Xu [LX18, Conjecture 1.2]. The conjecture predicts that there exists a valuation minimizing the normalized volume function and that the minimizer is unique up to scaling, quasi-monomial, has finitely generated associated graded ring, and induces a degeneration of the klt singularity to a K-semistable Fano cone singularity. These five statements were proven in [Blu18, LX18, Xu20, XZ21, XZ22]. In particular, the stable degeneration conjecture is now a theorem.

Using Theorem 1.1, we revisit Xu and Zhuang’s theorem stating that the minimizer of the normalized volume function is unique up to scaling [XZ21]. (The uniqueness was also previously proven in [LX18] under the assumption that the minimizer has finitely generated associated ring. The latter was recently shown in [XZ22].) In particular, we give a proof of the uniqueness result independent of the theory of K-semistability for valuations developed in [XZ21].

COROLLARY 1.3 (Uniqueness of minimizer). *If $x \in (X, D)$ is a klt singularity defined over an algebraically closed field of characteristic 0, then any minimizer of $\widehat{\text{vol}}_{X,D,x}$ (see Definition 2.8) is unique up to scaling.*

The new proof of Corollary 1.3 takes the following direct approach. Fix two valuations v_0 and v_1 that minimize $\widehat{\text{vol}}_{X,D,x}$ and consider the geodesic $(\mathbf{a}_{\bullet,t})_{t \in [0,1]}$ between $\mathbf{a}_{\bullet}(v_0)$ and $\mathbf{a}_{\bullet}(v_1)$. Using Theorem 1.1(2), a characterization of the infimum of the normalized volume function in terms of normalized multiplicities [Liu18, Theorem 27], and an inequality of log canonical thresholds [XZ21, Theorem 3.11], we show that $e(\mathbf{a}_{\bullet,t})$ is linear. Theorem 1.1(3) then implies $cv_0 = v_1$ for some $c > 0$.

1.2 Rees’s theorem

A theorem of Rees [Ree61] states that if $\mathfrak{a} \subset \mathfrak{b}$ are two \mathfrak{m} -primary ideals, then the following statements are equivalent.

- (1) $e(\mathfrak{a}) = e(\mathfrak{b})$.
- (2) $\bigoplus_{m \in \mathbb{N}} \overline{\mathfrak{a}^m} = \bigoplus_{m \in \mathbb{N}} \overline{\mathfrak{b}^m}$.
- (3) $\overline{\mathfrak{a}} = \overline{\mathfrak{b}}$.

The symbol $\overline{}$ in (2) denotes the algebraic closure in $R[t]$, while in (3) it denotes the integral closure of an ideal. The equivalence between (2) and (3) follows from definitions.

It is natural to ask for a generalization of the above result for \mathfrak{m} -filtrations $\mathbf{a}_{\bullet} \subset \mathbf{b}_{\bullet}$. In [Cut21], Cutkosky studies whether the two conditions

- (1) $e(\mathbf{a}_{\bullet}) = e(\mathbf{b}_{\bullet})$,
- (2) $\bigoplus_{m \in \mathbb{N}} \mathfrak{a}_m = \bigoplus_{m \in \mathbb{N}} \mathfrak{b}_m$

are equivalent. While (2) \Rightarrow (1) holds by [CSS19, Theorem 6.9], (1) \Rightarrow (2) can fail even in very simple examples (see Example 3.7). That said, (1) \Rightarrow (2) holds for special classes of \mathfrak{m} -filtrations [Cut21, Theorem 1.4].

To remedy this issue, we introduce the *saturation* $\widetilde{\mathbf{a}}_{\bullet}$ of an \mathfrak{m} -filtration \mathbf{a}_{\bullet} in § 3. (The definition may be viewed as a local analogue of a construction studied by Boucksom and Jonsson in [BJ21]; see § 6.) The saturation is defined using divisorial valuations, analogous to the valuative definition of the integral closure of an ideal. Using this notion, we prove a version of Rees’s theorem for filtrations.

THEOREM 1.4 (Rees’s theorem). *For \mathfrak{m} -filtrations $\mathbf{a}_{\bullet} \subset \mathbf{b}_{\bullet}$, $e(\mathbf{a}_{\bullet}) = e(\mathbf{b}_{\bullet})$ if and only if $\widetilde{\mathbf{a}}_{\bullet} = \widetilde{\mathbf{b}}_{\bullet}$.*

The above result can be explained as follows. The multiplicity of an \mathfrak{m} -filtration is determined by its valuative properties, not by the integral properties of its Rees algebra. These two properties coincide for ideals, but do not always coincide for filtrations by Example 3.7.

1.3 Minkowski inequality

By work of Teissier [Tei78], Rees and Sharp [RS78], and Katz [Kat88] in increasing generality, for two \mathfrak{m} -primary ideals \mathfrak{a} and \mathfrak{b} of R ,

$$e(\mathfrak{a}\mathfrak{b})^{1/n} \leq e(\mathfrak{a})^{1/n} + e(\mathfrak{b})^{1/n}$$

and the equality holds if and only if there exist $c, d \in \mathbb{Z}_{>0}$ such that $\overline{\mathfrak{a}^c} = \overline{\mathfrak{b}^d}$.

For two \mathfrak{m} -filtrations \mathbf{a}_{\bullet} and \mathbf{b}_{\bullet} , we let $\mathbf{a}_{\bullet}\mathbf{b}_{\bullet}$ denote the \mathfrak{m} -filtration $(\mathfrak{a}_{\lambda}\mathfrak{b}_{\lambda})_{\lambda \in \mathbb{R}_{>0}}$. Using the saturation of a filtration, we characterize when the Minkowski inequality for filtrations is an equality.

COROLLARY 1.5 (Minkowski inequality). For \mathfrak{m} -filtrations \mathfrak{a}_\bullet and \mathfrak{b}_\bullet with positive multiplicity,

$$e(\mathfrak{a}_\bullet, \mathfrak{b}_\bullet)^{1/n} \leq e(\mathfrak{a}_\bullet)^{1/n} + e(\mathfrak{b}_\bullet)^{1/n} \tag{1.1}$$

and the equality holds if and only if $\tilde{\mathfrak{a}}_\bullet = \tilde{\mathfrak{b}}_{c\bullet}$ for some $c \in \mathbb{R}_{>0}$.

The inequality statement of the above corollary is not new and is due to Mustață [Mus02], Kaveh and Khovanskii [KK14], and Cutkosky [Cut15] in increasing levels of generality. In the equality statement, the forward implication follows easily from [Cut21, Theorem 10.3] and the definition of the saturation, while the reverse implication relies on Theorem 1.4.

Remark 1.6 (Relation to work of Cutkosky). Cutkosky proved a version of the equality part of Corollary 1.5 in the special case when \mathfrak{a}_\bullet and \mathfrak{b}_\bullet are bounded filtrations [Cut21, Definition 1.3], which roughly means that their integral closure is induced by a finite collection of divisorial valuations [Cut21, Theorem 1.6]. For such filtrations, the integral closure and saturation agree by Lemma 3.20. Thus, Corollary 1.5 may be viewed as a generalization of the latter result. Similarly, Theorem 1.4 may be viewed as a generalization of [Cut21, Theorem 1.4].

This paper is organized as follows: In §2, we recall definitions and basic facts concerning filtrations, valuations, and multiplicities. In §3, we introduce the saturation of a filtration and prove Theorem 1.4. In §4, we define the geodesic between two filtrations and prove Theorem 1.1. In §5, we deduce Corollaries 1.2, 1.3, and 1.5 as consequences of results in the previous two sections. In §6, we discuss relations between the results in this paper and global results in the K-stability literature. The appendix of the paper is devoted to an alternate proof of a special case of Theorem 1.1 using the theory of Okounkov bodies.

2. Preliminaries

Throughout this section, $(R, \mathfrak{m}, \kappa)$ denotes an n -dimensional, analytically irreducible,² Noetherian, local domain. We set $X := \text{Spec}(R)$ and write $x \in X$ for the closed point corresponding to \mathfrak{m} .

2.1 Filtrations

DEFINITION 2.1. An \mathfrak{m} -filtration is a collection $\mathfrak{a}_\bullet = (\mathfrak{a}_\lambda)_{\lambda \in \mathbb{R}_{>0}}$ of \mathfrak{m} -primary ideals of R such that

- (1) $\mathfrak{a}_\lambda \subset \mathfrak{a}_\mu$ when $\lambda > \mu$,
- (2) $\mathfrak{a}_\lambda = \mathfrak{a}_{\lambda-\epsilon}$ when $0 < \epsilon \ll 1$, and
- (3) $\mathfrak{a}_\lambda \cdot \mathfrak{a}_\mu \subset \mathfrak{a}_{\lambda+\mu}$ for any $\lambda, \mu \in \mathbb{R}_{>0}$.

By convention, we set $\mathfrak{a}_0 := R$. The definition is a local analogue of a filtration of the section ring of a polarized variety in [BHJ17].

For $\lambda \in \mathbb{R}_{\geq 0}$, define $\mathfrak{a}_{>\lambda} := \bigcup_{\mu > \lambda} \mathfrak{a}_\mu$. If λ satisfies $\mathfrak{a}_{>\lambda} \subsetneq \mathfrak{a}_\lambda$, then we say λ is a *jumping number* of the filtration.

The *scaling* of an \mathfrak{m} -filtration \mathfrak{a}_\bullet by $c \in \mathbb{R}_{>0}$ is $\mathfrak{a}_{c\bullet} := (\mathfrak{a}_{c\lambda})_{\lambda \in \mathbb{R}_{>0}}$. The *product* of two \mathfrak{m} -filtrations \mathfrak{a}_\bullet and \mathfrak{b}_\bullet is $\mathfrak{a}_\bullet \mathfrak{b}_\bullet := (\mathfrak{a}_\lambda \mathfrak{b}_\lambda)_{\lambda \in \mathbb{R}_{>0}}$. Both are again \mathfrak{m} -filtrations.

DEFINITION 2.2. An \mathfrak{m} -filtration \mathfrak{a}_\bullet is *linearly bounded* if there exists a constant $c > 0$ such that $\mathfrak{a}_\lambda \subset \mathfrak{m}^{\lceil c\lambda \rceil}$ for all $\lambda \in \mathbb{R}_{>0}$.

² Recall that (R, \mathfrak{m}) is *analytically irreducible* if its completion \hat{R} is a domain. The latter holds, for example, when R is an excellent Noetherian local ring that is normal by [Sta, Lemma 0C23].

LEMMA 2.3. Let \mathfrak{a}_\bullet and \mathfrak{b}_\bullet be \mathfrak{m} -filtrations. If \mathfrak{a}_\bullet is linearly bounded, then there exists $c \in \mathbb{R}_{>0}$ such that $\mathfrak{a}_{c\lambda} \subset \mathfrak{b}_\lambda$ for all real numbers $\lambda \geq 1$

Proof. Since \mathfrak{a}_\bullet is linearly bounded, there exists $c_0 \in \mathbb{R}_{>0}$ such that $\mathfrak{a}_\lambda \subset \mathfrak{m}^{\lceil c_0\lambda \rceil}$ for all $\lambda > 0$. Since \mathfrak{b}_1 is an \mathfrak{m} -primary ideal, there exists $d \in \mathbb{Z}_{>0}$ such that $\mathfrak{m}^d \subset \mathfrak{b}_1$. If we set $c := 2d/c_0$, then

$$\mathfrak{a}_{c\lambda} \subset \mathfrak{m}^{\lceil 2d\lambda \rceil} \subset \mathfrak{m}^{d\lceil \lambda \rceil} \subset (\mathfrak{b}_1)^{\lceil \lambda \rceil} \subset \mathfrak{b}_{\lceil \lambda \rceil} \subset \mathfrak{b}_\lambda$$

for all $\lambda \geq 1$. (The second inclusion uses that $d\lceil \lambda \rceil \leq d(\lambda + 1) \leq 2d\lambda \leq \lceil 2d\lambda \rceil$). □

2.2 Valuations

Let (Γ, \geq) be a totally ordered abelian group. A Γ -valuation of R is a map $v : \text{Frac}(R)^\times \rightarrow \Gamma$ such that

- (1) $v(fg) = v(f) + v(g)$, and
- (2) $v(f + g) \geq \min\{v(f), v(g)\}$.

By convention, we set $v(0) := \infty$. We say v is centered at \mathfrak{m} if $v \geq 0$ on R and $v > 0$ on $\mathfrak{m} \subset R$.

A valuation v of R induces a valuation ring $R_v := \{f \in R \mid v(f) \geq 0\}$. We write \mathfrak{m}_v for the maximal ideal of R_v and $\kappa_v := R_v/\mathfrak{m}_v$.

2.2.1 *Real valuations.* When $\Gamma = \mathbb{R}$ with the usual order, we say that v is a real valuation. We denote by $\text{Val}_{R,\mathfrak{m}}$ the set of real valuations centered at \mathfrak{m} .³ In geometric settings, we will instead denote the set by $\text{Val}_{X,x}$.

For $v \in \text{Val}_{R,\mathfrak{m}}$ and $\lambda \in \mathbb{R}_{>0}$, we define the valuation ideal

$$\mathfrak{a}_\lambda(v) := \{f \in R \mid v(f) \geq \lambda\}$$

for each $\lambda \in \mathbb{R}$. Using (1) and (2), one can show $\mathfrak{a}_\bullet(v)$ is an \mathfrak{m} -filtration.

For $v \in \text{Val}_{R,\mathfrak{m}}$ and an ideal $\mathfrak{a} \subset R$, set $v(\mathfrak{a}) := \min\{v(f) \mid f \in \mathfrak{a}\}$. For an \mathfrak{m} -filtration \mathfrak{a}_\bullet , set

$$v(\mathfrak{a}_\bullet) := \lim_{\mathbb{N} \ni m \rightarrow \infty} \frac{v(\mathfrak{a}_m)}{m} = \inf_{m \in \mathbb{N}} \frac{v(\mathfrak{a}_m)}{m},$$

where the existence of the limit and second equality is [JM12, Proposition 2.3].

2.2.2 *Divisorial valuations.* A valuation $v \in \text{Val}_{R,\mathfrak{m}}$ is divisorial if

$$\text{tr.deg}_\kappa(\kappa_v) = n - 1.$$

We write $\text{DivVal}_{R,\mathfrak{m}} \subset \text{Val}_{R,\mathfrak{m}}$ for the set of such valuations.

Divisorial valuations appear geometrically. If $\mu : Y \rightarrow X$ is a proper birational morphism with Y normal and $E \subset Y$ a prime divisor, then there is an induced valuation $\text{ord}_E : \text{Frac}(R)^\times \rightarrow \mathbb{Z}$. If $\mu(E) = x$ and $c \in \mathbb{R}_{>0}$, then $c \cdot \text{ord}_E \in \text{DivVal}_{R,\mathfrak{m}}$. When R is excellent, all divisorial valuations are of this form; see, for example, [CS22, Lemma 6.5].

2.2.3 *Quasi-monomial valuations.* In the following construction, we always assume R contains a field. Let $\mu : Y := \text{Spec}(S) \rightarrow X = \text{Spec}(R)$ be a birational morphism with $R \rightarrow S$ finite type and $\eta \in Y$ a not necessarily closed point such that $\mathcal{O}_{Y,\eta}$ is regular and $\mu(\eta) = x$. Given a regular system of parameters y_1, \dots, y_r of $\mathcal{O}_{Y,\eta}$ and $\alpha = (\alpha_1, \dots, \alpha_r) \in \mathbb{R}_{\geq 0}^r \setminus \mathbf{0}$, we define a valuation v_α as follows. For $0 \neq f \in \mathcal{O}_{Y,\eta}$, we can write f in $\widehat{\mathcal{O}}_{Y,\eta} \simeq k(\eta)[[y_1, \dots, y_r]]$ as $\sum_{\beta \in \mathbb{Z}_{\geq 0}^r} c_\beta y^\beta$

³ In the body of this paper, we will only work with real valuations. In the Appendix, \mathbb{Z}^n -valued valuation, where \mathbb{Z}^n has the lexicographic order, will play a key role.

and set

$$v_{\alpha}(f) := \min\{\langle \alpha, \beta \rangle \mid c_{\beta} \neq 0\}.$$

A valuation of the above form is called *quasi-monomial*.

Let $D = D_1 + \dots + D_r$ be a reduced divisor on Y such that $y_i = 0$ locally defines D_i and $\mu(D_i) = x$ for each i . We call $\eta \in (Y, D)$ a *log smooth birational model* of X . We write $\text{QM}_{\eta}(Y, E) \subset \text{Val}_{X,x}$ for the set of quasi-monomial valuations that can be described at η with respect to y_1, \dots, y_r and note that $\text{QM}_{\eta}(Y, D) \simeq \mathbb{R}_{\geq 0}^r \setminus \mathbf{0}$.

2.2.4 Izumi’s inequality. The order function $R \setminus 0 \rightarrow \mathbb{N}$ is defined by

$$\text{ord}_{\mathfrak{m}}(f) := \max\{k \in \mathbb{Z}_{\geq 0} \mid f \in \mathfrak{m}^k\}.$$

The following version of Izumi’s inequality compares $\text{ord}_{\mathfrak{m}}$ to a fixed quasi-monomial valuation.

LEMMA 2.4. *Let $v \in \text{Val}_{R,\mathfrak{m}}$. If (i) v is divisorial or (ii) R contains a field and v is quasi-monomial, then there exists a constant $c > 0$ such that*

$$v(\mathfrak{m}) \cdot \text{ord}_{\mathfrak{m}}(f) \leq v(f) \leq c \cdot \text{ord}_{\mathfrak{m}}(f)$$

for all $f \in R$. In particular, $\mathfrak{a}_{\bullet}(v)$ is linearly bounded.

Proof. Since $f \in \mathfrak{m}^{\text{ord}_{\mathfrak{m}}(f)}$ by definition, $v(\mathfrak{m}) \cdot \text{ord}_{\mathfrak{m}}(f) = v(\mathfrak{m}^{\text{ord}_{\mathfrak{m}}(f)}) \leq v(f)$. It remains to prove the existence of $c > 0$ such that $v(f) \leq c \cdot \text{ord}_{\mathfrak{m}}(f)$ for all $f \in R$.

If (i) holds, the existence of c follows from Izumi’s theorem for divisorial valuations as phrased in [RS14, Remark 1.6].⁴ If (ii) holds, then there exists a log smooth birational model $\eta \in (Y, D)$ of $x \in X$ and $\alpha \in \mathbb{R}^r$ such that $v = v_{\alpha}$. Choose $\gamma \in \mathbb{Z}_{>0}^r$ such that $\alpha_i \leq \gamma_i$ for each $i = 1, \dots, r$ and consider the valuation $w := v_{\gamma}$. We claim that $w \in \text{DivVal}_{R,\mathfrak{m}}$. Assuming the claim, then (i) implies that there exists $c > 0$ such that $w(f) \leq c \cdot \text{ord}_{\mathfrak{m}}(f)$ for all $f \in R$. Since $v(f) \leq w(f)$ for all $f \in R$, (ii) then follows.

