

## STOCHASTIC GROWTH MODELS: BOUNDS ON CRITICAL VALUES

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### Abstract

We give upper bounds on the critical values for oriented percolation and some interacting particle systems by computing their behavior on small finite sets.

ORIENTED PERCOLATION; CONTACT PROCESS; PHASE TRANSITION

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### 1. Introduction

Many results have been proved for interacting particle systems by showing that when viewed on suitable length and time scales, the process dominates oriented percolation. Here, we use this comparison to give upper bounds on critical values for oriented site percolation, the contact process and a process with sexual reproduction.

(a) *Oriented site percolation.* We begin with an account of the necessary definitions. See Durrett (1984) or Durrett (1988), Chapter 5, for more details. Let  $\mathcal{L} = \{(x, n) \in \mathbb{Z}^2 : x + n \text{ is even}\}$ . The notation indicates that we are thinking of the first coordinate as space and the second as time. Independently each site  $(x, n) \in \mathcal{L}$  is designated as open ( $\eta(x, n) = 1$ ) or closed ( $\eta(x, n) = 0$ ) with probabilities  $p$  and  $1 - p$ . We say that  $(x_0, n), (x_1, n + 1), \dots, (x_m, n + m)$  is a path from  $(x, n)$  to  $(y, n + m)$  if  $x_0 = x, x_m = y$  and for  $1 \leq k \leq m, x_k \in \{x_{k-1} + 1, x_{k-1} - 1\}$  and  $(x_k, n + k)$  is open. Notice that  $(x, n)$  is not required to be open. We say that  $(y, n + m)$  can be reached from  $(x, n)$  inside  $A$  if there is a path from  $(x, n)$  to  $(y, n + m)$  that lies in  $A$ . If  $A = \mathcal{L}$  we drop the phrase ‘inside  $A$ ’ and write  $(x, n) \rightarrow (y, n + m)$ .

Let  $\mathcal{C}_0 = \{(x, n) : (0, 0) \rightarrow (x, n)\}$  and  $\Omega_\infty = \{|\mathcal{C}_0| = \infty\}$ .  $\mathcal{C}_0$  is the cluster containing the origin  $(0, 0)$ .  $\Omega_\infty$  is the event that ‘percolation’ occurs, i.e.,  $\mathcal{C}_0$  is infinite. Let  $p_c = \inf\{p : P_p(\Omega_\infty) > 0\}$ , where the subscript  $p$  indicates the parameter value.  $p_c$  is called the critical value for percolation because  $P_p(\Omega_\infty) = 0$  if  $p < p_c$  and  $P_p(\Omega_\infty) > 0$  if  $p > p_c$ . It is well known that

$$(1) \quad p_c < 80/81 \approx 0.987654321.$$

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This result was proved by a ‘contour argument.’ By more carefully counting the number of contours, Toom (1968) was able to show that  $p_c < 0.93$ .

To improve Toom’s bound we show that if the probabilities of events in certain boxes are large enough then percolation occurs. Suppose for concreteness,  $B = (-1, 1) \times [0, T]$ , and tile the plane with translates of  $B$ , i.e.,  $B_{m,n} = (6m, Tn) + B$  for  $(m, n) \in \mathcal{L}'$  where  $\mathcal{L}'$  is a copy of  $\mathcal{L}$ . Let  $J = \{2, 4, 6, 8\}$ , let  $I = J \times \{0\}$ , and translate  $I$  in a similar fashion:  $I_{m,n} = (6m, Tn) + I$ . For  $j \in J$  let  $G_T^j$  be the event that  $(j, 0)$  is connected to  $I_{-1,1}$  and to  $I_{1,1}$  inside  $B_{0,0}$ , i.e., to  $\{0, 2\} \times \{T\}$  and to  $\{8, 10\} \times \{T\}$ . (See Figure 1.) The boxes  $B_{m,n}$  are disjoint, so the occurrence of translates of  $G_T^j$  by  $(6m, nT)$  with  $(m, n) \in \mathcal{L}'$  are independent. It is for this reason we allow the first site in the path to be closed.

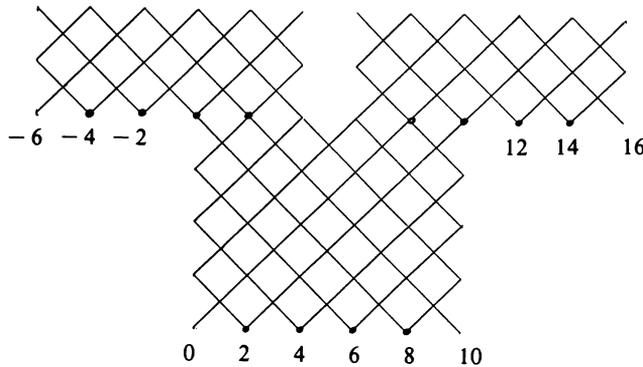


Figure 1

*Lemma 1. If site percolation with parameter  $\phi_T(p)$  percolates on  $\mathcal{L}'$  then site percolation with parameter  $p$  percolates on  $\mathcal{L}$ .*

The events are designed so that if  $G_T^j$  occurs in  $B_{m,n}$  then we get starting points for some  $G_T^k$  in  $B_{m+1,n+1}$  and some  $G_T^l$  in  $B_{m-1,n+1}$ . Let  $\phi_T(p) = \min_{j \in J} P_p(G_T^j)$ . Once enough notation is introduced, it is not hard to show the following result.

The second ingredient in deriving our new bounds is a result from reliability theory that appears on p. 211 of Barlow and Proschan (1965). To state their result we need some definitions. A *monotonic structure function* is the indicator function of an increasing event. That is, it is a function  $\psi: \{0, 1\}^n \rightarrow \{0, 1\}$  so that if  $x_i \leq y_i$  for  $1 \leq i \leq n$  then  $\psi(x) \leq \psi(y)$ . Intuitively, 0 means failed and 1 means working and  $\psi(x)$  gives the state of a machine when its  $n$  components have states given by  $x$ . The *reliability function* of a monotonic structure is  $h(p) = E\psi(X_p)$  where  $X_p$  is a vector in which the  $n$  components are independent and take the values 1 and 0 with probabilities  $p$  and  $1 - p$ . Let  $1_i x$  (or  $0_i x$ ) be the vectors obtained by setting the  $i$ th component of  $x$  to 1 (or 0). Component  $i$  is said to be *essential* if  $\psi(1_i x) \neq \psi(0_i x)$  for some  $x$  and *inessential* otherwise.

*Lemma 2. Let  $h(p)$  be the reliability function of a monotonic structure with at least two essential components. Then for  $p \in (0, 1)$*

$$h'(p) > \frac{h(p)(1 - h(p))}{p(1 - p)}$$

When  $h(p) = p$  this says  $h'(p) > 1$ , so the graph of  $h$  can only cross the line  $y = x$  from below to above. In particular if  $h(p) > p$  then  $h(q) > q$  for all  $q \in (p, 1)$ . Using the last conclusion with  $h = \phi_T$  it follows that if  $\phi_T(p) > p$  then iteration of  $\phi_T$  drives  $\phi_T^k(p) \uparrow 1$ , so repeatedly applying Lemma 1 and then using (1) gives the following result.

*Theorem 1.* If  $\phi_T(p) > p$  for some  $T > 0$  then  $P_p(\Omega_\infty) > 0$ .

The probabilities  $\phi_T(p)$  are tedious to compute by hand. However for a fixed value of  $p$ , it is not hard to evaluate  $\phi_T(p)$  on a computer. For fixed  $j$ ,  $W_n = \{x : (j, 0) \rightarrow (x, n) \text{ in } B\}$  is a Markov chain whose state at even times is a subset of  $\{0, 2, 4, 6, 8, 10\}$  and at odd times is a subset of  $\{1, 3, 5, 7, 9\}$ , and it is easy to write a formula for the transition probability. A little experimentation shows that  $\phi_{12}(0.901) = 0.90194$ , and hence  $p_c < 0.901$ .

There is nothing special about the choice of  $B$  in the argument above. By using larger boxes and intervals one gets better results as shown in Table 1.

TABLE 1

Sites	$B$	$I$	Bound
6/5	$(-1, 11) \times [0, 12]$	$[2, 8]$	0.901
8/7	$(-1, 15) \times [0, 18]$	$[2, 12]$	0.842
10/9	$(-1, 19) \times [0, 25]$	$[2, 16]$	0.819

The first column indicates the number of sites at even and odd times. For a given width, the interval and time have been chosen to optimize the result. The sequence ends with strips of width 20 because in this case each value of  $p$  investigated took about 35 minutes on a personal computer running at 12 Mhz, and the next case (12/11) would take at least 16 times as long. One could of course turn to a bigger, faster computer but our experimentation showed that the next bound would not be a big improvement ( $> 0.805$ ). In our exploratory runs we saved work by just checking that  $P_p(G_T^j) > p$ , trusting in the fact that the worst case is  $j = 2$ . Because the bound

$$(2) \quad p_c < 0.819$$

is crucial for the developments below, we have verified our intuition in this case:

$j$	2	4	6	8
$P_{0.819}(G_T^j)$	0.82062	0.86012	0.87084	0.87380

The answers for 16, 14, 12, and 10 are the same, by symmetry.