To verify that $w \in \text{DivVal}_{R,\mathfrak{m}}$, note that $w : \text{Frac}(R)^{\times} \rightarrow \mathbb{R}$ is the composition

$$\text{Frac}(R)^{\times} \hookrightarrow \text{Frac}(\widehat{\mathcal{O}}_{Y,\eta})^{\times} \xrightarrow{\widehat{w}} \mathbb{R},$$

where \widehat{w} is the valuation that sends $\sum_{\beta \in \mathbb{Z}_{\geq 0}^r} c_{\beta} y^{\beta}$ to $\min\{\langle \alpha, \beta \rangle \mid c_{\beta} \neq 0\}$. Using that w is \mathbb{Z} -valued, a computation (see, for example, [JM12, Proposition 3.7]) shows \widehat{w} is a divisorial valuation of $\widehat{\mathcal{O}}_{Y,\eta}$. Hence, w is a divisorial valuation of $\mathcal{O}_{Y,\eta}$ by [HS06, Proposition 9.3.5]. Now, we compute

$$\text{tr.deg}_{\kappa}(\kappa_w) = \text{tr.deg}_{\kappa}(k(\eta)) + \text{tr.deg}_{k(\eta)}(\kappa_v) = \text{tr.deg}_{\kappa}(k(\eta)) + (\dim \mathcal{O}_{Y,\eta} - 1) = \dim R.$$

To see that the last equality holds, note that the complete local ring \widehat{R} is a domain by assumption and thus equidimensional. Therefore R is universally catenary by [Sta, Tag 0AW6] and so the dimension formula [Sta, Tag 02II] gives the last equality. Therefore, w is divisorial as desired. \square

2.3 Intersection numbers

The theory of intersection numbers of line bundles on a proper scheme over an algebraically closed fields was developed in [Kle66]. We will use a more general framework developed in [Kle05, Appendix B].

2.3.1 Definition. Let Z be a proper scheme over an Artinian ring Λ . For line bundles $\mathcal{L}_1, \dots, \mathcal{L}_r$ on Z , the function

$$\mathbb{Z}^r \ni (m_1, \dots, m_r) \rightarrow \sum_{i \geq 0} (-1)^i \ell_{\Lambda}(H^i(Z, \mathcal{L}_1^{m_1} \otimes \dots \otimes \mathcal{L}_r^{m_r})) \in \mathbb{Z}$$

⁴ The application of Izumi’s Theorem uses our assumption throughout the paper that R is analytically irreducible.

is a polynomial of degree $\leq \dim Z$ [Kle05, Theorem B.7]. The *intersection number* $(\mathcal{L}_1 \cdots \mathcal{L}_r)$ is defined to be the coefficient of $m_1 \cdots m_r$ in the above polynomial. When the choice of Λ is unclear, we will write $(\mathcal{L}_1 \cdots \mathcal{L}_r)_\Lambda$.

2.3.2 Intersections of exceptional divisors. Let $\mu : Y \rightarrow X = \text{Spec}(R)$ be a proper birational morphism with Y normal. For Cartier divisors F_1, \dots, F_{n-1} on Y and a Weil divisor $D := \sum_{i=1}^r a_i D_i$ on Y with support contained in Y_κ , we set

$$F_1 \cdots F_{n-1} \cdot D := \sum_{i=1}^r a_i (\mathcal{O}_Y(F_1)|_{D_i} \cdots \mathcal{O}_Y(F_{n-1})|_{D_i})_\kappa.$$

This is well defined, since each prime divisor D_i is proper over $\text{Spec}(\kappa)$.

PROPOSITION 2.5. *Assume R is complete. If F_1, \dots, F_n are Cartier divisors on Y with support contained in Y_κ , then $F_1 \cdots F_n$ is independent of the ordering of the F_i .*

Since the intersection product in § 2.3.1 is symmetric, $F_1 \cdots F_n$ is independent of the order of F_1, \dots, F_{n-1} . To deduce the full result, we rely on intersection theory [Ful98].

A subtle issue is that the results in [Ful98, § 1-18] are stated for schemes of finite type over a field and, hence, do not immediately apply to Y . Fortunately, the Chow group of a scheme of finite type over a regular base scheme can be defined and the results of [Ful98, § 2] extend to this setting by [Ful98, § 20.1] (see also [Sta, Chapter 02P3] for the results in an even more general setting).

Proof. By Cohen’s structure theorem, there exists a surjective map $A \twoheadrightarrow R$, where A is a regular local ring. Since the composition $Y \rightarrow X := \text{Spec}(R) \rightarrow \text{Spec}(A)$ is finite type, the framework of [Ful98, § 20.1] applies. Using intersection theory on Y and its subschemes, we compute

$$\begin{aligned} F_1 \cdots F_n &= \sum_{i=1}^r a_i \int_{D_i} c_1(\mathcal{M}_1) \cap \cdots \cap c_1(\mathcal{M}_{n-1}) \cap [D_i] \\ &= \int_{Y_\kappa} c_1(\mathcal{M}_1) \cap \cdots \cap c_1(\mathcal{M}_{n-1}) \cap [F_n] = \int_{Y_\kappa} F_1 \cdots F_{n-1} \cdot [F_n], \end{aligned}$$

where \int_{D_i} and \int_{Y_κ} denote the degree maps induced by the proper morphisms $D_i \rightarrow \text{Spec}(\kappa)$ and $Y_\kappa \rightarrow \text{Spec}(\kappa)$ [Ful98, Definition 1.4] and $\mathcal{M}_i := \mathcal{O}_Y(F_i)$. The first equality holds by [Ful98, Example 18.3.6] with the fact that D_i is a proper scheme over κ , the second by the equality $[F_n] = \sum_{i=1}^r a_i [D_i]$, and the last by the definition of the first Chern class. Since

$$F_1 \cdots F_{n-1} \cdot [F_n] = F_1 \cdots F_n \cdot [Y] \in A_*(Y_\kappa)$$

and the latter is independent of the order of the F_i by [Ful98, Corollary 2.4.2], the proposition holds. □

2.4 Multiplicity

2.4.1 Multiplicity of an ideal. The *multiplicity* of an \mathfrak{m} -primary ideal \mathfrak{a} is

$$e(\mathfrak{a}) := \lim_{m \rightarrow \infty} \frac{\ell(R/\mathfrak{a}^m)}{m^n/n!}.$$

The following intersection formula for multiplicities commonly appears in the literature when $x \in X$ is a closed point on a quasi-projective variety [Laz04, p. 92]. In the generality stated below, it follows from [Ram73].

PROPOSITION 2.6. Let $\mathfrak{a} \subset R$ be an \mathfrak{m} -primary ideal and $Y \rightarrow \text{Spec}(R)$ a proper birational morphism with Y normal. If $\mathfrak{a} \cdot \mathcal{O}_Y = \mathcal{O}_Y(-E)$ for a Cartier divisor $E = \sum_i a_i D_i$, then

$$e(\mathfrak{a}) = (-E)^{n-1} \cdot E.$$

Proof. By [Ram73, Theorem and Remark 1], $e(\mathfrak{a}) = (\mathcal{L}^{n-1})_{R/\mathfrak{a}}$, where $\mathcal{L} := \mathcal{O}_Y(-E)|_E$. Observe that

$$(\mathcal{L}^{n-1})_{R/\mathfrak{a}} = \sum \ell_{\mathcal{O}_{E,\xi_i}}(\mathcal{O}_{E,\xi_i})(\mathcal{L}_i^{n-1})_{R/\mathfrak{a}} = \sum a_i (\mathcal{L}_i^{n-1})_{R/\mathfrak{a}},$$

where $E = \sum a_i D_i$ and $\mathcal{L}_i := \mathcal{L}|_{D_i}$. Indeed, the first equality is [Kle05, Lemma B.12] and the second equality holds, since \mathcal{O}_{Y,ξ_i} is a DVR and $\mathcal{O}_{E,\xi_i} = \mathcal{O}_{Y,\xi_i}/(\pi^{a_i})$, where π is a uniformizer of the DVR.

To finish the proof, notice that D_i is defined over κ , since there is a natural inclusion $D_i \hookrightarrow Y_{\kappa,\text{red}} \rightarrow Y_\kappa$. Since the residue field of the local ring R/\mathfrak{a} is κ , any finite-length κ -module M satisfies $\ell_{R/\mathfrak{a}}(M) = \ell_\kappa(M)$ by [Sta, Lemma 02M0]. Thus,

$$\sum_{i=1}^r a_i (\mathcal{L}_i^{n-1})_{R/\mathfrak{a}} = \sum_{i=1}^r a_i (\mathcal{L}_i^{n-1})_\kappa = (-E)^{n-1} \cdot E,$$

which completes the proof. □

2.4.2 *Multiplicity of a filtration.* Following [ELS03], the *multiplicity* of a graded sequence of \mathfrak{m} -primary ideals \mathfrak{a}_\bullet is

$$e(\mathfrak{a}_\bullet) := \lim_{m \rightarrow \infty} \frac{\ell(R/\mathfrak{a}_m)}{m^n/n!} \in [0, \infty).$$

By [ELS03, Mus02, LM09, Cut13, Cut14] in increasing generality, the above limit exists and

$$e(\mathfrak{a}_\bullet) = \lim_{m \rightarrow \infty} \frac{e(\mathfrak{a}_m)}{m^n} = \inf_m \frac{e(\mathfrak{a}_m)}{m^n}; \tag{2.1}$$

see, in particular, [Cut14, Theorem 6.5]. Also defined in [ELS03], the *volume* of a valuation $v \in \text{Val}_{R,\mathfrak{m}}$ is

$$\text{vol}(v) := e(\mathfrak{a}_\bullet(v)).$$

We set

$$\text{Val}_{R,\mathfrak{m}}^+ := \{v \in \text{Val}_{R,\mathfrak{m}} \mid \text{vol}(v) > 0\} \subset \text{Val}_{R,\mathfrak{m}}.$$

PROPOSITION 2.7. Let $v \in \text{Val}_{R,\mathfrak{m}}$. If (i) v is divisorial or (ii) R contains a field and v is quasi-monomial, then $\text{vol}(v) > 0$.

Proof. By Lemma 2.4, there exists $c > 0$ such that $\mathfrak{a}_\bullet(v) \subset \mathfrak{m}^{c\bullet}$. Thus, $\text{vol}(v) \geq e(\mathfrak{m}^{c\bullet}) = c^{-n}e(\mathfrak{m}) > 0$. □

2.5 Normalized volume

Assume R is the local ring of a closed point on an algebraic variety defined over an algebraically closed field of characteristic 0. We say $x \in (X, \Delta)$ is a *klt singularity* if Δ is an \mathbb{R} -divisor on X such that $K_X + \Delta$ is \mathbb{R} -Cartier, and (X, Δ) is klt as defined in [KM98].

The following invariant was first introduced in [Li18] and plays an important role in the study of K-semistable Fano varieties and Fano cone singularities.

DEFINITION 2.8. For a klt singularity $x \in (X, \Delta)$, the *normalized volume function* $\widehat{\text{vol}}_{(X, \Delta), x} : \text{Val}_{X, x} \rightarrow (0, +\infty]$ is defined by

$$\widehat{\text{vol}}_{(X, \Delta), x}(v) := \begin{cases} A_{X, \Delta}(v)^n \cdot \text{vol}(v), & \text{if } A_{X, \Delta}(v) < +\infty, \\ +\infty, & \text{if } A_{X, \Delta}(v) = +\infty, \end{cases}$$

where $A_{X, \Delta}(v)$ is the log discrepancy of v as defined in [JM12, BdFFU15]. The *local volume* of a klt singularity $x \in (X, \Delta)$ is defined as

$$\widehat{\text{vol}}(x, X, \Delta) := \inf_{v \in \text{Val}_{X, x}} \widehat{\text{vol}}_{(X, \Delta), x}(v).$$

The above infimum is indeed a minimum by [Blu18, Xu20].

3. Saturation

Throughout this section, $(R, \mathfrak{m}, \kappa)$ denotes an n -dimensional, analytically irreducible, Noetherian, local domain.

3.1 Definition of the saturation

Let $\mathfrak{a} \subset R$ be an \mathfrak{m} -primary ideal. Recall that the *integral closure* $\bar{\mathfrak{a}}$ of \mathfrak{a} can be characterized valuatively by

$$\bar{\mathfrak{a}} = \{f \in R \mid v(f) \geq v(\mathfrak{a}) \text{ for all } v \in \text{DivVal}_{R, \mathfrak{m}}\}.$$

See, for example, [HS06, Theorem 6.8.3] and [Laz04, Example 9.6.8]. We define the saturation of an \mathfrak{m} -filtration in a similar manner.

DEFINITION 3.1. The *saturation* $\tilde{\mathfrak{a}}_\bullet$ of an \mathfrak{m} -filtration \mathfrak{a}_\bullet is defined by

$$\tilde{\mathfrak{a}}_\lambda := \{f \in \mathfrak{m} \mid v(f) \geq \lambda \cdot v(\mathfrak{a}_\bullet) \text{ for all } v \in \text{DivVal}_{R, \mathfrak{m}}\}$$

for each $\lambda \in \mathbb{R}_{>0}$. We say \mathfrak{a}_\bullet is *saturated* if $\mathfrak{a}_\bullet = \tilde{\mathfrak{a}}_\bullet$.

Remark 3.2. Proposition 3.19 shows that it is equivalent to define the saturation using all positive volume valuations, rather than only divisorial valuations.

Remark 3.3. The saturation is a local analogue of the maximal norm of a multiplicative norm of the section ring of a polarized variety, which was defined and studied in [BJ21]. See § 6.

Remark 3.4. Definition 3.1 differs from the definition of *saturation* used in [Mus02, § 2] for monomial ideals, which coincides with the ideals in Lemma 3.6 defined using the integral closure of the Rees algebra.

Remark 3.5. After the first version of this paper was posted on the arXiv, Cutkosky and Praharaĳ introduced an operation on \mathbb{R} -filtrations that is defined using certain asymptotic Hilbert–Samuel functions (see the definition in [CP22, Theorem 1.3]). As shown by [CP22, Example 7.2], their operation does not always coincide with the saturation.

LEMMA 3.6 [Cut21, Lemma 5.6]. *If \mathfrak{a}_\bullet be an \mathfrak{m} -filtration, then the integral closure of $\text{Rees}(\mathfrak{a}_\bullet) := \bigoplus_{m \in \mathbb{N}} \mathfrak{a}_m t^m$ in $R[t]$ is given by $\bigoplus_{m \in \mathbb{N}} \mathfrak{a}'_m t^m$, where*

$$\mathfrak{a}'_m = \{f \in R \mid \text{there exists } r > 0 \text{ such that } f^r \in \bar{\mathfrak{a}}_{rm}\}.$$

Example 3.7. In general, the ideals $\tilde{\mathfrak{a}}_m$ and \mathfrak{a}'_m appearing above do not coincide. For example, let $R := k[x]_{(x)}$ and $\mathfrak{m} = (x)$. Consider the \mathfrak{m} -filtration \mathfrak{a}_\bullet defined by $\mathfrak{a}_\lambda := (x^{\lceil \lambda + 1 \rceil})$. Using that

$\text{ord}_m(\mathfrak{a}_\lambda) = \lceil \lambda + 1 \rceil$ and $\text{ord}_m(\mathfrak{a}_\bullet) = 1$, we compute

$$\mathfrak{a}'_m = (x^{m+1}) \subsetneq (x^m) = \tilde{\mathfrak{a}}_m$$

for each $m \in \mathbb{Z}_{>0}$. In particular, $\text{Rees}(\mathfrak{a}_\bullet)$ is integrally closed, but \mathfrak{a}_\bullet is not saturated.

The following lemma states basic properties of the saturation.

LEMMA 3.8. *For any \mathfrak{m} -filtration \mathfrak{a}_\bullet , the following statements hold:*

- (1) $\mathfrak{a}_\bullet \subset \tilde{\mathfrak{a}}_\bullet$;
- (2) $v(\mathfrak{a}_\bullet) = v(\tilde{\mathfrak{a}}_\bullet)$ for all $v \in \text{DivVal}_{R,\mathfrak{m}}$; and
- (3) $\tilde{\mathfrak{a}}_\bullet$ is saturated.

Proof. For any $\lambda \in \mathbb{R}_{>0}$,

$$v(\mathfrak{a}_\lambda) = \lim_{m \rightarrow \infty} \frac{v(\mathfrak{a}_\lambda^m)}{m} \geq \lim_{m \rightarrow \infty} \left(\frac{v(\mathfrak{a}_{\lfloor \lambda m \rfloor})}{\lfloor \lambda m \rfloor} \cdot \frac{\lfloor \lambda m \rfloor}{m} \right) = \lambda v(\mathfrak{a}_\bullet),$$

where the inequality uses that $(\mathfrak{a}_\lambda)^m \subset \mathfrak{a}_{\lambda m} \subset \mathfrak{a}_{\lfloor \lambda m \rfloor}$. Therefore, $\mathfrak{a}_\lambda \subset \tilde{\mathfrak{a}}_\lambda$, which is (1).

For (2), note that $v(\mathfrak{a}_\bullet) \leq v(\tilde{\mathfrak{a}}_m)/m \leq v(\mathfrak{a}_m)/m$ for each $m \in \mathbb{Z}_{>0}$, where the first inequality follows from the definition of $\tilde{\mathfrak{a}}_m$ and the second from (1). Sending $m \rightarrow \infty$ gives $v(\mathfrak{a}_\bullet) \leq v(\tilde{\mathfrak{a}}_\bullet) \leq v(\mathfrak{a}_\bullet)$, which implies (2). Statement (3) follows immediately from (2). \square

We say two \mathfrak{m} -filtrations \mathfrak{a}_\bullet and \mathfrak{b}_\bullet are *equivalent* if $\tilde{\mathfrak{a}}_\bullet = \tilde{\mathfrak{b}}_\bullet$. The following proposition gives a characterization of when two filtrations are equivalent after possible scaling.

PROPOSITION 3.9. *Let \mathfrak{a}_\bullet and \mathfrak{b}_\bullet be \mathfrak{m} -filtrations and $c \in \mathbb{R}_{>0}$. The following statements are equivalent.*

- (1) $\tilde{\mathfrak{a}}_\bullet = \tilde{\mathfrak{b}}_{c\bullet}$.
- (2) $v(\mathfrak{a}_\bullet) = cv(\mathfrak{b}_\bullet)$ for all $v \in \text{DivVal}_{R,\mathfrak{m}}$.

Proof. First, note that $v(\mathfrak{b}_{c\bullet}) = cv(\mathfrak{b}_\bullet)$ for all $v \in \text{Val}_{R,\mathfrak{m}}$. Therefore, (1) implies (2) follows from Lemma 3.8(2), while (2) implies (1) follows from the definition of the saturation. \square

3.2 Saturation and completion

Let $(\hat{R}, \hat{\mathfrak{m}})$ denote the \mathfrak{m} -adic completion of (R, \mathfrak{m}) and write $\varphi : R \rightarrow \hat{R}$ for the natural morphism. (Note that \hat{R} is a domain by the assumption throughout the paper that R is analytically irreducible.) For an \mathfrak{m} -filtration \mathfrak{a}_\bullet , we set $\mathfrak{a}_\bullet \hat{R} := (\mathfrak{a}_\lambda \hat{R})_{\lambda > 0}$, which is an $\hat{\mathfrak{m}}$ -filtration.