The bounds just given stop far short of the critical value, which is estimated to be 0.7058 (see Kinzel and Yeomans (1981)). A little thought reveals that even in very large

boxes we can have  $\phi_T(p) > p$  only if  $P_p(\Omega_\infty) > p$  so the bounds, as they are formulated now, cannot converge to the critical value. This defect could be remedied by replacing the  $G_T^j$  by events that start with more than one occupied site, but for the boxes considered above this does not give a better bound.

(b) *Contact process.* Again we begin with an account of the necessary definitions. This time the reader can find more details in Chapter VI of Liggett (1985) or Chapter 4 of Durrett (1988). In the (basic one-dimensional) contact process  $\xi_t \subset \mathbb{Z}$  and the system evolves as follows:

- (i) Particles die at rate 1; i.e., if  $x \in \xi_t$  then  $P(x \notin \xi_{t+s} \mid \xi_t) = s + o(s)$  as  $s \rightarrow 0$ .
- (ii) Particles are born at rate  $\lambda$  times the number of occupied neighbors; i.e. if  $x \notin \xi_t$  then

$$P(x \in \xi_{t+s} \mid \xi_t) = \lambda \mid \xi_t \cap \{x - 1, x + 1\} \mid s + o(s) \quad \text{as } s \rightarrow 0.$$

In this model the empty set is an absorbing state and attention focuses on  $\Omega_\infty = \{\xi_t^0 \neq \emptyset \text{ for all } t\}$ , i.e. the event that the process does not die out starting from a single particle at 0.  $P_\lambda(\Omega_\infty)$  is a non-decreasing function of  $\lambda$ , so there is a critical value  $\lambda_c = \inf\{\lambda : P_\lambda(\Omega_\infty) > 0\}$ . Comparing with a branching process shows  $\lambda_c \geq 1/2$ , but it is much harder to show that  $\lambda_c < \infty$ . Harris (1974) was the first to do this. He compared the contact process with oriented site percolation, and used (1) to conclude that  $\lambda_c < \infty$ . Harris did not compute an explicit upper bound, but a slight improvement of his argument given on pp. 87–89 of Durrett (1988) shows  $\lambda_c < 1328$ . One of the problems with this result is that (1) is crude. Replacing  $p_c < 80/81$  by our new bound (and taking  $\delta = 1/12$  in the argument in Durrett (1988)) reduces the bound to

$$(3) \quad \lambda_c < 52.$$

Harris’s argument is based on comparison events for which it is easy to compute the probability. By using events similar to the ones employed in Part (a), the result can be improved further. Let  $A = \{0, 1, \dots, 2K - 1\}$  and let  $\bar{\xi}_t$  be a version of the contact process in which births outside  $A$  are not allowed. Let  $J = \{1, \dots, 2K - 2\}$  and let  $J_m = mK + J$ . For each  $j \in J$  let  $G_T^j$  be the event that  $\bar{\xi}_T^{(j)}$  contains points in  $J_1$  and  $J_{-1}$  at time  $T$ . Here and in what follows the superscript indicates the sites that are occupied at time 0. Let  $\phi_T(\lambda) = \min_{j \in J} P(G_T^j)$ . An argument similar to the proof of Lemma 1 shows

$$(4) \quad \text{if } \phi_T(\lambda) \geq 0.819 \text{ for some } T > 0 \text{ then } \lambda > \lambda_c.$$

Using (4) we get the bounds shown in Table 2. The methods used to obtain them are explained in Section 3.

The number of sites is  $2K$ . The number of flips is  $(2\lambda T)$  times the number of sites. The sources column indicates that for 6 or 8 sites we did what was advertised above, but for 10 sites we start with two particles. That is, for  $\{i, j\} \subset \{1, \dots, K - 1\}$ , we let  $G_T^{i,j}$  be the event that  $\bar{\xi}_T^{(i,j)}$  contains at least two points in  $J_1$  and in  $J_{-1}$ . We only consider sources in one half of the interval because only half of  $J_1$  and  $J_{-1}$  lie in  $A$ .

TABLE 2

Sites	Sources	Flips	Bound
6	1	236	6.20
8	1	420	4.95
10	2	690	3.95

The first of these results is already better than the bound  $\lambda_c < 7$  that Gray and Griffeath (1982) get from a ‘continuous time’ contour argument. However, none of these bounds come close to the remarkable result of Holley and Liggett (1978):

$$(5) \quad \lambda_c \leq 2,$$

which is only 20 per cent larger than Brower et al.’s (1978) estimate  $\lambda_c \approx 1.645$ . Indeed, computer simulations of large systems indicate that we cannot do better than (5) with less than 60 sites.