PROPOSITION 3.10. *If \mathfrak{a}_\bullet is an \mathfrak{m} -filtration and $\mathfrak{b}_\bullet := \mathfrak{a}_\bullet \hat{R}$, then $\tilde{\mathfrak{a}}_\bullet \hat{R} = \tilde{\mathfrak{b}}_\bullet$.*

The proposition shows that completion and saturation commute, as is the case with the integral closure of ideals [HS06, Proposition 1.6.2]. As a consequence of the proposition, many results regarding saturations reduce to the case when R is a complete local domain.

Proof. By [HS06, Theorem 9.3.5], there is a bijective map $\varphi_* : \text{DivVal}_{\hat{R}, \hat{\mathfrak{m}}} \rightarrow \text{DivVal}_{R, \mathfrak{m}}$ that sends \hat{v} to the valuation v defined by composition

$$\text{Frac}(R)^\times \hookrightarrow \text{Frac}(\hat{R})^\times \xrightarrow{\hat{v}} \mathbb{R}.$$

Note that $\hat{v}(\mathfrak{a} \hat{R}) = v(\mathfrak{a})$ for any ideal $\mathfrak{a} \subset R$.

Fix $\lambda \in \mathbb{R}_{>0}$. Using the previous observation twice and the definition of the saturation, we compute

$$\hat{v}(\tilde{\mathfrak{a}}_\lambda \hat{R}) = v(\tilde{\mathfrak{a}}_\lambda) \geq \lambda v(\mathfrak{a}_\bullet) = \lambda \hat{v}(\mathfrak{b}_\bullet).$$

Thus, $\tilde{\mathfrak{a}}_\lambda \hat{R} \subset \tilde{\mathfrak{b}}_\lambda$. To prove the reverse inclusion, note that $\mathfrak{c}R = \tilde{\mathfrak{b}}_\lambda$ for some ideal $\mathfrak{c} \subset R$, since $\tilde{\mathfrak{b}}_\lambda$ is \hat{m} -primary. As before, we compute

$$v(\mathfrak{c}) = \hat{v}(\tilde{\mathfrak{b}}_\lambda) \geq \lambda \hat{v}(\mathfrak{b}_\bullet) = \lambda v(\mathfrak{a}_\bullet).$$

Therefore, $\tilde{\mathfrak{b}}_\lambda = \mathfrak{c}\hat{R} \subset \tilde{\mathfrak{a}}_\lambda \hat{R}$. □

PROPOSITION 3.11. *If \mathfrak{a}_\bullet is an \mathfrak{m} -filtration and $\mathfrak{b}_\bullet = \mathfrak{a}_\bullet \hat{R}$, then $e(\mathfrak{a}_\bullet) = e(\mathfrak{b}_\bullet)$.*

Proof. Since R/\mathfrak{a}_λ and $\hat{R}/\tilde{\mathfrak{b}}_\lambda$ are isomorphic as Artinian rings, $\ell(R/\mathfrak{a}_\lambda) = \ell(\hat{R}/\tilde{\mathfrak{b}}_\lambda)$ for each $\lambda > 0$. Therefore, the equality of multiplicities holds. □

3.3 Saturation and multiplicity

In this section we prove Theorem 1.4 and a number of corollaries. The theorem is a consequence of the following two propositions.

PROPOSITION 3.12. *Let $\mathfrak{a}_\bullet \subset \mathfrak{b}_\bullet$ be \mathfrak{m} -filtrations. If $e(\mathfrak{a}_\bullet) = e(\mathfrak{b}_\bullet)$, then $v(\mathfrak{a}_\bullet) = v(\mathfrak{b}_\bullet)$ for all $v \in \text{Val}_{R,\mathfrak{m}}^+$.*

The proposition was shown when v is divisorial in [Cut21, Theorem 7.3] using Okounkov bodies. The proof below instead follows the strategy of [LX20, Proposition 2.7] and [XZ21, Lemma 3.9].

Proof. Suppose the statement is false. Then there exists $v \in \text{Val}_{R,\mathfrak{m}}^+$ such that $v(\mathfrak{b}_\bullet) < v(\mathfrak{a}_\bullet)$. After scaling v , we may assume $v(\mathfrak{a}_\bullet) = 1$. Therefore, there exists $l \in \mathbb{Z}_{>0}$ so that $v(\mathfrak{b}_l)/l < v(\mathfrak{a}_\bullet) = 1$. Hence, we may choose $f \in \mathfrak{b}_l \setminus \mathfrak{a}_l$ such that $k := v(f) < l$.

Now, consider the map

$$\phi : R \rightarrow \mathfrak{b}_{lm}/\mathfrak{a}_{lm} \quad \text{sending } g \mapsto f^m g.$$

We claim $\ker(\phi) \subset \mathfrak{a}_{(l-k)m}(v)$. Indeed, if $g \in \ker \phi$, then $f^m g \in \mathfrak{a}_{lm}$. Hence,

$$v(g) = v(f^m g) - v(f^m) \geq v(\mathfrak{a}_{lm}) - km \geq lm - km = (l - k)m$$

as desired. Using the claim, we deduce

$$\ell(\mathfrak{b}_{lm}/\mathfrak{a}_{lm}) \geq \ell(R/\ker \phi) \geq \ell(R/\mathfrak{a}_{(l-k)m}(v)). \tag{3.1}$$

Finally, we compute

$$e(\mathfrak{a}_\bullet) - e(\mathfrak{b}_\bullet) = \lim_{m \rightarrow \infty} \frac{\ell(\mathfrak{b}_{lm}/\mathfrak{a}_{lm})}{(lm)^n/n!} \geq \lim_{m \rightarrow \infty} \frac{\ell(R/\mathfrak{a}_{(l-k)m}(v))}{(lm)^n/n!} = \frac{(l - k)^n \text{vol}(v)}{l^n} > 0,$$

where the first inequality is by (3.1) and the second uses that v has positive volume. This contradicts our assumption that $e(\mathfrak{a}_\bullet) = e(\mathfrak{b}_\bullet)$. □

PROPOSITION 3.13. *Let \mathfrak{a}_\bullet and \mathfrak{b}_\bullet be \mathfrak{m} -filtrations. Assume (R, \mathfrak{m}) is complete. If $v(\mathfrak{b}_\bullet) \leq v(\mathfrak{a}_\bullet)$ for all $v \in \text{DivVal}_{R,\mathfrak{m}}$, then $e(\mathfrak{b}_\bullet) \leq e(\mathfrak{a}_\bullet)$.*

The proposition and its proof are local analogues of [Szé15, Lemma 22], which concerns the volumes of graded linear series of projective varieties.

Remark 3.14. The assumption that R is complete in Proposition 3.13 implies that R is Nagata [Sta, Lemma 032W] (see also [Sta, Definition 033S]). The latter property will be used in the proof of Proposition 3.13 to ensure certain normalization morphisms are proper.

Proof. We claim that there exists a sequence of proper birational morphisms

$$\cdots \rightarrow X_3 \rightarrow X_2 \rightarrow X_1 \rightarrow X_0 = X$$

such that the each X_i is normal and the sheafs $\mathfrak{a}_i \cdot \mathcal{O}_{X_m}$ and $\mathfrak{b}_i \cdot \mathcal{O}_{X_m}$ are line bundles when $l \leq m$. Such a sequence can be constructed inductively as follows. Let $X_i \rightarrow X_{i-1}$ be defined by the composition

$$X_i \rightarrow X_{i,2} \rightarrow X_{i,1} \rightarrow X_{i-1},$$

where $X_{i,1}$ is the blowup of $\mathfrak{a}_i \cdot \mathcal{O}_{X_{i-1}}$, $X_{i,2}$ is the blowup of $\mathfrak{b}_i \cdot \mathcal{O}_{X_{i,1}}$, and X_i the normalization of $X_{i,2}$. Note that $X_{i,2} \rightarrow X$ is proper, since $X_{i,2} \rightarrow X_{i-1}$ is a blowup and $X_{i-1} \rightarrow X$ is proper by our inductive assumption. Since X is Nagata and $X_{i,2} \rightarrow X$ is finite type, $X_{i,2}$ is Nagata by [Sta, Lemma 035A]. Therefore, $X_i \rightarrow X_{i,2}$ is finite by the definition of Nagata, and we conclude the composition $X_i \rightarrow X_{i-1}$ is proper, which completes the proof of the claim.

Next, consider a proper birational morphism $Y \rightarrow X$ with Y normal and factoring as

$$Y \xrightarrow{\pi_m} X_m \rightarrow X,$$

For $l \leq m$, there exist Cartier divisors G_l and G'_l on Y such that

$$\mathfrak{a}_l \cdot \mathcal{O}_Y = \mathcal{O}_Y(G_l) \quad \text{and} \quad \mathfrak{b}_l \cdot \mathcal{O}_Y = \mathcal{O}_Y(G'_l).$$

Set $F_l := G_l/l$ and $F'_l := G'_l/l$, which are relatively nef over X and satisfy

$$F_l = - \sum_{E \subset Y} \frac{\text{ord}_E(\mathfrak{a}_l)}{l} E \quad \text{and} \quad F'_l = - \sum_{E \subset Y} \frac{\text{ord}_E(\mathfrak{b}_l)}{l} E,$$

where the sums run through prime divisors $E \subset Y$. Throughout the proof, we will without mention replace Y with higher birational models so it factors through certain $X_m \rightarrow X$.

Given $\epsilon > 0$, by Proposition 2.6 and (2.1), there exists $m_0 > 0$ such that

$$-F_{m_0}^{n-1} \cdot F_{m_0} = e(\mathfrak{a}_{m_0})/m_0^n < e(\mathfrak{a}_\bullet) + \epsilon. \tag{3.2}$$

For a multiple m_1 of m_0 ,

$$-(F_{m_0})^{n-1} \cdot F_{m_1} < e(\mathfrak{a}_\bullet) + \epsilon,$$

since $F_{m_0} \leq F_{m_1}$ and F_{m_0} is nef over X . Now, we compute

$$\begin{aligned} -(F_{m_0})^{n-1} \cdot F_{m_1} + (F_{m_0})^{n-1} \cdot F'_{m_1} &= \sum_{E \subset Y} (-\text{ord}_E(F_{m_1}) + \text{ord}_E(F'_{m_1}))(F_{m_0})^{n-1} \cdot E \\ &= \sum_{E \subset X_{m_0}} (-\text{ord}_E(F_{m_1}) + \text{ord}_E(F'_{m_1}))(\pi_{m_0*} F_{m_0})^{n-1} \cdot E, \end{aligned}$$

where the second equality uses that F_{m_0} is the pullback of $\pi_{m_0*} F_{m_0}$ and the projection formula [Kle05, Proposition B.16]. Since $\pi_{m_0*} F_{m_0}$ is nef over X and

$$\lim_{m \rightarrow \infty} (-\text{ord}_E(F_m) + \text{ord}_E(F'_m)) = \text{ord}_E(\mathfrak{a}_\bullet) - \text{ord}_E(\mathfrak{b}_\bullet) \geq 0,$$

we may choose m_1 sufficiently large and divisible by m_0 such that

$$-(F_{m_0})^{n-1} \cdot F'_{m_1} \leq -F_{m_0}^{n-1} \cdot F_{m_1} + \epsilon.$$

Therefore,

$$-(F_{m_0})^{n-2} \cdot F'_{m_1} \cdot F_{m_0} = -(F_{m_0})^{n-1} \cdot F'_{m_1} \leq e(\mathfrak{a}_\bullet) + 2\epsilon,$$

where the first equality is Proposition 2.5.

For any multiple m_2 of m_1 ,

$$-(F_{m_0})^{n-2} \cdot F'_{m_1} \cdot F_{m_2} \leq e(\mathbf{a}_\bullet) + 2\epsilon,$$

since $F_{m_0} \leq F_{m_2}$ and the terms F'_{m_1} and F_{m_0} are nef over X . Similarly to the previous paragraph, we compute

$$\begin{aligned} & -(F_{m_0})^{n-2} \cdot F'_{m_1} \cdot F_{m_2} - (F_{m_0})^{n-2} \cdot F'_{m_1} \cdot F'_{m_2} \\ &= \sum_{E \subset X_{m_1}} (-\text{ord}_E(F_{m_2}) + \text{ord}_E(F'_{m_2})) ((\pi_{m_1*} F_{m_0})^{n-1} \cdot \pi_{m_1*} F'_{m_1}) \cdot E. \end{aligned}$$

and, hence, we may choose m_2 sufficiently large and divisible by m_1 so that

$$-(F_{m_0})^{n-2} \cdot F'_{m_1} \cdot F'_{m_2} < -(F_{m_0})^{n-2} \cdot F'_{m_1} \cdot F_{m_2} + \epsilon.$$

Thus,

$$-(F_{m_0})^{n-2} \cdot F'_{m_1} \cdot F'_{m_2} < e(\mathbf{a}_\bullet) + 3\epsilon.$$

Repeating in this way gives

$$-F'_{m_1} \cdot F'_{m_2} \cdots F'_{m_n} < e(\mathbf{a}_\bullet) + (n + 1)\epsilon,$$

where each m_i divides m_{i+1} . Since each F'_{m_i} is nef over X and $F'_{m_i} \leq F'_{m_{i+1}}$, we see that

$$-(F'_{m_n})^n < e(\mathbf{a}_\bullet) + n\epsilon.$$

Using that $e(\mathbf{b}_\bullet) = \inf_m e(\mathbf{b}_m)/m^n = \inf_m -(F'_m)^n$ by Lemma 2.6 and (2.1), we see that $e(\mathbf{b}_\bullet) < e(\mathbf{a}_\bullet) + (n + 1)\epsilon$. Therefore, $e(\mathbf{b}_\bullet) \leq e(\mathbf{a}_\bullet)$. \square

Remark 3.15. When $x \in X$ is an isolated singularity on a normal variety, Proposition 3.13 follows easily from the intersection theory for nef b -divisors developed in [BdFF12]. Indeed, the assumption $v(\mathbf{a}_\bullet) \leq v(\mathbf{b}_\bullet)$ implies $Z(\mathbf{a}_\bullet) \geq Z(\mathbf{b}_\bullet)$, where $Z(\mathbf{a}_\bullet)$ is the nef b -Weil \mathbb{R} -divisor associated to \mathbf{a}_\bullet . The proposition then follows from [BdFF12, Remark 4.17]. However, when x is not an isolated singularity, a satisfactory intersection theory for nef b -divisors seems missing from the literature.

We are now ready to prove Theorem 1.4 and a number of its corollaries.

Proof of Theorem 1.4. By Propositions 3.10 and 3.11, we may assume (R, \mathfrak{m}) is complete. This condition will be needed below to apply Proposition 3.13.

By Propositions 3.12 and 3.13, $e(\mathbf{a}_\bullet) = e(\mathbf{b}_\bullet)$ if and only if $v(\mathbf{a}_\bullet) = v(\mathbf{b}_\bullet)$ for all $v \in \text{DivVal}_{R, \mathfrak{m}}$. By Proposition 3.9, the latter condition holds if and only if $\tilde{\mathbf{a}}_\bullet = \tilde{\mathbf{b}}_\bullet$. \square

COROLLARY 3.16. *If \mathbf{a}_\bullet is an \mathfrak{m} -filtration, then $e(\mathbf{a}_\bullet) = e(\tilde{\mathbf{a}}_\bullet)$.*

Proof. By Lemma 3.8, $\mathbf{a}_\bullet \subset \tilde{\mathbf{a}}_\bullet$ and $\tilde{\mathbf{a}}$ is saturated. Thus, Theorem 1.4 implies $e(\mathbf{a}_\bullet) = e(\tilde{\mathbf{a}}_\bullet)$. \square

COROLLARY 3.17. *Let \mathbf{a}_\bullet and \mathbf{b}_\bullet be \mathfrak{m} -filtrations. The following statements are equivalent.*

- (1) $\tilde{\mathbf{a}}_\bullet = \tilde{\mathbf{b}}_\bullet$.
- (2) $e(\mathbf{a}_\bullet) = e(\mathbf{a}_\bullet \cap \mathbf{b}_\bullet) = e(\mathbf{b}_\bullet)$.

Proof of Corollary 3.17. Assume (1) holds. Observe that

$$e(\mathbf{a}_\bullet) = e(\tilde{\mathbf{a}}_\bullet) = e(\tilde{\mathbf{b}}_\bullet) = e(\mathbf{b}_\bullet),$$

where the first and third equality follow from Corollary 3.16. Thus, it remains to show that $e(\mathbf{a}_\bullet \cap \mathbf{b}_\bullet) = e(\mathbf{b}_\bullet)$. First, note that $e(\mathbf{a}_\bullet \cap \mathbf{b}_\bullet) \geq e(\mathbf{b}_\bullet)$ holds trivially, since $\mathbf{a}_\bullet \cap \mathbf{b}_\bullet \subset \mathbf{b}_\bullet$. For the

reverse inequality, we compute

$$\begin{aligned} e(\mathfrak{a}_\bullet \cap \mathfrak{b}_\bullet) &= \lim_{m \rightarrow \infty} \frac{\ell(R/(\mathfrak{a}_m \cap \mathfrak{b}_m))}{m^n/n!} = \lim_{m \rightarrow \infty} \left(\frac{\ell(R/\mathfrak{a}_m)}{m^n/n!} + \frac{\ell(R/\mathfrak{b}_m)}{m^n/n!} - \frac{\ell(R/(\mathfrak{a}_m + \mathfrak{b}_m))}{m^n/n!} \right) \\ &= e(\mathfrak{a}_\bullet) + e(\mathfrak{b}_\bullet) - \lim_{m \rightarrow \infty} \frac{\ell(R/(\mathfrak{a}_m + \mathfrak{b}_m))}{m^n/n!} \\ &\leq e(\mathfrak{a}_\bullet) + e(\mathfrak{b}_\bullet) - \lim_{m \rightarrow \infty} \frac{\ell(R/\tilde{\mathfrak{a}}_m)}{m^n/n!} \\ &= e(\mathfrak{a}_\bullet) + e(\mathfrak{b}_\bullet) - e(\tilde{\mathfrak{a}}_\bullet) \\ &= e(\mathfrak{b}_\bullet), \end{aligned}$$

where the inequality uses that $\mathfrak{a}_m + \mathfrak{b}_m \subset \tilde{\mathfrak{a}}_m + \tilde{\mathfrak{b}}_m = \tilde{\mathfrak{a}}_m + \tilde{\mathfrak{a}}_m = \tilde{\mathfrak{a}}_m$ by the assumption that (1) holds. Therefore, (2) holds.

Conversely, assume (2) holds. Applying Theorem 1.4 to both $(\mathfrak{a}_\bullet \cap \mathfrak{b}_\bullet) \subset \mathfrak{a}_\bullet$ and $(\mathfrak{a}_\bullet \cap \mathfrak{b}_\bullet) \subset \mathfrak{b}_\bullet$, we see that $\mathfrak{a}_\bullet \cap \mathfrak{b}_\bullet = \tilde{\mathfrak{a}}_\bullet$ and $\mathfrak{a}_\bullet \cap \mathfrak{b}_\bullet = \tilde{\mathfrak{b}}_\bullet$, which implies that (1) holds. \square

The following result was proven when R is regular in [Mus02, Theorem 1.7.2].

COROLLARY 3.18. *An \mathfrak{m} -filtration \mathfrak{a}_\bullet is linearly bounded if and only if $e(\mathfrak{a}_\bullet) > 0$.*

Proof. If \mathfrak{a}_\bullet is linearly bounded, then there exists $c > 0$ such that $\mathfrak{a}_\lambda \subset \mathfrak{m}^{\lceil c\lambda \rceil}$ for all $\lambda > 0$. Thus, $e(\mathfrak{a}_\bullet) \geq c^n e(\mathfrak{m}) > 0$ as desired.

Next, assume $e(\mathfrak{a}_\bullet) > 0$. We claim that there exists $v \in \text{DivVal}_{R,\mathfrak{m}}$ such that $v(\mathfrak{a}_\bullet) > 0$. If not, then $\tilde{\mathfrak{a}}_\lambda = \mathfrak{m}$ for all $\lambda > 0$. Using Corollary 3.16, we then see $e(\mathfrak{a}_\bullet) = e(\tilde{\mathfrak{a}}_\bullet) = 0$, which is a contradiction. Now, fix $v \in \text{DivVal}_{R,\mathfrak{m}}$ with $v(\mathfrak{a}_\bullet) > 0$. Using that $\mathfrak{a}_\bullet \subset \mathfrak{a}_\bullet(v)$ and Proposition 2.7, we conclude $e(\mathfrak{a}_\bullet) \geq e(\mathfrak{a}_\bullet(v)) > 0$. \square

3.4 Saturation and finite-volume valuations

Using results from the previous section, we show that the saturation can be defined using positive volume valuations, rather than only divisorial valuations.

PROPOSITION 3.19. *If \mathfrak{a}_\bullet is an \mathfrak{m} -filtration and $\lambda \in \mathbb{R}_{>0}$, then*

$$\tilde{\mathfrak{a}}_\lambda = \{f \in \mathfrak{m} \mid v(f) \geq \lambda \cdot v(\mathfrak{a}_\bullet) \text{ for all } v \in \text{Val}_{R,\mathfrak{m}}^+\}.$$

The proposition will be deduced from the following lemma.

LEMMA 3.20. *If $\{v_i\}_{i \in I}$ is a collection of valuations in $\text{Val}_{R,\mathfrak{m}}^+$ and $\{c_i\}_{i \in I}$ of non-negative real numbers, then the \mathfrak{m} -filtration \mathfrak{a}_\bullet defined by*

$$\mathfrak{a}_\lambda := \{f \in \mathfrak{m} \mid v_i(f) \geq \lambda c_i \text{ for all } i \in I\}$$

is saturated. Hence, $\mathfrak{a}_\bullet(v)$ is saturated for any $v \in \text{Val}_{R,\mathfrak{m}}^+$.

Proof. Suppose the statement is false. Then there exists some $\lambda \in \mathbb{R}_{>0}$ such that $\mathfrak{a}_\lambda \subsetneq \tilde{\mathfrak{a}}_\lambda$. Thus, there exist $f \in \tilde{\mathfrak{a}}_\lambda$ and $i \in I$ such that $v_i(f) < \lambda c_i$. We claim $v_i(\tilde{\mathfrak{a}}_\bullet) < v_i(\mathfrak{a}_\bullet)$. Indeed,

$$v_i(\tilde{\mathfrak{a}}_\bullet) = \lim_{m \rightarrow \infty} \frac{v_i(\tilde{\mathfrak{a}}_{\lfloor \lambda m \rfloor})}{\lfloor \lambda m \rfloor} \leq \lim_{m \rightarrow \infty} \frac{v_i(f^m)}{\lfloor \lambda m \rfloor} = \frac{v_i(f)}{\lambda} < c_i,$$

where the second equality uses that $f^m \in (\tilde{\mathfrak{a}}_\lambda)^m \subset \tilde{\mathfrak{a}}_{\lambda m} \subset \tilde{\mathfrak{a}}_{\lfloor \lambda m \rfloor}$. Since $v_i(\mathfrak{a}_\bullet) \geq c_i$, the claim holds.

By Proposition 3.12 and the claim, $e(\tilde{\mathfrak{a}}_\bullet) < e(\mathfrak{a}_\bullet)$. The latter contradicts Theorem 1.4. \square

Proof of Proposition 3.19. Let \mathfrak{b}_\bullet denote the \mathfrak{m} -filtration defined by

$$\mathfrak{b}_\lambda := \{f \in \mathbb{R} \mid v(f) \geq \lambda \cdot v(\mathfrak{a}_\bullet) \text{ for all } v \in \text{Val}_{R,\mathfrak{m}}^+\}.$$

Notice that $\mathfrak{a}_\bullet \subset \mathfrak{b}_\bullet \subset \tilde{\mathfrak{a}}_\bullet$, where the second inclusion uses that all divisorial valuations have positive volume. Taking saturations gives $\tilde{\mathfrak{a}}_\bullet \subset \tilde{\mathfrak{b}}_\bullet \subset \tilde{\tilde{\mathfrak{a}}}_\bullet$. Since \mathfrak{b}_\bullet and $\tilde{\mathfrak{a}}_\bullet$ are saturated by Lemma 3.20, we conclude $\tilde{\mathfrak{a}}_\bullet \subset \mathfrak{b}_\bullet \subset \tilde{\mathfrak{a}}_\bullet$. □

4. Multiplicity and geodesics

In this section, we prove Theorem 1.1 on the convexity of the multiplicity function along geodesics. Throughout, $(R, \mathfrak{m}, \kappa)$ denotes an n -dimensional, analytically irreducible, Noetherian local domain containing a field of arbitrary characteristic.

4.1 Geodesics

Fix two finite-multiplicity \mathfrak{m} -filtrations $\mathfrak{a}_{\bullet,0}$ and $\mathfrak{a}_{\bullet,1}$.

DEFINITION 4.1. For each $t \in (0, 1)$, we define an \mathfrak{m} -filtration $\mathfrak{a}_{\bullet,t}$ by setting

$$\mathfrak{a}_{\lambda,t} = \sum_{\lambda=(1-t)\mu+t\nu} \mathfrak{a}_{\mu,0} \cap \mathfrak{a}_{\nu,1},$$

where the sum runs through all $\mu, \nu \in \mathbb{R}$ satisfying $\lambda = (1 - t)\mu + t\nu$. We call $(\mathfrak{a}_{\bullet,t})_{t \in [0,1]}$ the *geodesic* between $\mathfrak{a}_{\bullet,0}$ and $\mathfrak{a}_{\bullet,1}$.

This definition is a local analog of the geodesic between two filtrations of the section ring of a polarized variety [BLXZ23, Reb22]. See Section 6 for details.

Example 4.2. Let $R := k[x, y]_{(x,y)}$. For $\alpha = (\alpha_1, \alpha_2) \in \mathbb{R}_{>0}^2$, let $v_\alpha: \text{Frac}(R)^\times \rightarrow \mathbb{R}$ be the monomial valuation with weight α_1 and α_2 with respect to x and y , that is,

$$v\left(\sum_{m,n \geq 0} c_{m,n}x^m y^n\right) = \min\{\alpha_1 m + \alpha_2 n \mid c_{m,n} \neq 0\}.$$

Note that $\mathfrak{a}_\lambda(v_\alpha) = \{x^m y^n \mid m\alpha_1 + n\alpha_2 \geq \lambda\}$. Now, fix $\alpha, \beta \in \mathbb{R}_{\geq 0}^2$ and consider the geodesic $\mathfrak{a}_{\bullet,t}$ between $\mathfrak{a}_{\bullet,0} := \mathfrak{a}_\bullet(v_\alpha)$ and $\mathfrak{a}_{\bullet,1} := \mathfrak{a}_\bullet(v_\beta)$. A short computation shows

$$\mathfrak{a}_{\bullet,t} = \mathfrak{a}_\bullet(v_{(1-t)\alpha+t\beta}).$$

This computation unfortunately does not generalize to the case of quasi-monomial valuations. See [LXZ22, § 4] for a study of this failure in the global setting.

To prove Theorem 1.1, we define a measure on \mathbb{R}^2 that encodes the multiplicities along the geodesic. The argument may be viewed as a local analogue of the construction in [BLXZ23, § 3.1] for the geodesic between two filtrations of the section ring of a polarized variety. The latter global construction is motivated by [BC11] and [BHJ17, Theorem 4.3], which constructs a measure on \mathbb{R} associated to a single filtration of the section ring of polarized variety.

Before proceeding with the proof, fix integers $C > 0$ and $D > 0$ such that

$$\begin{aligned} \mathfrak{m}^{[C\lambda]} &\subset \mathfrak{a}_{\lambda,0} \cap \mathfrak{a}_{\lambda,1}, \\ \mathfrak{a}_{D\lambda,0} &\subset \mathfrak{a}_{\lambda,1} \quad \text{and} \quad \mathfrak{a}_{D\lambda,1} \subset \mathfrak{a}_{\lambda,0}. \end{aligned}$$

for all $\lambda \geq 1$. The existence of C follows from Lemma 2.3, while the existence of D follows from Lemma 2.3 and the fact that $\mathfrak{a}_{\bullet,0}$ and $\mathfrak{a}_{\bullet,1}$ are both linearly bounded by Corollary 3.18.

4.2 Sequences of measures

For each $m \in \mathbb{Z}_{>0}$, consider the function $H_m: \mathbb{R}^2 \rightarrow \mathbb{R}$ defined by

$$H_m(x, y) := \frac{\ell(R/(\mathfrak{a}_{mx,0} \cap \mathfrak{a}_{my,1}))}{m^n/n!}.$$

Notice that H_m is non-decreasing and left continuous in each variable. Using the sequence $(H_m)_{m \geq 1}$, we define a sequence of measures on \mathbb{R}^2 .

PROPOSITION 4.3. *The distributional derivative $\mu_m := -(\partial^2 H_m / \partial x \partial y)$ is a discrete measure on \mathbb{R}^2 and has support contained in $\{(1/D)x \leq y \leq Dx\} \cup \{0 \leq x \leq 1/m\} \cup \{0 \leq y \leq 1/m\}$.*

Before proving the proposition, we prove the following lemma allowing us to reduce to the case of complete local rings.

LEMMA 4.4. *Let $(\widehat{R}, \widehat{\mathfrak{m}})$ denote the completion of (R, \mathfrak{m}) , $\mathfrak{b}_{\bullet,i} = \mathfrak{a}_{\bullet,i} \cdot \widehat{R}$ for $i = 0, 1$, and $(\mathfrak{b}_{t,\bullet})_{t \in [0,1]}$ the geodesic between $\mathfrak{b}_{\bullet,0}$ and $\mathfrak{b}_{\bullet,1}$. Then the following statements hold,*

- (1) $\ell(R/\mathfrak{a}_{a,0} \cap \mathfrak{a}_{b,1}) = \ell(\widehat{R}/\mathfrak{b}_{a,0} \cap \mathfrak{b}_{b,1})$ for each $a, b \geq 0$.
- (2) $\ell(R/\mathfrak{a}_{a,t}) = \ell(\widehat{R}/\mathfrak{b}_{a,t})$ for each $a \geq 0$ and $t \in [0, 1]$.

Proof. For \mathfrak{m} -primary ideals $\mathfrak{c} \subset R$ and $\mathfrak{d} \subset R$,

$$(\mathfrak{c} + \mathfrak{d})\widehat{R} = \mathfrak{c}\widehat{R} + \mathfrak{d}\widehat{R} \quad \text{and} \quad (\mathfrak{c} \cap \mathfrak{d})\widehat{R} = \mathfrak{c}\widehat{R} \cap \mathfrak{d}\widehat{R}.$$

Therefore,

$$(\mathfrak{a}_{a,0} \cap \mathfrak{a}_{b,1})\widehat{R} = \mathfrak{b}_{a,0} \cap \mathfrak{b}_{b,1} \quad \text{and} \quad \mathfrak{a}_{a,t}\widehat{R} = \mathfrak{b}_{a,t}.$$

Statements (1) and (2) follow from these equalities. □

Proof of Proposition 4.3. By Lemma 4.4, it suffices to prove the statement when (R, \mathfrak{m}) is complete. Since R contains a field by assumption and is complete, there is an inclusion $\kappa \hookrightarrow R$ such that the composition $\kappa \hookrightarrow R \rightarrow \kappa$ is the identity. This will be helpful, since any finite-length R -module is naturally a finite-dimensional κ -vector space via restriction of scalars.

Fix an integer $N > 0$ and consider the finite-dimensional κ -vector space $V := R/\mathfrak{m}^{NCm}$. The \mathfrak{m} -filtrations $\mathfrak{a}_{\bullet,0}$ and $\mathfrak{a}_{\bullet,1}$ induce decreasing filtrations \mathcal{F}_0^\bullet and \mathcal{F}_1^\bullet of V defined by

$$\mathcal{F}_0^\lambda V := \text{im}(\mathfrak{a}_{\lambda,0} \rightarrow V) \quad \text{and} \quad \mathcal{F}_1^\lambda V := \text{im}(\mathfrak{a}_{\lambda,1} \rightarrow V).$$

Observe that when $x < N$ and $y < N$,

$$\ell(R/(\mathfrak{a}_{mx,0} \cap \mathfrak{a}_{my,1})) = \dim_\kappa (V/(\mathcal{F}_0^{mx,0} V \cap \mathcal{F}_1^{my,1} V)) = \dim_\kappa V - \dim_\kappa (\mathcal{F}_0^{mx,0} V \cap \mathcal{F}_1^{my,1} V),$$

where the first equality uses that $\mathfrak{m}^{NCm} \subset \mathfrak{a}_{mx,0} \cap \mathfrak{a}_{my,1}$.

To analyze the dimension of $\mathcal{F}_0^{mx} V \cap \mathcal{F}_1^{my} V$, we use that any finite-dimensional vector space with two filtrations admits a basis simultaneously diagonalizing both filtrations (see [BE21, Proposition 1.14] and [AZ22, Lemma 3.1]). This means there exists a basis (s_1, \dots, s_ℓ) for V such that each $\mathcal{F}_j^\lambda V$ is the span of some subset of the basis elements. Hence, if we set $\lambda_{i,j} := \sup\{\lambda \in \mathbb{R} \mid s_i \in \mathcal{F}_j^\lambda V\}$, then

$$\mathcal{F}_0^\lambda V = \text{span}\langle s_i \mid \lambda_{i,0} \geq \lambda \rangle \quad \text{and} \quad \mathcal{F}_1^\lambda V = \text{span}\langle s_i \mid \lambda_{i,1} \geq \lambda \rangle.$$

Using that $\dim_\kappa \mathcal{F}_0^{mx} V \cap \mathcal{F}_1^{my} V = \#\{i \mid \lambda_{i,0} \geq mx \text{ and } \lambda_{i,1} \geq my\}$, we compute

$$\frac{\partial^2}{\partial x \partial y} \dim_\kappa (\mathcal{F}_0^{mx} V \cap \mathcal{F}_1^{my} V) = \sum_{i=1}^{\ell} \delta_{(m^{-1}\lambda_{i,0}, m^{-1}\lambda_{i,1})}.$$

Therefore, the restriction of μ_m to $(-\infty, N) \times (-\infty, N)$ equals

$$\mu_m|_{(-\infty, N) \times (-\infty, N)} = \frac{1}{m^n/n!} \sum_{i|\lambda_{i,0}, \lambda_{i,1} < mN} \delta_{(m^{-1}\lambda_{i,0}, m^{-1}\lambda_{i,1})}. \tag{4.1}$$

Since $N > 0$ was arbitrary, μ_m is a discrete measure on \mathbb{R}^2 .

It remains to analyze the support of μ_m . Note that $\mathcal{F}_0^{\lambda D}V \subset \mathcal{F}_1^\lambda V$ and $\mathcal{F}_1^{\lambda D}V \subset \mathcal{F}_0^\lambda V$ for all $\lambda \geq 1$. Therefore, the following statements hold.

- (i) If $\lambda_{i,1} > 1$, then $\lambda_{i,0} \leq D\lambda_{i,1}$.
- (ii) If $\lambda_{i,0} > 1$, then $\lambda_{i,1} \leq D\lambda_{i,0}$.

Using (i), (ii), and (4.1), we see that $\text{supp}(\mu_m) \subset \{(1/D)x \leq y \leq Dx\} \cup \{0 \leq x \leq 1/m\} \cup \{0 \leq y \leq 1/m\}$. □

Using (4.1), we show that μ_m encodes the colengths of $\mathfrak{a}_{\bullet,t}$.

PROPOSITION 4.5. For each $t \in [0, 1]$ and $a, b \in \mathbb{R}_{\geq 0}$,

$$\begin{aligned} \mu_m(\{(1-t)x + ty < a\}) &= \frac{\ell(R/\mathfrak{a}_{ma,t})}{m^n/n!}, \\ \mu_m(\{x < a\} \cup \{y < b\}) &= \frac{\ell(R/(\mathfrak{a}_{ma,0} \cap \mathfrak{a}_{mb,1}))}{m^n/n!}. \end{aligned}$$

Proof. By Lemma 4.4, it suffices to prove the result when (R, \mathfrak{m}) is complete. In this case we continue with the notation from the previous proof, but additionally fix $N > \max\{a/(1-t), b/t\}$. Now, consider the filtration \mathcal{F}_t^\bullet of V defined by $\mathcal{F}_t^\lambda V := \text{im}(\mathfrak{a}_{\lambda,t} \rightarrow V)$ for each $\lambda \in \mathbb{R}$. Observe that

$$\mathcal{F}_t^{ma}V = \sum_{ma=(1-t)\mu + t\nu} \mathcal{F}_0^\mu V \cap \mathcal{F}_1^\nu V = \text{span}\langle s_i \mid (1-t)\lambda_{i,0} + t\lambda_{i,1} \geq ma \rangle, \tag{4.2}$$

where the sum runs through all $\mu, \nu \in \mathbb{R}$ satisfying $ma = (1-t)\mu + t\nu$. Now, we compute

$$\begin{aligned} \ell(R/\mathfrak{a}_{ma,t}) &= \dim_\kappa V/\mathcal{F}_t^{ma}V = \dim_\kappa V - \dim_\kappa \mathcal{F}_t^{ma}V \\ &= \#\{i \mid (1-t)\lambda_{i,0} + t\lambda_{i,1} < ma\} = \frac{m^n}{n!} \mu_m(\{(1-t)x + ty < a\}) \end{aligned} \tag{4.3}$$

where the first equality uses that $\mathfrak{m}^{NCm} \subseteq \mathfrak{a}_{ma/(1-t),0} + \mathfrak{a}_{mb/t,1} \subset \mathfrak{a}_{ma,t}$, the third (4.2), and the fourth (4.1). Therefore, the first formula in the proposition holds.

To verify the second formula, we fix $N > \max\{a, b\}$ and similarly compute

$$\begin{aligned} \dim_\kappa R/(\mathfrak{a}_{ma,0} \cap \mathfrak{a}_{mb,1}) &= \dim_\kappa V/(\mathcal{F}_0^{ma}V \cap \mathcal{F}_1^{mb}V) = \dim_\kappa V - \dim_\kappa \mathcal{F}_0^{ma}V \cap \mathcal{F}_1^{mb}V \\ &= \#\{j \mid \lambda_{j,0} < ma \text{ or } \lambda_{j,1} < mb\} = \frac{m^n}{n!} \mu_m(\{x < a\} \cup \{y < b\}), \end{aligned}$$

where the first equality uses that $\mathfrak{m}^{NCm} \subset \mathfrak{a}_{ma,0} \cap \mathfrak{a}_{mb,1}$. □

4.3 Limit measure

We now construct a limit of the sequence of measures $(\mu_m)_{m \geq 1}$ that encodes the multiplicity along the geodesic.

Consider the function $H: \mathbb{R}^2 \rightarrow \mathbb{R}$ defined by

$$H(x, y) = e(\mathfrak{a}_{x\bullet,0} \cap \mathfrak{a}_{y\bullet,1}).$$

Above, we use the convention that $e(R) = 0$, which occurs when $x \leq 0$ and $y \leq 0$.

PROPOSITION 4.6. *The distributional derivative $\mu := -(\partial^2 H/\partial x \partial y)$ is a measure on \mathbb{R}^2 and the sequence of measures $(\mu_m)_{m \geq 1}$ converges weakly to μ as $m \rightarrow \infty$.*

Proof. We claim that H_m converges to H in $L^1_{\text{loc}}(\mathbb{R}^2)$. Assuming the claim, then H_m converges to H as distributions, and, hence, μ_m converges to $\mu := -(\partial^2 H/\partial x \partial y)$ as distributions as well. Since each μ_m is a measure, [Hör03, Theorem 2.1.9] implies μ is a measure and μ_m converges to μ weakly as measures.

It remains to prove the above claim. First, observe that H_m converges to H pointwise by the definition of the multiplicity as a limit. Next, fix an integer $N > 0$. For $-N \leq x, y < N$, observe that

$$0 \leq H_m(x, y) \leq \frac{\ell(R/m^{NCm})}{m^n/n!} \leq \sup_{m \geq 1} \frac{\ell(R/m^{NCm})}{m^n/n!} < +\infty.$$

Therefore, the dominated convergence theorem implies that

$$\lim_{m \rightarrow \infty} \int_{(-N, N) \times (-N, N)} |H_m - H| = 0.$$

Since $N > 0$ was arbitrary, H_m converges to H in $L^1_{\text{loc}}(\mathbb{R}^2)$ as desired. □

PROPOSITION 4.7. *For each $t \in [0, 1]$ and $a, b \in \mathbb{R}_{\geq 0}$,*

- (1) $a^n e(\mathbf{a}_{\bullet, t}) = e(\mathbf{a}_{a\bullet, t}) = \mu(\{x(1-t) + yt < a\})$,
- (2) $e(\mathbf{a}_{a\bullet, 0} \cap \mathbf{a}_{b\bullet, 1}) = \mu(\{x < a\} \cup \{y < b\})$.

Proof. Fix $\epsilon > 0$. Since $(\mu_m)_{m \geq 1}$ converges to μ weakly as measures by Proposition 4.6,

$$\begin{aligned} \limsup \mu_m(\{x(1-t) + yt < a - \epsilon\}) &\leq \mu(\{x(1-t) + yt < a\}) \\ &\leq \liminf \mu_m(\{x(1-t) + yt < a\}) \end{aligned}$$

Using Proposition 4.5(1) to compute the lim sup and lim inf, we deduce

$$(a - \epsilon)^n e(\mathbf{a}_{\bullet, t}) = e(\mathbf{a}_{(a-\epsilon)\bullet, t}) \leq \mu(\{x(1-t) + yt < a\}) \leq e(\mathbf{a}_{a\bullet, t}) = a^n e(\mathbf{a}_{\bullet, t}).$$

Sending $\epsilon \rightarrow 0$ completes the proof of (1). The proof of the second is similar, but uses Proposition 4.5(2). □

LEMMA 4.8. *For any real number $c > 0$, the following statements are equivalent.*

- (1) $\text{supp}(\mu) \subset \{cx = y\}$.
- (2) $e(\mathbf{a}_{\bullet, 0}) = e(\mathbf{a}_{\bullet, 0} \cap \mathbf{a}_{c\bullet, 1}) = e(\mathbf{a}_{c\bullet, 1})$.

Proof. If (1) holds, then

$$\mu(\{x < 1\}) = \mu(\{x < 1\} \cup \{y < c\}) = \mu(\{y < c\}).$$

Proposition 4.7 then implies (2) holds.

For the reverse implication, assume (2) holds. We claim

$$H(x, y) = \begin{cases} e(\mathbf{a}_{x\bullet, 0}), & \text{if } cx \geq y, \\ e(\mathbf{a}_{y\bullet, 1}), & \text{if } y \geq cx. \end{cases}$$

Indeed, if $cx \geq y$, then

$$e(\mathbf{a}_{x\bullet, 0}) \leq H(x, y) \leq e(\mathbf{a}_{x\bullet, 0} \cap \mathbf{a}_{cx\bullet, 1}) = e(\mathbf{a}_{x\bullet, 0}).$$

Thus, the claim holds when $cx \geq y$ and similar reasoning treats the case when $cx < y$. Using the claim, we conclude (1) holds. □

PROPOSITION 4.9. *The following statements hold.*

- (1) *The support of μ is contained in $\{(1/D)x \leq y \leq Dx\}$.*
- (2) *μ is homogeneous of degree n (i.e., $\mu(c \cdot A) = c^n \mu(A)$ for any Borel set $A \subset \mathbb{R}^2$ and real number $c > 0$).*

Proof. By Propositions 4.3 and 4.6, $\text{supp}(\mu) \subset \{(1/D)x \leq y \leq Dx\} \cup \{x = 0\} \cup \{y = 0\}$. Note that

$$\mu(\{x = 0\}) = \lim_{a \rightarrow 0^+} \mu(\{0 \leq x < a\}) = \lim_{a \rightarrow 0^+} e(\mathfrak{a}_{a\bullet,0}) = \lim_{a \rightarrow 0^+} a^n e(\mathfrak{a}_{\bullet,0}) = 0,$$

where the second equality is by Proposition 4.7. Similar reasoning shows $\mu(\{y = 0\}) = 0$. Thus, (1) holds. Statement (2) follows from the fact that H is homogeneous of degree n , meaning $H(cx, cy) = c^n H(x, y)$ for each $c > 0$ and $(x, y) \in \mathbb{R}^2$. □

4.4 Measure on the segment

To prove Theorem 1.1, it will be convenient to work with a measure induced by μ on the interval $[0, 1]$.

Consider the embedding

$$j: [0, 1] \hookrightarrow \mathbb{R}^2 \quad \text{defined by} \quad j(z) := (z, 1 - z).$$

Note that $j([0, 1])$ is the line segment between $(0, 1)$ and $(1, 0)$ in \mathbb{R}^2 . For $A \subset [0, 1]$, set

$$\text{Cone}(A) := \mathbb{R}_{\geq 0} \cdot j(A).$$

We define a measure $\tilde{\mu}$ on $[0, 1]$ by setting

$$\tilde{\mu}(A) := \mu(\text{Cone}(A) \cap \{x + y < 1\})$$

for any Borel set $A \subset [0, 1]$. Note that $\tilde{\mu}$ is indeed a measure, since μ is a measure by Proposition 4.6 and $\mu(\{0\}) = 0$, which follows from Proposition 4.7. Additionally, $\text{supp}(\tilde{\mu}) \subset [1/(D + 1), D/(D + 1)]$, since $\text{supp}(\mu) \subset \{(1/D)x \leq y \leq Dx\}$.

LEMMA 4.10. *For any real number $c > 0$, the following statements are equivalent.*

- (1) $\text{supp}(\tilde{\mu}) = 1/(c + 1)$.
- (2) $e(\mathfrak{a}_{\bullet,0}) = e(\mathfrak{a}_{\bullet,0} \cap \mathfrak{a}_{c\bullet,1}) = e(\mathfrak{a}_{c\bullet,1})$.

Proof. Using that $\mathbb{R} \cdot j(1/(c + 1)) = \{cx = y\}$ and the definition of $\tilde{\mu}$, we see that (1) holds if and only if $\text{supp}(\mu|_{\{0 \leq x + y < 1\}}) \subset \{cx = y\}$. Since μ is homogeneous by Proposition 4.9(2), $\text{supp}(\mu|_{\{0 \leq x + y < 1\}}) \subset \{xc = y\}$ if and only if $\text{supp}(\mu) \subset \{cx = y\}$. By Lemma 4.8, the latter condition holds if and only if (2) holds. □

LEMMA 4.11. *For $t \in [0, 1]$,*

$$e(\mathfrak{a}_{\bullet,t}) = \int ((1 - t)z + t(1 - z))^{-n} d\tilde{\mu}.$$

Proof. For simplicity, we assume $t \leq \frac{1}{2}$, which ensures $g_t(z) := (1 - t)z + t(1 - z)$ is non-decreasing. (The case when $t > \frac{1}{2}$ can be proved similarly, but using left-hand approximations below.) For each $m \geq 1$, consider the simple function

$$g_{t,m}(z) = \sum_{k=1}^{2^m} g_t(k/2^m) \cdot 1_{I_{k,m}}$$

where $I_{1,m} := [0, 1/2^m]$ and $I_{k,m} := ((k - 1)/2^m, k/2^m]$ when $1 < k \leq 2^m$. Since $g_{t,m} \geq g_{t,m+1}$ and $g_{t,m} \rightarrow g$ pointwise as $m \rightarrow \infty$, the monotone convergence theorem implies

$$\int g_t(z)^{-n} d\tilde{\mu} = \lim_{m \rightarrow \infty} \int g_{t,m}(z)^{-n} d\tilde{\mu}.$$

(Above we are using that $g_t(z)^{-n}$ is defined and continuous on $(z, t) \in \text{supp}(\tilde{\mu}) \times [0, 1]$.) Now, set $A_{k,m} := g_t(k/2^m)^{-1} \cdot (\text{Cone}(I_{k,m}) \cap \{x + y < 1\}) \subset \mathbb{R}^2$ and $A_m = \cup_{k=1}^{2^m} A_{k,m}$. We compute

$$\int g_{t,m}(z)^{-n} d\tilde{\mu} = \sum_{k=1}^{2^m} g_t(k/2^m)^{-n} \tilde{\mu}(I_{k,m}) = \sum_{k=1}^{2^m} \mu(A_{k,m}) = \mu(A_m),$$

where the second equality uses Proposition 4.9(2). Next, notice that $A_m \subset A_{m+1}$ and

$$\begin{aligned} \bigcup_{m \geq 1} A_m &= \bigcup_{m \geq 1} \bigcup_{z \in [0,1]} g_{t,m}(z)^{-1} \cdot (\text{Cone}(\{z\}) \cap \{x + y < 1\}) \\ &= \bigcup_{z \in [0,1]} g_t(z)^{-1} \cdot (\text{Cone}(\{z\}) \cap \{x + y < 1\}) = \mathbb{R}_{>0}^2 \cap \{(1 - t)x + ty < 1\}. \end{aligned}$$

Combining the previous two computations, we conclude

$$\int g_t(z)^{-n} d\tilde{\mu} = \lim_{m \rightarrow \infty} \mu(A_m) = \mu(\mathbb{R}_{>0}^2 \cap \{(1 - t)x + ty < 1\}) = e(\mathbf{a}_\bullet, t),$$

where the final equality is Proposition 4.7(1). □

4.5 Proof of Theorem 1.1

Proof of Theorem 1.1. Set $g(z, t) := (1 - t)z + t(1 - z)$. By Lemma 4.11,

$$E(t) = \int g(z, t)^{-n} d\tilde{\mu}.$$

Since $g(z, t) = 0$ if and only if $z = t/(2t - 1)$, there exists $\epsilon > 0$ such that

$$g(z, t) \neq 0 \quad \text{for all } (z, t) \in \left[\frac{1}{D+1} - \epsilon, \frac{D}{D+1} + \epsilon \right] \times [-\epsilon, 1 + \epsilon].$$

Since $g(z, t)$ is also smooth on \mathbb{R}^2 , $g(z, t)^{-n}$ is smooth and has bounded derivatives on $(1/(D + 1) - \epsilon, D/(D + 1) + \epsilon) \times (-\epsilon, 1 + \epsilon)$. Using that $\text{supp } \tilde{\mu} \subset [1/(D + 1), D/(D + 1)]$, the Leibniz integral formula implies $\int g(z, t)^{-n} d\tilde{\mu}$ is smooth on $(-\epsilon, 1 + \epsilon)$, which proves (1). The formula additionally implies

$$E'(t) = n \int (2z - 1)g(z, t)^{-n-1} d\tilde{\mu} \quad \text{and} \quad E''(t) = n(n + 1) \int (2z - 1)^2 g(z, t)^{-n-2} d\tilde{\mu}.$$

To prove (2), it suffices to show $(d^2/dt^2)(E(t)^{-1/n}) \leq 0$ for all $t \in (0, 1)$. Note that

$$\frac{d^2}{dt^2} (E(t)^{-1/n}) = \frac{(n + 1)E'(t)^2 - nE(t)E''(t)}{n^2 E(t)^{1/n+2}}.$$

To show the latter quantity is non-positive, we compute

$$\begin{aligned} nE(t)E''(t) &= n^2(n + 1) \left(\int |g(z, t)^{-n}| d\tilde{\mu} \right) \left(\int |(2z - 1)^2 g(z, t)^{-n-2}| d\tilde{\mu} \right) \\ &\geq n^2(n + 1) \left(\int |(2z - 1)g(z, t)^{-n-1}| d\tilde{\mu} \right)^2 \geq (n + 1)E'(t)^2, \end{aligned} \tag{4.4}$$

where the first inequality is by the Cauchy–Schwarz inequality. Therefore, $(d^2/dt^2)(E(t)^{-1/n}) \leq 0$ for all $t \in (0, 1)$, which proves (2).

To prove (3), note that $E(t)$ is linear if and only if $(d^2/dt^2)(E(t)^{-1/n}) = 0$ for all $t \in (0, 1)$. The latter holds if and only if the inequalities in (4.4) are all equalities. The inequalities are equalities if and only if $(2z - 1)g(z, t)^{-(n+2)/2}$ and $g(z, t)^{-n/2}$ are linearly dependent in $L^1(\tilde{\mu})$. Equivalently, $2z - 1$ and $g(z, t)$ are linearly dependent in $L^1(\tilde{\mu})$. The linear dependence holds exactly when $\tilde{\mu}$ is supported at a single point. By Lemma 4.10 and Corollary 3.17, the latter condition is equivalent to the existence of $c \in \mathbb{R}$ such that $\tilde{\mathbf{a}}_{\bullet,0} = \tilde{\mathbf{a}}_{c\bullet,1}$. \square

5. Applications

In this section, we prove Corollaries 1.2, 1.3, and 1.5 using results from previous sections.

5.1 Convexity of the volume of a valuation

Proof of Corollary 1.2. Let $(\mathbf{a}_{\bullet,t})_{t \in [0,1]}$ denote the geodesic between $\mathbf{a}_{\bullet,0} := \mathbf{a}_{\bullet}(v_{\alpha})$ and $\mathbf{a}_{\bullet,1} := \mathbf{a}_{\bullet}(v_{\beta})$. Since $v_{(1-t)\alpha+t\beta}(f) \geq (1-t)v_{\alpha}(f) + tv_{\beta}(f)$ for any $f \in \mathcal{O}_{X,x}$, $\mathbf{a}_{\bullet,t} \subset \mathbf{a}_{\bullet}(v_{(1-t)\alpha+t\beta})$. We compute

$$\begin{aligned} \text{vol}(v_{(1-t)\alpha+t\beta})^{-1/n} &:= e(\mathbf{a}_{\bullet}(v_{(1-t)\alpha+t\beta}))^{-1/n} \geq e(\mathbf{a}_{\bullet,t})^{-1/n} \\ &\geq (1-t)e(\mathbf{a}_{\bullet,0})^{-1/n} + te(\mathbf{a}_{\bullet,1})^{-1/n} = (1-t)\text{vol}(v_0)^{-1/n} + t\text{vol}(v_1)^{-1/n}, \end{aligned}$$

where the second inequality is Theorem 1.1(2). Additionally, by Theorem 1.1(3) and Lemma 3.20, the second inequality is strict unless there exists $c \in \mathbb{R}_{>0}$ such that $\mathbf{a}_{\bullet,0} = \mathbf{a}_{c\bullet,1}$. Since $\mathbf{a}_{\bullet,0} = \mathbf{a}_{c\bullet,1}$ if and only if $c\alpha_0 = \alpha_1$, the result follows. \square

5.2 Uniqueness of normalized volume minimizer

Throughout this section, we assume R is essentially of finite type over a field of characteristic 0.

To prove Corollary 1.3, we need the following proposition, which will be deduced from [XZ21, Theorem 3.11].

PROPOSITION 5.1. *Let $x \in (X, D)$ be a klt singularity. If $\mathbf{a}_{\bullet,0}$ and $\mathbf{a}_{\bullet,1}$ are \mathfrak{m} -filtrations, then*

$$\text{lct}(X, D; \mathbf{a}_{\bullet,t}) \leq (1-t) \cdot \text{lct}(X, D; \mathbf{a}_{\bullet,0}) + t \cdot \text{lct}(X, D; \mathbf{a}_{\bullet,1}),$$

for all $t \in (0, 1)$.

Proof. We claim $v(\mathbf{c}_{\bullet}) = v(\mathbf{a}_{\bullet,t})$ for all $v \in \text{Val}_{R,\mathfrak{m}}$, where \mathbf{c}_{\bullet} is the \mathfrak{m} -filtration defined by $\mathbf{c}_{\lambda} := \sum_{i=0}^{\lceil \lambda \rceil} (\mathbf{a}_{(\lceil \lambda \rceil - i)/(1-t),0} \cap \mathbf{a}_{i/t,0})$. Assuming the claim holds,

$$\begin{aligned} \text{lct}(X, D; \mathbf{a}_{\bullet,t}) &= \text{lct}(X, D; \mathbf{c}_{\bullet}) \\ &\leq \text{lct}(X, D; \mathbf{a}_{\bullet/(1-t),0}) + \text{lct}(X, D; \mathbf{a}_{\bullet/t,1}) = (1-t)\text{lct}(X, D; \mathbf{a}_{\bullet,0}) + t\text{lct}(X, D; \mathbf{a}_{\bullet,1}), \end{aligned}$$

where the inequality is [XZ21, Theorem 3.11]. It remains to verify the claim. Since $\mathbf{c}_m \subset \mathbf{a}_{t,m}$ for each integer $m > 0$, $v(\mathbf{c}_{\bullet}) \geq v(\mathbf{a}_{t,\bullet})$ for all $v \in \text{Val}_{R,\mathfrak{m}}$. Next fix $\mu, \nu \in \mathbb{R}$ such that $\mu(1-t) + \nu t = m$. Set $r := \mu(1-t) = m - \nu t$. For each $\ell \in \mathbb{Z}_{>0}$,

$$\begin{aligned} (\mathbf{a}_{\mu,0} \cap \mathbf{a}_{\nu,1})^{\ell} &\subset (\mathbf{a}_{\ell\mu,0} \cap \mathbf{a}_{\ell\nu,1}) = (\mathbf{a}_{\ell r/(1-t),1} \cap \mathbf{a}_{\ell(m-r)/t,1}) \subset (\mathbf{a}_{\lceil \ell r \rceil/(1-t),0} \\ &\quad \cap \mathbf{a}_{(\ell m - 1 - \lceil \ell r \rceil)/t,1}) \subset \mathbf{c}_{\ell m - 1}. \end{aligned}$$

Using the previous inclusions, we see that

$$v(\mathbf{a}_{\mu,0} \cap \mathbf{a}_{\nu,1}) = \lim_{\ell \rightarrow \infty} \frac{v((\mathbf{a}_{\mu,0} \cap \mathbf{a}_{\nu,1})^\ell)}{\ell} \geq \lim_{\ell \rightarrow \infty} \frac{v(\mathbf{c}_{\ell m-1})}{\ell} = v(\mathbf{c}_\bullet).$$

We can now compute

$$v(\mathbf{a}_{\bullet,t}) = \lim_{m \rightarrow \infty} \frac{v(\mathbf{a}_{m,t})}{m} = \lim_{m \rightarrow \infty} \min_{\mu(1-t)+\nu t=m} \left\{ \frac{v(\mathbf{a}_{\mu,0} \cap \mathbf{a}_{\nu,1})}{m} \right\} \geq v(\mathbf{c}_\bullet).$$

Therefore, $v(\mathbf{a}_{\bullet,t}) = v(\mathbf{c}_\bullet)$ as desired. □

Proof of Corollary 1.3. Let $v_0, v_1 \in \text{Val}_{R,m}$ be minimizers of $\widehat{\text{vol}}_{(X,D),x}$. By rescaling, we may assume $A_{X,\Delta}(v_i) = 1$ and, hence, $\text{vol}(v_i) = \widehat{\text{vol}}(x, X, D)$ for $i = 0, 1$. We seek to show $v_0 = v_1$.

Consider the geodesic $(\mathbf{a}_{t,\bullet})_{t \in [0,1]}$ between $\mathbf{a}_{\bullet,0} := \mathbf{a}_\bullet(v_0)$ and $\mathbf{a}_{\bullet,1} := \mathbf{a}_\bullet(v_1)$. For $t \in (0, 1)$,

$$\begin{aligned} \text{lct}(X, D; \mathbf{a}_{\bullet,t}) &\leq (1-t)\text{lct}(X, D; \mathbf{a}_{\bullet,0}) + t\text{lct}(X, D; \mathbf{a}_{\bullet,1}) \\ &\leq (1-t)A_{X,D}(v_0) + tA_{X,D}(v_1) = 1, \end{aligned} \tag{5.1}$$

where the first inequality is Proposition 5.1 and the second uses that $v_i(\mathbf{a}_{\bullet,i}) = 1$ by [Blu18, Lemma 3.5]. Observe that

$$\begin{aligned} \widehat{\text{vol}}(x, X, D)^{1/n} &\leq \text{lct}(\mathbf{a}_{\bullet,t})e(\mathbf{a}_{\bullet,t})^{1/n} \leq e(\mathbf{a}_{\bullet,t})^{1/n} \\ &\leq ((1-t)e(\mathbf{a}_{\bullet,0})^{-1/n} + te(\mathbf{a}_{\bullet,1})^{-1/n})^{-1} = \widehat{\text{vol}}(X, D, x)^{1/n} \end{aligned}$$

where the first inequality holds by [Liu18, Theorem 27], the second by (5.1), and the third by Theorem 1.1(2). Therefore, every inequality above, in particular the third one, is an equality. By Theorem 1.1(3) and Lemma 3.20, there exists $c > 0$ such that $\mathbf{a}_{\bullet,0} = \mathbf{a}_{c\bullet,1}$, which translates to $\mathbf{a}_\bullet(cv_0) = \mathbf{a}_\bullet(v_1)$. Therefore, $cv_0 = v_1$. Since $c = A_{X,D}(cv_0) = A_{X,D}(v_1) = 1$, we see that $v_0 = v_1$. □

Remark 5.2. [Liu18, Theorem 27] was originally proved for \mathbb{Q} -coefficients, but the proof works for \mathbb{R} -coefficients with little change.

5.3 Minkowski inequality

Lastly, we prove Corollary 1.5. In the proof we use the following result.

PROPOSITION 5.3 [Cut21, Theorem 10.3]. *Let \mathbf{a}_\bullet and \mathbf{b}_\bullet be \mathfrak{m} -filtrations with positive multiplicity. If $e(\mathbf{a}_\bullet \mathbf{b}_\bullet)^{1/n} = e(\mathbf{a}_\bullet)^{1/n} + e(\mathbf{b}_\bullet)^{1/n}$, then there exists $c > 0$ such that $v(\mathbf{a}_\bullet) = cv(\mathbf{b}_\bullet)$ for all $v \in \text{DivVal}_{R,m}$.*

Proof of Corollary 1.5. By [Cut15, Theorem 3.2],

$$e(\mathbf{a}_\bullet \mathbf{b}_\bullet)^{1/n} \leq e(\mathbf{a}_\bullet)^{1/n} + e(\mathbf{b}_\bullet)^{1/n}. \tag{5.2}$$

Thus, it remains to analyze when the inequality is an equality.

If (5.2) is an equality, then Propositions 5.3 and 3.9 imply there exists $c > 0$ such that $\widetilde{\mathbf{a}}_\bullet = \widetilde{\mathbf{b}}_{c\bullet}$. Conversely, assume there exists $c > 0$ such that $\widetilde{\mathbf{a}}_\bullet = \widetilde{\mathbf{b}}_{c\bullet}$. Note that

$$v(\mathbf{a}_\bullet \mathbf{b}_\bullet) = v(\mathbf{a}_\bullet) + v(\mathbf{b}_\bullet) = (c+1)v(\mathbf{b}_\bullet)$$

where the first equality uses that $v(\mathbf{a}_m \mathbf{b}_m)/m = v(\mathbf{a}_m)/m + v(\mathbf{b}_m)/m$ for all $m \in \mathbb{N}$ and the second is by Proposition 3.9. Therefore, Proposition 3.9 implies $\widetilde{\mathbf{a}}_\bullet \mathbf{b}_\bullet = \widetilde{\mathbf{b}}_{(c+1)\bullet}$. We now compute

$$e(\mathbf{a}_\bullet \mathbf{b}_\bullet)^{1/n} = e(\mathbf{b}_{(c+1)\bullet})^{1/n} = e(\mathbf{b}_{c\bullet})^{1/n} + e(\mathbf{b}_\bullet)^{1/n} = e(\mathbf{a}_\bullet)^{1/n} + e(\mathbf{b}_\bullet)^{1/n},$$

where the first and third equality hold by Corollary 3.17. □

6. Relation to global results

In this section, we explain the relationship between the local constructions in this paper and certain global constructions in the K-stability literature.

Throughout, let X be an n -dimensional normal projective variety over a field k and L be an ample line bundle on X . The section ring of (X, L) is

$$R(X, L) := \bigoplus_{m \in \mathbb{N}} R_m := \bigoplus_{m \in \mathbb{N}} H^0(X, mL).$$

6.1 Filtrations and multiplicity

The analogue of an \mathbb{R} -filtration of a local ring in the global setting is the following definition [BHJ17, § 1.1], which plays an important role in the K-stability literature.

DEFINITION 6.1. A filtration \mathcal{F} of $R(X, L)$ is the data of vector subspaces $\mathcal{F}^\lambda R_m \subset R_m$ for each $\lambda \in \mathbb{R}$ and $m \in \mathbb{N}$ such that

- (1) $\mathcal{F}^\lambda R_m \subset \mathcal{F}^\mu R_m$ when $\lambda > \mu$,
- (2) $\mathcal{F}^\lambda R_m = \mathcal{F}^{\lambda-\epsilon} R_m$ when $0 < \epsilon \ll 1$,
- (3) $\mathcal{F}^\lambda R_m \cdot \mathcal{F}^\mu R_n \subset \mathcal{F}^{\lambda+\mu}$,
- (4) $\mathcal{F}^\lambda R_m = R_m$ when $\lambda \ll 0$, and
- (5) $\mathcal{F}^\lambda R_m = 0$ when $\lambda \gg 0$.

A filtration \mathcal{F} is linearly bounded if there exists $C > 0$ such that $\mathcal{F}^{Cm} R_m = 0$ for all $m > 0$.

The data of a filtration \mathcal{F} can be encoded as a norm $\chi : R(X, L) \rightarrow \mathbb{R} \cup \{+\infty\}$ by setting

$$\chi(s) := \max\{\lambda \in \mathbb{R} \mid s \in \mathcal{F}^\lambda R_m\}$$

when $s \in R_m$ and $\chi(\sum_m s_m) := \min\{\chi(s_m)\}$ when $\sum_m s_m \in \bigoplus_m R_m$; see [BJ21, § 1.1] for details. Following [BJ21, § 1.1], we write $\mathcal{N}_{\mathbb{R}}$ for the set of such norms $\chi : R(X, L) \rightarrow \mathbb{R} \cup \{+\infty\}$ that arise from linearly bounded \mathbb{R} -filtrations of $R(X, L)$.

DEFINITION 6.2. The energy of a linearly bounded filtration \mathcal{F} of $R(X, L)$ is

$$E(\mathcal{F}) := \lim_{m \rightarrow \infty} \frac{\sum_{\lambda \in \mathbb{R}} \dim_k \text{gr}_{\mathcal{F}}^\lambda R_m}{m^n/n!},$$

where $\text{gr}_{\mathcal{F}}^\lambda R_m = \mathcal{F}^\lambda R_m / \mathcal{F}^{\lambda+\epsilon} R_m$ and $0 < \epsilon \ll 1$. The limit in the definition exists as a consequence [BHJ17, Theorem 5.3].

The energy measures the size of a filtration and is an analogue of the multiplicity of an \mathbb{R} -filtration in the global setting. The invariant appears under different names in the K-stability literature such as the energy in [BHJ17] and the S -invariant in [Xu21]. For a norm $\chi \in \mathcal{N}_{\mathbb{R}}$ of $R(X, L)$, $\text{vol}(\chi)$ denotes the energy of the associated filtration in [BJ21].

6.2 Saturation and Rees’s theorem

Rees’s theorem for \mathbb{R} -filtrations (Theorem 1.4) says that two \mathbb{R} -filtrations $\mathbf{a}_\bullet \subset \mathbf{b}_\bullet$ have equal volume if and only if their saturations are equal. As we will explain, an analogue of this result was previously proven by Boucksom and Jonsson in the global setting.

For a norm $\chi \in \mathcal{N}_{\mathbb{R}}$, Boucksom and Jonsson introduce the notion of its maximal norm χ^{\max} [BJ21, Definition 6.16]. Similar to the saturation in Definition 3.1, it is defined using divisorial valuations. When $\text{char}(k) = 0$, they prove that two norms $\chi, \chi' \in \mathcal{N}_{\mathbb{R}}$ with $\chi \leq \chi'$ satisfy $\text{vol}(\chi) = \text{vol}(\chi')$ if and only if $\chi^{\max} = \chi'^{\max}$ [BJ21, Lemma 3.11 and Theorem 6.22]. The proof relies on results from non-Archimedean pluripotential developed in [BJ22].

6.3 Geodesics

The notion of a geodesic between two \mathbb{R} -filtrations (Definition 4.1) is inspired by the following definition in the global setting.

DEFINITION 6.3 [BLXZ23, Reb22]. For two linearly bounded filtrations \mathcal{F}_0 and \mathcal{F}_1 of $R(X, L)$ and $t \in (0, 1)$, we define a filtration \mathcal{F}_t of $R(X, L)$ by setting

$$\mathcal{F}_t^\lambda R_m = \sum_{\mu+\nu=\lambda} \mathcal{F}_0^\mu R_m \cap \mathcal{F}_1^\nu R_m.$$

We call $(\mathcal{F}_t)_{t \in [0,1]}$ the *geodesic* between \mathcal{F}_0 and \mathcal{F}_1 .

This definition was introduced by the first two authors, Xu, and Zhuang in [BLXZ23, §3.1.2] to prove uniqueness results for certain optimal destabilizations of Fano varieties that arise from limits of Kähler–Ricci flow [BLXZ23, §3]. Independently, Reboulet introduced an equivalent definition phrased in the language of norms on the section ring, rather than filtrations, and used it to define geodesics between metrics on a line bundle in non-Archimedean pluripotential theory [Reb22].

A different and possibly more intuitive way to understand the above definition is in terms of a well-chosen basis. By [AZ22, Lemma 3.1], there exists a basis (s_1, \dots, s_{N_m}) of R_m that is compatible with both \mathcal{F}_0 and \mathcal{F}_1 . Here, ‘compatible’ means that $\mathcal{F}_j^\lambda R_m$ is the span of some subset of (s_1, \dots, s_{N_m}) for each $j \in \{0, 1\}$ and $\lambda \in \mathbb{R}$. If we set $\lambda_{i,j} := \max\{\lambda \in \mathbb{R} \mid s_i \in \mathcal{F}_j^\lambda R_m\}$, then a computation shows

$$\mathcal{F}_t^\lambda R_m := \text{span}(s_i \mid \lambda_{i,0}(1 - t) + \lambda_{i,1}t \geq \lambda).$$

(A similar computation is made in the proof of Proposition 4.3.) Imprecisely, this expression shows that $(\mathcal{F}_t)_{t \in [0,1]}$ interpolates between \mathcal{F}_0 and \mathcal{F}_1 .

As shown in [BLXZ23, §3.2] and [Reb23, §4.7], various functionals on the space of filtrations are convex along geodesics. These convexity results are proven in [BLXZ23, §3.2] using a measure on \mathbb{R}^2 (similarly to the proof of Theorem 1.1) and imply the uniqueness of minimizers of the h -functional on the space of valuations on a Fano variety [BLXZ23, §3.3]. Thus, Theorem 1.1 and Corollary 1.3 can be viewed as local counterparts to these global results.

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CONFLICTS OF INTEREST

None.

Appendix A. Okounkov bodies and geodesics

In this appendix, we give an alternative proof of a weaker version of Theorem 1.1 (see Theorem A.7) using the theory of Okounkov bodies.

Throughout, X is an n -dimensional normal variety over an algebraically closed field \mathbb{k} and $x \in X$ is a closed point. We set $R := \mathcal{O}_{X,x}$ and write $\mathfrak{m} \subset R$ for the maximal ideal. Note that the assumptions on R are more restrictive than in the earlier sections of this paper.

A.1 Good valuations

DEFINITION A.1 [KK14, Definition 8.3]. We equip a total order \preceq on \mathbb{Z}^n respecting addition. Let $v : R \setminus \{0\} \rightarrow \mathbb{Z}^n$ be a valuation. We say that v is a *good valuation* if it satisfies the following conditions.

- (1) v has one-dimensional leaves, that is, for any $f, g \in R$ with $v(f) = v(g)$, there exists $\lambda \in \mathbb{k}$ such that $v(f + \lambda g) > v(f)$.
- (2) The value semigroup $\mathcal{S} = v(R \setminus \{0\}) \cup \{0\}$ generates \mathbb{Z}^n as a group, and its associated cone $C(\mathcal{S})$, which is the closure of the convex hull of \mathcal{S} , is a strongly convex cone. The latter is equivalent to the existence of a linear function $\xi : \mathbb{R}^n \rightarrow \mathbb{R}$ such that $C(\mathcal{S}) \setminus \{0\} \subset \xi_{>0}$.
- (3) There exist $r_0 > 0$ and a linear function ξ as above such that

$$\text{ord}_m(f) \geq r_0 \xi(v(f))$$

for any $f \in R \setminus \{0\}$.

For every quasi-monomial valuation $w \in \text{Val}_{X,x}$, we can construct a good valuation v on X associated to w as follows. The construction is similar to [KK14, Example 8.5], where the characteristic of \mathbb{k} is 0. We provide a proof for the general case here for the reader’s convenience.

LEMMA A.2. *If $w \in \text{Val}_{X,x}$ is a quasi-monomial valuation, then there exist a vector $\alpha \in \mathbb{R}^n$ and a good \mathbb{Z}^n -valued valuation v on R such that*

$$w(f) = \langle \alpha, v(f) \rangle$$

for any $f \in R \setminus \{0\}$. In addition, we can take $\xi = \langle \alpha, \cdot \rangle$ for condition (3) of a good valuation.

Proof. Since w is quasi-monomial, there is a birational morphism of varieties $\mu : Y \rightarrow X$, a point $\eta \in Y$, and a divisor D on Y that is simple normal crossing (snc) at η such that $w \in \text{QM}_\eta(Y, D)$. We may further assume $\eta \in Y$ is a closed point and D is locally defined at η by a regular system of a parameters $y_1, \dots, y_n \in \mathcal{O}_{Y,\eta}$. (Indeed, since the locus where (Y, D) is snc is open, we may replace η with closed point in $\bar{\eta}$ and possibly add components to D so that the latter holds.) By replacing Y with a higher birational model as in the proof of [JM12, Lemma 3.6], we may assume $w = v_\alpha$, where $\alpha := (\alpha_1, \dots, \alpha_r, 0, \dots, 0)$ and $\alpha_1, \dots, \alpha_r$ are linearly independent over \mathbb{Q} .

Choose $\alpha'_{r+1}, \dots, \alpha'_n > 0$ linearly independent over \mathbb{Q} and define $\alpha' := (0, \dots, 0, \alpha'_{r+1}, \dots, \alpha'_n)$. Then we can define a quasi-monomial valuation u on $k[[y_1, \dots, y_n]] \cong \widehat{\mathcal{O}}_{Y,z}$ given by the weight vector α' . Define $\mathbf{u} : \widehat{\mathcal{O}}_{Y,z} \setminus \{0\} \rightarrow (\mathbb{R}^2, \leq_{\text{lex}})$ by

$$\mathbf{u}(f) := (w(f), u(f)).$$

Then it is easy to check that \mathbf{u} is an \mathbb{R}^2 -valued valuation. Moreover, it induces a well-order \preceq on \mathbb{N}^n by

$$\beta_1 \preceq \beta_2 \text{ if and only if } (\langle \alpha, \beta_1 \rangle, \langle \alpha', \beta_1 \rangle) \leq_{\text{lex}} (\langle \alpha, \beta_2 \rangle, \langle \alpha', \beta_2 \rangle).$$

Now define $v : \widehat{\mathcal{O}}_{Y,z} \setminus \{0\} \rightarrow (\mathbb{N}^n, \preceq)$ by $v(f) := \min\{\beta \mid c_\beta \neq 0\}$. Then v is a \mathbb{Z}^n -valued valuation on X with center x .

Following the calculation in [Cut13, § 4], we see that the value semigroup \mathcal{S} contains β' and $\beta' + e_i$ for some β' and $1 \leq i \leq n$, where $e_i \in \mathbb{Z}^n$ is the i th standard basis vector. Hence, \mathcal{S} generates \mathbb{Z}^n as a group.

By definition we have $w(f) = \langle \alpha, v(f) \rangle$. Let $\xi := \langle \alpha, \cdot \rangle : \mathbb{R}^n \rightarrow \mathbb{R}$. Then we know that $\xi(\beta) \geq \min\{\alpha_i \mid 1 \leq i \leq r\} > 0$ for any $0 \neq \beta = v(f) \in \mathcal{S}$. So $C(\mathcal{S}) \setminus \{0\} \subset \xi_{>0}$ and condition (2) is satisfied.

By Izumi’s inequality Lemma 2.4, there exists $r > 0$ such that $w(f) \leq r \cdot \text{ord}_m(f)$ for any $f \in R \setminus \{0\}$. So if we set $r_0 := r^{-1}$, then

$$\text{ord}_m(f) \geq r_0 w(f) = r_0 \xi(v(f)).$$

Thus, condition (3) is satisfied. Condition (1) is straightforward to verify. So we conclude v is a good \mathbb{Z}^n -valued valuation. □

A.2 A volume formula for convex bodies

The following lemma will be useful in proving the convexity of multiplicities. It is a slight generalization of [Izm14, Lemma 4.4] which follows from the same argument, so we omit the proof.

LEMMA A.3 [Izm14]. *Let $C \subset \mathbb{R}^n$ be a strongly convex cone of dimension n , and $h : \text{Int}(C) \rightarrow \mathbb{R}_{\geq 0}$ a continuous function homogeneous of degree 1. Then*

$$n! \text{vol}(h_{\leq 1}) = \int_C e^{-h} d\mu,$$

where $h_{\leq 1} := \{x \in \text{Int}(C) \mid h(x) \leq 1\}$ and μ is the Lebesgue measure on \mathbb{R}^n .

A.3 The cutting function induced by a filtration

To proceed, we fix a quasi-monomial valuation $w \in \text{Val}_{X,x}$ and a good \mathbb{Z}^n -valued valuation v associated to w given by Lemma A.2. Let $\mathcal{S} = v(R \setminus \{0\}) \cup \{0\} \subset \mathbb{Z}^n$ be the value semigroup of v . Let \mathfrak{a}_\bullet be a linearly bounded \mathfrak{m} -filtration. The goal of this section is to construct a cutting function on the strongly convex cone $C(\mathcal{S})$ satisfying the conditions in Lemma A.3.

The following lemma is an easy generalization of Izumi’s inequality to filtrations. For a family version, see Lemma A.10.

LEMMA A.4. *With notation as above, for any $m \in \mathbb{R}_{\geq 0}$, there exists $\beta_0 \in \mathcal{S}$ such that if $f \in R \setminus \{0\}$ satisfies $v(f) \succeq \beta_0$, then $f \in \mathfrak{a}_{>m}$.*

Proof. By Lemma 2.4, there exists $r > 0$ such that $\xi(v(f)) = w(f) \leq r \cdot \text{ord}_m(f)$. By assumption, there exists $d \in \mathbb{Z}_{>0}$ such that $\mathfrak{m}^d \subset \mathfrak{a}_1$. So if we choose $\beta_0 \in \mathcal{S}$ such that $\xi(\beta_0) \geq rd(\lceil m \rceil + 1)$, then for $v(f) \succeq \beta_0$, we have $\text{ord}_m(f) \geq d(\lceil m \rceil + 1)$. Hence,

$$f \in \mathfrak{m}^{d(\lceil m \rceil + 1)} \subset \mathfrak{a}_1^{\lceil m \rceil + 1} \subset \mathfrak{a}_{>m}. \quad \square$$

We define a function $m_{\mathfrak{a}_\bullet} : \mathcal{S} \rightarrow \mathbb{R}_{\geq 0}$ associated to \mathfrak{a}_\bullet by

$$m_{\mathfrak{a}_\bullet}(\beta) := \sup\{m \in \mathbb{R}_{\geq 0} \mid v^{-1}(\beta) \cap \mathfrak{a}_m \neq \emptyset\} \tag{A.1}$$

for any $\beta \in \mathcal{S}$. Since \mathfrak{a}_\bullet is left continuous, the above supremum is indeed a maximum. Now we choose an element $f_{\mathfrak{a}_\bullet, \beta} \in v^{-1}(\beta) \cap \mathfrak{a}_{m_{\mathfrak{a}_\bullet}(\beta)}$. We will write $m(\beta)$ and f_β respectively, if there is no chance of ambiguity.

We first prove some properties of $m(\cdot)$ and construct a function h on \mathcal{S} using it, which will play a key role in estimating the multiplicities.

PROPOSITION A.5. *Let $m = m_{\mathfrak{a}_\bullet} : \mathcal{S} \rightarrow \mathbb{R}_{\geq 0}$ be the function associated to a linearly bounded \mathfrak{m} -filtration \mathfrak{a}_\bullet as above. Then the following statements hold.*

- (1) m is superadditive, that is,

$$m(\beta_1 + \beta_2) \geq m(\beta_1) + m(\beta_2),$$

for any $\beta_1, \beta_2 \in \mathcal{S}$.

- (2) There exists $M_2 > 0$ such that for any $\beta \in \mathcal{S}$, we have $m(\beta) \leq M_2\xi(\beta)$.
- (3) If $\mathfrak{a}_\bullet = \mathfrak{a}_\bullet(u)$ for some valuation $u \in \text{Val}_{X,x}$, then we have

$$u(f) \leq \lim_{k \rightarrow \infty} \frac{m(kv(f))}{k}$$

for all $f \in R \setminus \{0\}$, and the equality holds if $u = w$.

- (4) There is a continuous, concave function $h_{\mathfrak{a}_\bullet} : \text{Int}(C(\mathcal{S})) \rightarrow \mathbb{R}_{\geq 0}$ homogeneous of degree 1, such that for any $\beta \in \mathcal{S}$, $h_{\mathfrak{a}_\bullet}(\beta) = \lim_{k \rightarrow \infty} (m_{\mathfrak{a}_\bullet}(k\beta)/k)$. As before, we will drop \mathfrak{a}_\bullet from the notation of $h(\beta)$ if there is no ambiguity.
- (5) For $\lambda > 0$, we have

$$h_{\mathfrak{a}_{\lambda\bullet}} = h_{\mathfrak{a}_\bullet}/\lambda. \tag{A.2}$$

Proof. (1) For any $\beta_1, \beta_2 \in \mathcal{S}$, we have

$$f_{\beta_1} f_{\beta_2} \in v^{-1}(\beta_1 + \beta_2) \cap \mathfrak{a}_{m(\beta_1)} \mathfrak{a}_{m(\beta_2)} \subset v^{-1}(\beta_1 + \beta_2) \cap \mathfrak{a}_{m(\beta_1)+m(\beta_2)}.$$

Hence, $m(\beta_1 + \beta_2) \geq m(\beta_1) + m(\beta_2)$.

(2) By Lemma 2.4 there exists $r > 0$ such that $\text{ord}_{\mathfrak{m}}(f) < rw(f) = r\xi(v(f))$. Hence, if we choose $f \in v^{-1}(\beta)$, then we know that $\text{ord}_{\mathfrak{m}}(f) < r\xi(\beta)$, that is, $f \notin \mathfrak{m}^{\lceil r\xi(\beta) \rceil}$. Since \mathfrak{a}_\bullet is linearly bounded by \mathfrak{m} , there exists $c > 0$ such that $\mathfrak{a}_{ck} \subset \mathfrak{m}^k$ for any $k \in \mathbb{Z}_{>0}$. In particular, we get $f \notin \mathfrak{a}_{c\lceil r\xi(\beta) \rceil}$. So $m(\beta) \leq M_2\xi(\beta)$, where $M_2 := cr$.

(3) Assume that $\mathfrak{a}_\bullet = \mathfrak{a}_\bullet(u)$. Then for any $k \in \mathbb{N}$, we have $f^k \in v^{-1}(kv(f)) \cap \mathfrak{a}_{ku(f)}(u)$ which implies that

$$m(kv(f)) \geq ku(f).$$

Note that $\lim_{k \rightarrow \infty} (m(kv(f))/k) \leq M_2$ exists by (1) and (2). Letting $k \rightarrow \infty$, we get the inequality. If $u = w$, then by definition we have $m(\beta) = \langle \alpha, \beta \rangle$, which implies the equality immediately.

(4) Choose finitely generated subsemigroups $\mathcal{S}_1 \subset \mathcal{S}_2 \subset \dots$ of \mathcal{S} such that $\bigcup_{l=1}^\infty \mathcal{S}_l = \mathcal{S}$ and each \mathcal{S}_l generates \mathbb{Z}^n as a group. By [Kho92, §3, Proposition 3], there exists $\gamma_l \in \mathcal{S}_l$ such that

$$(C(\mathcal{S}_l) + \gamma_l) \cap \mathbb{N}^n \subset \mathcal{S}_l.$$

For any $\beta \in \text{Int}(C(\mathcal{S}_l))$, there exists k_0 such that for any $k \geq k_0$, we have

$$\lfloor k\beta \rfloor \in (C(\mathcal{S}_l) + \gamma_l) \cap \mathbb{N}^n \subset \mathcal{S}_l.$$

So we can define

$$h(\beta) := \limsup_{k \rightarrow \infty} \frac{m(\lfloor k\beta \rfloor)}{k}. \tag{A.3}$$

We first show that this is indeed a limit. Since $\beta \in \text{Int}(C(\mathcal{S}_l))$, there exists $d \in \mathbb{Z}_{>0}$ such that $B(d\beta, 3) \subset C(\mathcal{S}_l) + \gamma_l$, where we use the maximum norm on \mathbb{R}^n . Hence, for any $k_1, k_2 \geq k_0$, we have

$$\lfloor (k_1 + k_2 + d)\beta \rfloor - (\lfloor k_1\beta \rfloor + \lfloor k_2\beta \rfloor) \in C(\mathcal{S}_l) + \gamma_l \subset \mathcal{S}_l.$$

By (1) we have

$$m(\lfloor (k_1 + k_2 + d)\beta \rfloor) \geq m(\lfloor k_1\beta \rfloor) + m(\lfloor k_2\beta \rfloor).$$

This shows that the function $k \mapsto m(\lfloor (k - d)\beta \rfloor)$ is sup-additive for $k \geq k_0 + d$. So by (2) and Fekete's lemma we know that there exists

$$\lim_{k \rightarrow \infty} \frac{m(\lfloor k\beta \rfloor)}{k} = \lim_{k \rightarrow \infty} \frac{m(\lfloor (k - d)\beta \rfloor)}{k} \leq M_2.$$

Since the above definition does not depend on l , we have a well-defined function h on $\text{Int}(C(\mathcal{S})) = \cup_{l=1}^{\infty} \text{Int}(C(\mathcal{S}_l))$. Clearly h is homogeneous of degree 1.

Next we prove the continuity of h . It suffices to prove the continuity of h in $\text{Int}(C(\mathcal{S}_l))$. We first show that for any $\beta, \beta_1 \in \text{Int}(C(\mathcal{S}_l))$, we have

$$h(\beta + \beta_1) \geq h(\beta). \tag{A.4}$$

Indeed, there exists k_0 such that for any $k > k_0$, we have $B(k\beta, 2) \subset C(\mathcal{S}_l) + \gamma_l$. Hence,

$$\lfloor k(\beta + \beta_1) \rfloor - \lfloor k\beta \rfloor \in \mathcal{S}_l,$$

and by (1) we get $m(\lfloor k(\beta + \beta_1) \rfloor) \geq m(\lfloor k\beta \rfloor)$. Dividing by k and letting $k \rightarrow \infty$, we get (A.4). Now for any $\epsilon \in (0, 1)$ and $\beta \in \text{Int}(C(\mathcal{S}_l))$, there exists $\rho > 0$ such that $B(\epsilon\beta, \rho) \subset \text{Int}(C(\mathcal{S}_l))$. Hence, for any $\beta' \in B(\beta, \rho)$, we have $\epsilon\beta \pm (\beta - \beta') \in \text{Int}(C(\mathcal{S}_l))$. By (A.4) and the homogeneity of h , we get

$$(1 - \epsilon)h(\beta) = h((1 - \epsilon)\beta) \leq h(\beta') \leq h((1 + \epsilon)\beta) = (1 + \epsilon)h(\beta).$$

Hence, h is continuous on $\text{Int}(C(\mathcal{S}_l))$.

The concavity of h follows easily from its continuity, homogeneity, and (1).

(5) By definition we have

$$m_{\mathbf{a}_\bullet}(\beta) = \lambda \cdot m_{\mathbf{a}_{\lambda\bullet}}(\beta)$$

for any $\beta \in \mathcal{S}$. So we have

$$h_{\mathbf{a}_\bullet}(\beta) = \lim_{m \rightarrow \infty} \frac{m_{\mathbf{a}_\bullet}(k\beta)}{k} = \lambda \cdot h_{\mathbf{a}_{\lambda\bullet}}(\beta)$$

for any $\beta \in \mathcal{S}$. Now (A.2) follows easily from the homogeneity and continuity of h . □

The following proposition asserts that f_β gives a basis for $\mathbf{a}_m/\mathbf{a}_{>m}$.

PROPOSITION A.6. *For any $m \in \mathbb{R}_{\geq 0}$, the set $\{[f_\beta] \mid m(\beta) = m\}$ forms a basis for the \mathbb{k} -vector space $\mathbf{a}_m/\mathbf{a}_{>m}$.*

Proof. We may assume that m is a jumping number of \mathbf{a}_\bullet . Let

$$\mathfrak{B}_m := \{\beta \in \mathcal{S} \mid m(\beta) = m\}.$$

We first show that the set $\{[f_\beta]\}$ is linearly independent. Assume to the contrary that there exist $c_\beta \in \mathbb{k}$, not all zero, such that

$$g := \sum_{\beta \in \mathfrak{B}_m} c_\beta f_\beta \in \mathbf{a}_{>m}.$$

Then we have $v(g) = \beta_0 := \min\{\beta \in \mathfrak{B}_m \mid c_\beta \neq 0\}$, which implies $m(\beta_0) > m$, a contradiction.

It remains to prove that for any $f \in \mathbf{a}_m \setminus \mathbf{a}_{>m}$, there exist $c_\beta \in \mathbb{k}$ for each $\beta \in \mathfrak{B}_m$ such that

$$f - \sum_{\beta \in \mathfrak{B}_m} c_\beta f_\beta \in \mathbf{a}_{>m}. \tag{A.5}$$

By condition (1) of Definition A.1, there exists a unique sequence $\{a_\beta\}_{\beta \in \mathcal{S}}$ such that for any $\beta' \in \mathcal{S}$, we have

$$w\left(f - \sum_{\beta \preceq \beta'} a_\beta f_\beta\right) > \langle \alpha, \beta' \rangle.$$

Let β_0 be as defined in Lemma A.4. In particular, we have $\beta \prec \beta_0$ for any $\beta \in \mathfrak{B}_m$. Write

$$f - \sum_{\beta \in \mathfrak{B}_m} a_\beta f_\beta = \left(f - \sum_{\beta \prec \beta_0} a_\beta f_\beta \right) + \sum_{\beta \notin \mathfrak{B}_m, \beta \prec \beta_0} a_\beta f_\beta =: g_1 + g_2.$$

By the choice of a_β , we know that $v(g_1) \geq \beta_0$. Hence, $g_1 \in \mathfrak{a}_{>m}$ by the choice of β_0 .

CLAIM. For any $\beta \preceq \beta_0$ with $\beta \notin \mathfrak{B}_m$, if $a_\beta \neq 0$, then $m(\beta) > m$.

We prove the claim by induction. Let $\beta \preceq \beta_0$ and assume that the claim is true for any $\beta' \prec \beta$. Then $f_{\beta'} \in \mathfrak{a}_m$ for $\beta' \prec \beta$, and hence $f - \sum_{\beta' \prec \beta} a_{\beta'} f_{\beta'} \in \mathfrak{a}_m$. Now $a_\beta \neq 0$, so we have

$$v\left(f - \sum_{\beta' \prec \beta} a_{\beta'} f_{\beta'}\right) = v(a_\beta f_\beta) = \beta.$$

Hence, $v^{-1}(\beta) \cap \mathfrak{a}_m \neq \emptyset$ and $m(\beta) \geq m$. Since $m(\beta) \neq m$ by assumption, we have strict inequality and the claim is proved.

By the claim we know that $g_2 \in \mathfrak{a}_{>m}$. Thus we conclude that

$$f - \sum_{\beta \in \mathfrak{B}_m} a_\beta f_\beta \in \mathfrak{a}_{>m},$$

that is, (A.5) holds with $c_\beta = a_\beta$. □

A.4 Multiplicities and Okunkov bodies

In this section we apply the strategy as in [LM09, KK14] to estimate the multiplicities.

Let $\mathfrak{a}_{\bullet,i}$ be two linearly bounded \mathfrak{m} -filtrations for $i = 0, 1$. Recall that the geodesic $(\mathfrak{a}_{\bullet,t})_{t \in [0,1]}$ between $\mathfrak{a}_{\bullet,0}$ and $\mathfrak{a}_{\bullet,1}$ is defined as

$$\mathfrak{a}_{\lambda,t} := \sum_{\lambda=(1-t)\mu+t\nu} (\mathfrak{a}_{\mu,0} \cap \mathfrak{a}_{\nu,1}).$$

Then $\mathfrak{a}_{\bullet,t}$ is also a linearly bounded \mathfrak{m} -filtration.

THEOREM A.7. Let $w \in \text{Val}_{X,x}$ be a quasi-monomial valuation. Denote $\mathfrak{a}_{\bullet,0} := \mathfrak{a}_{\bullet}(w)$. Let $\mathfrak{a}_{\bullet,1}$ be a linearly bounded \mathfrak{m} -filtration. Let $(\mathfrak{a}_{\bullet,t})_{t \in [0,1]}$ be the geodesic between $\mathfrak{a}_{\bullet,0}$ and $\mathfrak{a}_{\bullet,1}$. The function $E(t) : [0, 1] \rightarrow \mathbb{R}$ defined by $E(t) := e(\mathfrak{a}_{\bullet,t})$ satisfies the following properties.

- (1) $E(t)$ is smooth;
- (2) $E(t)^{-1/n}$ is concave, meaning

$$E(t)^{-1/n} \geq (1-t)E(0)^{-1/n} + tE(1)^{-1/n} \quad \text{for all } t \in [0, 1].$$

- (3) Suppose, in addition, $\mathfrak{a}_{\bullet,1} = \mathfrak{a}_{\bullet}(w')$ for some valuation $w' \in \text{Val}_{X,x}$. Then $E(t)^{-1/n}$ is linear if and only if $w' = cw$ for some $c \in \mathbb{R}_{>0}$.

From now on, we follow the notation of Theorem A.7. Let v be a good valuation associated to w given by Lemma A.2.

LEMMA A.8. Let $\mathfrak{a}_{\bullet,t}$ be as above for $t \in [0, 1]$. Let $m^{(t)} = m_{\mathfrak{a}_{\bullet,t}} : \mathcal{S} \rightarrow \mathbb{R}_{\geq 0}$ be the function defined in (A.1). Then we have

$$m^{(t)}(\beta) = (1-t)m^{(0)}(\beta) + tm^{(1)}(\beta)$$

for any $t \in [0, 1]$ and any $\beta \in \mathcal{S}$.

Proof. Fix an arbitrary element $\beta \in \mathcal{S}$. For simplicity, denote $m_t := m^{(t)}(\beta)$ for $t \in [0, 1]$.

We first show that $m_t \geq (1 - t)m_0 + tm_1$. By assumption, we have $w(f) = \langle \alpha, v(f) \rangle$ for any $f \in R \setminus \{0\}$. Thus

$$m_0 = m^{(0)}(\beta) = \max\{w(f) \mid f \in R \setminus \{0\} \text{ and } v(f) = \beta\} = \langle \alpha, \beta \rangle.$$

Pick $g \in v^{-1}(\beta) \cap \mathfrak{a}_{m_1,1}$. Then clearly $w(g) = \langle \alpha, v(g) \rangle = \langle \alpha, \beta \rangle = m_0$. Thus $g \in \mathfrak{a}_{m_0,0}$ which implies that $v^{-1}(\beta) \cap \mathfrak{a}_{m_1,1} \cap \mathfrak{a}_{m_0,0}$ is non-empty. Since $\mathfrak{a}_{m_1,1} \cap \mathfrak{a}_{m_0,0} \subset \mathfrak{a}_{(1-t)m_0+tm_1,t}$, we know that $v^{-1}(\beta) \cap \mathfrak{a}_{(1-t)m_0+tm_1,t} \neq \emptyset$ which implies $m_t \geq (1 - t)m_0 + tm_1$ by (A.1).

Next, we show that $m_t \leq (1 - t)m_0 + tm_1$. Choose $f \in v^{-1}(\beta) \cap \mathfrak{a}_{m_t,t}$. We may write $f = \sum_i f_i$ as a finite sum such that $f_i \in \mathfrak{a}_{\mu_i,0} \cap \mathfrak{a}_{\nu_i,1}$, where $(1 - t)\mu_i + t\nu_i = m_t$ for every i . Since $f_i \in \mathfrak{a}_{\mu_i,0} = \mathfrak{a}_{\mu_i}(w)$, we have $w(f_i) \geq \mu_i$. Hence, after replacing (μ_i, ν_i) by $(w(f_i), t^{-1}(m_t - (1 - t)w(f_i)))$ the assumption $f_i \in \mathfrak{a}_{\mu_i,0} \cap \mathfrak{a}_{\nu_i,1}$ still holds. Furthermore, if we have $\mu_i = \mu_j$ for some $i \neq j$ then we may replace (f_i, f_j) by $f_i + f_j$. After finitely many steps of replacements and permutations, we obtain a decomposition $f = \sum_{i=1}^l f_i$ such that $f_i \in \mathfrak{a}_{\mu_i,0} \cap \mathfrak{a}_{\nu_i,1}$ where $(1 - t)\mu_i + t\nu_i = m_t$ and $\mu_i = w(f_i)$ for every i , and $\mu_1 < \mu_2 < \dots < \mu_l$. Denote $\beta_i := v(f_i)$. Since $\mu_i = \langle \alpha, \beta_i \rangle$, we know that $\beta_1 \prec \beta_2 \prec \dots \prec \beta_l$. As a result, we have

$$\beta = v(f) = v\left(\sum_{i=1}^l f_i\right) = v(f_1) = \beta_1.$$

Thus, $f_1 \in v^{-1}(\beta) \cap \mathfrak{a}_{\nu_1,1}$, which implies that $m_1 \geq \nu_1$ by (A.1). Meanwhile, we have

$$\mu_1 = w(f_1) = \langle \alpha, \beta \rangle = m_0.$$

Therefore,

$$m_t = (1 - t)\mu_1 + t\nu_1 \leq (1 - t)m_0 + tm_1,$$

which completes the proof. □

As an immediate corollary to Lemma A.8, we get the following result.

COROLLARY A.9. *Let $h_t := h_{\alpha_\bullet,t} : \text{Int}(C(\mathcal{S})) \rightarrow \mathbb{R}_{>0}$ be defined as in (A.3) for $t \in [0, 1]$. Then we have $h_t = (1 - t)h_0 + th_1$.*

The following lemma can be viewed as an Izumi-type estimate in our setting of filtrations.

LEMMA A.10. *There exists $M > 0$ depending only on w and $\mathfrak{a}_{\bullet,1}$ such that for any $f \in R \setminus \{0\}$ with $\xi(v(f)) \geq M$ and $t \in [0, 1]$, we have*

$$f \in \mathfrak{a}_{M^{-1} \cdot \xi(v(f)),t}.$$

Proof. Recall that we may define $\xi : \mathbb{R}^n \rightarrow \mathbb{R}$ in Definition A.1 by $\xi(v(f)) = w(f)$. Moreover, there exists $r > 0$ such that $\text{ord}_{\mathfrak{m}}(f) \geq r\xi(v(f))$. Take $d \in \mathbb{Z}_{>0}$ such that $\mathfrak{m}^d \subset \mathfrak{a}_{1,i}$ for $i = 0, 1$. Set $M' := 2d \cdot r^{-1}$. For any $f \in R$ with $\xi(v(f)) \geq M'$, we have $\text{ord}_{\mathfrak{m}}(f) \geq r \cdot \xi(v(f)) \geq 2d$. Hence,

$$f \in \mathfrak{m}^{\lceil r\xi(v(f)) \rceil} \subset \mathfrak{a}_{\lceil r\xi(v(f))/d \rceil, i} \subset \mathfrak{a}_{M'^{-1} \cdot \xi(v(f)), i},$$

for $i = 0, 1$, where we used the inequality $\lceil \lambda \rceil \geq \lambda/2$ for $\lambda \geq 2$. Hence, when $\lambda := M'^{-1} \cdot \xi(v(f)) \geq 2$, we have

$$f \in \mathfrak{a}_{\lambda,0} \cap \mathfrak{a}_{\lambda,1} \subset \mathfrak{a}_{\lambda,t},$$

and the lemma is proved with $M = 2M'$. □

In view of the above lemma, we define

$$\Gamma := \{(\beta, m) \in \mathcal{S} \times \mathbb{N} \mid \xi(\beta) \leq Mm\} \quad \text{and} \quad \Gamma_m := \Gamma \cap (\mathcal{S} \times \{m\}).$$

For $\beta \in \mathcal{S}$ and $t \in [0, 1]$, recall that $m^{(t)}(\beta) := m_{\mathfrak{a}_{m,t}}(\beta)$ and choose an element $f_\beta^{(t)} := f_{\mathfrak{a}_{m,t},\beta} \in v^{-1}(\beta) \cap \mathfrak{a}_{m^{(t)}(\beta),t}$. Then we can define

$$\Gamma^{(t)} := \{(\beta, m) \in \Gamma \mid f_\beta^{(t)} \in \mathfrak{a}_{m,t}\} \quad \text{and} \quad \Gamma_m^{(t)} := \Gamma^{(t)} \cap (\mathcal{S} \times \{m\}).$$

We will use these sets to estimate the multiplicities of $\mathfrak{a}_{\bullet,t}$. The following property for $\Gamma^{(t)}$ is one of the main ingredients for our proof of Theorem A.7.

PROPOSITION A.11. *Let $m \in \mathbb{N}$. Then*

$$\ell(R/\mathfrak{a}_{m,t}) = \#(\Gamma_m \setminus \Gamma_m^{(t)})$$

for any $t \in [0, 1]$.

Proof. Note that the jumping numbers of the \mathfrak{m} -filtration $\mathfrak{a}_{\bullet,t}$ form a discrete set as for any $\mu > \lambda > 0$, $\ell(\mathfrak{a}_{\lambda,t}/\mathfrak{a}_{\mu,t}) < \infty$. By Proposition A.6, the quotient ring $R/\mathfrak{a}_{m,t}$ has a basis $\{[f_\beta^{(t)}] \mid f_\beta^{(t)} \notin \mathfrak{a}_{m,t}\}$. Note that if $\xi(v(f_\beta^{(t)})) \geq Mm$ for some $\beta \in \mathcal{S}$, then $f_\beta^{(t)} \in \mathfrak{a}_{M^{-1}\xi(v(f_\beta^{(t)})),t} \subset \mathfrak{a}_{m,t}$ by Lemma A.10. Hence, for any β satisfying $f_\beta^{(t)} \notin \mathfrak{a}_{m,t}$, we have $\xi(\beta) = \xi(v(f_\beta^{(t)})) \leq Mm$, that is, $\beta \in \Gamma_m$. Thus

$$\Gamma_m \setminus \Gamma_m^{(t)} = \{\beta \mid f_\beta^{(t)} \notin \mathfrak{a}_{m,t}\},$$

and the proposition follows. □

We now check that the Γ and $\Gamma^{(t)}$ are semigroups satisfying conditions (2.3-5) of [LM09].

LEMMA A.12. *Let $t \in [0, 1]$. Then*

- (1) Γ and $\Gamma^{(t)}$ are semigroups,
- (2) $\Gamma_0 = \Gamma_0^{(t)} = \{0\}$,
- (3) there exist finitely many vectors $(v_j, 1)$ spanning a semigroup $B \subset \mathbb{N}^{n+1}$ such that

$$\Gamma^{(t)} \subset \Gamma \subset B,$$

and

- (4) if we replace M by a proper multiple of it in the definition of Γ , then $\Gamma^{(t)}$ and Γ contain a set of generators of \mathbb{Z}^{n+1} as a group.

Proof. (1) This follows from the sup-additivity of $m(\cdot)$, Proposition A.5(1).

(2) This is clear.

(3) Choose $b \in \mathbb{N}$ such that $b \cdot \min\{\alpha_i \mid 1 \leq i \leq r\} > M$. Then it is easy to see that Γ is contained in the semigroup B generated by $\{(\beta_1, \dots, \beta_n, 1) \mid 0 \leq \beta_i \leq b\}$.

(4) It suffices to show that $\Gamma^{(t)}$ generates \mathbb{Z}^{n+1} . To this end, recall that as in the proof of Lemma A.2, \mathcal{S} contains some β' and $\beta' + e_i$ for $1 \leq i \leq n$. We may assume that $\xi(\beta') \geq 2M$. Let $m := \lfloor M^{-1} \cdot \xi(\beta') \rfloor - 1 > 0$. Then by Lemma A.10 we have

$$f_{\beta'} \in \mathfrak{a}_{M^{-1}\xi(\beta'),t} \subset \mathfrak{a}_{m+1,t} \subset \mathfrak{a}_{m,t}$$

and

$$f_{\beta'+e_i} \in \mathfrak{a}_{M^{-1}\xi(\beta'+e_i),t} \subset \mathfrak{a}_{m,t}.$$

If we replace M by $M' = 2M$ in the definition of Γ , then

$$\xi(\beta') = M' \cdot \frac{M^{-1} \cdot \xi(\beta')}{2} \leq M' \cdot \lfloor M^{-1} \cdot \xi(\beta') \rfloor = M'm,$$

where we used the inequality $\lfloor \lambda \rfloor \geq \lambda/2$ for $\lambda \geq 2$. Now by definition, $\Gamma^{(t)}$ contains the vectors (β', m) , $(\beta' + e_i, m)$, $1 \leq i \leq n$ and $(\beta', m + 1)$, which implies that $\Gamma^{(t)}$ generates \mathbb{Z}^{n+1} as a group. □

From now on we assume that Γ and $\Gamma^{(t)}$ satisfy the conditions of Lemma A.12. As in [LM09, (2.1)], we define Δ as the closed convex hull of $\bigcup_m \Gamma_m/m$, and define $\Delta^{(t)}$ as the closed convex hull of $\bigcup_m \Gamma_m^{(t)}/m$ for $t \in [0, 1]$. Then $\Delta^{(t)} \subset \Delta$. By Lemma A.12, Proposition A.11 and [LM09, Proposition 2.1], we have the following result.

COROLLARY A.13. *With notation as above,*

$$e(\mathbf{a}_{\bullet,t}) = n!(\text{vol}(\Delta) - \text{vol}(\Delta^{(t)}))$$

for $t \in [0, 1]$.

Next, we show that the $\Gamma^{(t)}$ can be characterized by the function h_t defined as in Corollary A.9.

PROPOSITION A.14. *Let $h_t := h_{\mathbf{a}_{\bullet,t}}$ for $t \in [0, 1]$, where h is defined by (A.3). Then*

- (1) $\text{Int}(\Delta^{(t)}) = \text{Int}(\Delta) \cap (h_t)_{>1}$, and
- (2) $\text{Int}(\Delta) \setminus \Delta^{(t)} = \text{Int}(C(\mathcal{S})) \cap (h_t)_{<1}$.

Proof. (1) First we show that $\text{Int}(\Delta^{(t)}) \subset \text{Int}(\Delta) \cap (h_t)_{>1}$. It suffices to show that for any $\beta \in \Delta^{(t)} \cap \text{Int}(C(\mathcal{S}))$ we have $h_t(\beta) \geq 1$. Recall that $\Delta^{(t)}$ is the closed convex hull of $\bigcup_m \Gamma_m^{(t)}/m$. Hence, by the concavity of h_i (Proposition A.5(4)), it suffices to show that for any $(\beta, m) \in \Gamma^{(t)} \cap \text{Int}(C(\mathcal{S}))$ we have $h_t(\beta) \geq m$. But this follows immediately from (A.3) and the definition of $\Gamma^{(t)}$.

To prove the converse, assume that $\beta \in \text{Int}(\Delta) \cap (h_t)_{>1}$. Let $\epsilon := (h_t(\beta) - 1)/3 > 0$. Applying the proof of Proposition A.5(4), we know that there exists m_1 such that for any $m \geq m_1$, there exists $\beta_m \in \Gamma_m$ with $\lim_m(\beta_m/m) = \beta$. Since h_t is continuous, we may choose $m_2 \geq m_1$ such that for any $m \geq m_2$, we have $h_t(\beta_m/m) > 1 + 2\epsilon$. By (A.3), for each $m \geq m_2$, we may choose k_m such that $(1 + \epsilon)m^{(i)}(k_m\beta_m) \geq h_i(\beta_m)$ for $i = 0, 1$. Then for $m \geq m_2$, we have

$$m^{(t)}(k_m\beta_m) \geq \frac{1}{1 + \epsilon} h_t(k_m\beta_m) \geq \frac{1 + 2\epsilon}{1 + \epsilon} k_m m > k_m m.$$

This shows that $k_m\beta_m \in \Gamma_{k_m m}^{(t)}$, hence

$$\beta = \lim_{m \rightarrow \infty} \frac{k_m\beta_m}{k_m m} \in \Delta^{(t)}.$$

(2) By (1) we have $\text{Int}(\Delta) \setminus \Delta^{(t)} = \text{Int}(\Delta) \cap (h_t)_{<1}$, so it suffices to show that for any $\beta \in \text{Int}(C(\mathcal{S})) \setminus \Delta$, we have $h_t(\beta) > 1$. By definition we have

$$\Delta = C(\Gamma) \cap (\mathbb{R}^n \times \{1\}) = C(\mathcal{S}) \cap \xi_{\leq M}.$$

Hence, by Proposition A.5 and Lemma A.10, we have

$$\begin{aligned} h_t(\beta) &= \lim_k \frac{m^{(t)}(\lfloor k\beta \rfloor)}{k} \geq \lim_k \frac{M^{-1} \cdot \xi(\lfloor k\beta \rfloor)}{k} \\ &= M^{-1} \cdot \xi(\beta) > 1, \end{aligned}$$

where in the first inequality and the second equality we used the fact that for any $k \in \mathbb{Z}_{>0}$,

$$k\xi(\beta) \geq \xi(\lfloor k\beta \rfloor) = \langle \alpha, \lfloor k\beta \rfloor \rangle \geq k\xi(\beta) - \|\alpha\|_1.$$

The proof is finished. □

Next we prove Theorem A.7.

Proof of Theorem A.7. By Lemma A.3, Corollary A.13 and Proposition A.14, we have

$$E(t) = n! \operatorname{vol}(\Delta \setminus \Delta^{(t)}) = n! \operatorname{vol}(\operatorname{Int}(C(\mathcal{S})) \cap (h_t)_{<1}) = \int_{C(\mathcal{S})} e^{-h_t} d\mu. \tag{A.6}$$

By Corollary A.9 we have $h_t = (1 - t)h_0 + th_1$, hence $(d/dt)h_t = h_1 - h_0$. By Proposition A.5(4), we can differentiate under the integral sign, and hence $E(t)$ is a smooth function. Thus (1) is proved.

To prove (2), we use a slight variant of the formula (A.6). Let $h := h_1 - h_0$. Then by homogeneity we have

$$\begin{aligned} E''(t) &= \int_{C(\mathcal{S})} h^2 e^{-h_t} d\mu \\ &= \int_0^\infty d\lambda \int_{C(\mathcal{S}) \cap (h_t)_{=\lambda}} h^2 e^{-h_t} d\mu_{n-1} \\ &= \int_0^\infty \lambda^{n+1} e^{-\lambda} d\lambda \int_{C(\mathcal{S}) \cap (h_t)_{=1}} h^2 d\mu_{n-1} \\ &= (n + 1)! \int_{C(\mathcal{S}) \cap (h_t)_{=1}} h^2 d\mu_{n-1}, \end{aligned}$$

where μ_{n-1} is the Lebesgue measure on \mathbb{R}^{n-1} . Similarly, we have

$$E'(t) = n! \int_{C(\mathcal{S}) \cap (h_t)_{=1}} h d\mu_{n-1}$$

and

$$E(t) = (n - 1)! \int_{C(\mathcal{S}) \cap (h_t)_{=1}} d\mu_{n-1}.$$

Hence, by Cauchy–Schwarz we have

$$E''(t)E(t) \geq \frac{n + 1}{n} E'(t)^2 \geq E'(t)^2, \tag{A.7}$$

which implies (2).

To prove (3), first assume that the equality in (2) holds. By (A.7) and Cauchy–Schwarz we know that $h_1 - h_0 = C_t$ is a constant on $(h_t)_{=1}$. In particular, $h_1 - h_0 = C_0$ on $(h_0)_{=1}$. Since h_1 and h_0 are 1-homogeneous and positive on $C(\mathcal{S})$, we know that $h_0 = ch_1$, where $c := 1/(C_0 + 1) > 0$. By Proposition A.5(5), we know that

$$h_0 = h_{\mathbf{a}_{\bullet,0}} = h_{\mathbf{a}_{c-1,\bullet,1}}. \tag{A.8}$$

By Lemma A.3, Corollary A.13 and Proposition A.14 we have

$$\operatorname{vol}(w) = e(\mathbf{a}_{\bullet,0}) = e(\mathbf{a}_{c-1,\bullet,1}) = \operatorname{vol}(cw').$$

Moreover, for any $f \in R \setminus \{0\}$, let $\beta := v(f)$. Then by Proposition A.5(3) we have $h_0(\beta) = w(f)$ and $h_{\mathbf{a}_{c-1,\bullet,1}}(\beta) \geq cw'(f)$. Combined with (A.8), we get $cw'(f) \leq w(f)$. So by [LX20, Proposition 2.7], we conclude that $w = cw'$.

Conversely, if $w = cw'$, then for any $\lambda \in \mathbb{R}_{>0}$, by a direct calculation we have

$$\mathbf{a}_{\lambda,t} = \sum_{(1-t)\mu + t\nu = \lambda} \mathbf{a}_{\mu}(cw') \cap \mathbf{a}_{\nu}(w') = \mathbf{a}_{f(t,\lambda)}(w'),$$

where $f(t) = \lambda/(t + c(1 - t))$. So $E(t) = (1/(t + c(1 - t)))^n \text{vol}(w')$, and the proof is finished. \square

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Harold Blum blum@math.utah.edu

Department of Mathematics, University of Utah, Salt Lake City, UT 84112, USA

Yuchen Liu yuchenl@northwestern.edu

Department of Mathematics, Northwestern University, Evanston, IL 60208, USA

Lu Qi luq@princeton.edu

Department of Mathematics, Princeton University, Princeton, NJ 08544, USA

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