As in the case of oriented percolation we saved work by only checking the worst case, i.e. for 10 sites we checked that  $P(G_T^{(1,2)}) \geq 0.819$ . It seems intuitively clear that moving one or both particles closer to the middle improves the probability, but we do not know how to show this. Since our best bound is worse than (5), we have not bothered to verify (as we did for (2)) that the desired event has probability  $\geq 0.819$  for all initial configurations.

(c) *Sexual reproduction.* This model is a variation of the contact process in which two particles are needed to produce a new one for the state at time  $t$ ,  $\xi_t \subset \mathbb{Z}$  and the system evolves as follows:

- (i) Particles die at rate 1.
  - (ii) If  $x$  and  $x + 1$  are occupied then new particles are produced at rate  $2\lambda$  and are sent with equal probability to  $x - 1$  and  $x + 2$ .
  - (iii) If the site to which the particle is sent is occupied the two particles coalesce to 1.
- If (ii) were changed to:

- (ii’) If  $x$  is occupied then new particles are produced at rate  $2\lambda$  and are sent with equal probability to  $x - 1$  and  $x + 1$ ,

we would have the contact process. As in that model, the empty set is an absorbing state, attention focuses on  $\Omega_\infty = \{\xi^{(0,1)} \neq \emptyset \text{ for all } t\}$ , and there is a critical value  $\lambda_c = \inf\{\lambda : P_\lambda(\Omega_\infty) > 0\}$ . Comparing with a branching process shows  $\lambda_c \geq 1$ . (When there are  $k$  particles the death rate is  $k$ , and the birth rate is  $\leq 2\lambda[k/2]$ , where  $[x]$  is the greatest integer  $\leq x$ .) By looking at the rightmost occupied site (see Section 4), it is not hard to improve the last result to

$$(6) \quad \lambda_c \geq \sqrt{3} = 1.732.$$

As usual, upper bounds are harder to come by. Using our new improvement of Harris’s argument (i.e. the proof of (3)) we can show

$$(7) \quad \lambda_c \leq 151.$$

Again details will be given in Section 4.

The bound (7) can be improved dramatically by using the methods we applied to the contact process. Let  $A = \{0, 1, \dots, 2K - 1\}$  and let  $\xi_t$  be a version of sexual reproduction in which births outside  $A$  are not allowed. Let  $J = \{1, \dots, 2K - 2\}$ , and  $J_m = mK + J$ . For  $1 \leq j \leq K - 2$  let  $G_T^j$  be the event that  $\xi_T^{(j, j+1)}$  contains a pair of sites in  $J_1$  and a pair in  $J_{-1}$ . Let

$$\phi_T(\lambda) = \inf_{1 \leq j \leq K-2} P(G_T^j).$$

An argument similar to the proof of Lemma 1 shows that

$$(8) \quad \text{if } \phi_T(\lambda) \geq 0.819 \text{ for some } T > 0 \text{ then } \lambda > \lambda_c.$$

Using this result gives the bounds shown in Table 3.

TABLE 3

Sites	Flips	Bound
6	108	32
8	230	23
10	380	15.1
12	625	13.7

As in the two previous cases we saved work by only computing  $P(G_T^1)$ . To verify the bound for 12 sites we have computed the probabilities for the other cases:

$j$	1	2	3	4
$P(G_T^j)$	0.82053	0.84606	0.85298	0.85521

The sexual reproduction process has not been studied in the physics literature, so we are left on our own to determine the critical value. Let  $\xi_t$  denote the process starting from  $\xi_0 = \{0, -1, -2, \dots\}$  and let  $r_t = \sup \xi_t$ . As explained in Section 4, if  $Er_t < 0$  then  $\lambda < \lambda_c$ . To investigate  $Er_t$  we simulated  $\xi_t$ , a modification of  $\xi_t$  in which all sites  $\leq -500$  were always occupied. When  $\lambda = 3.6$ , the average value of  $\sup \xi_{500}$  for 100 runs was  $-25.52$  indicating that  $\lambda_c > 3.6$ . For a ‘bound’ in the other direction we simulated the process with  $\lambda = 3.8$  on an interval of length 10 000 with periodic boundary conditions starting from all sites occupied. Figure 2 shows the fraction of occupied sites plotted against time. The graph suggests that the density is converging exponentially fast to a limiting value about 0.55. Combining the last two observations suggests that  $\lambda_c \approx 3.7$ .

## 2. Proof of Lemma 1

We begin by recalling some definitions from the introduction. Let  $\mathcal{L} = \{(m, n) \in \mathbb{Z} : m + n \text{ is even}\}$  and let  $\mathcal{L}'$  be a copy of  $\mathcal{L}$ . Let  $B = (-1, 2K - 1) \times [0, T]$

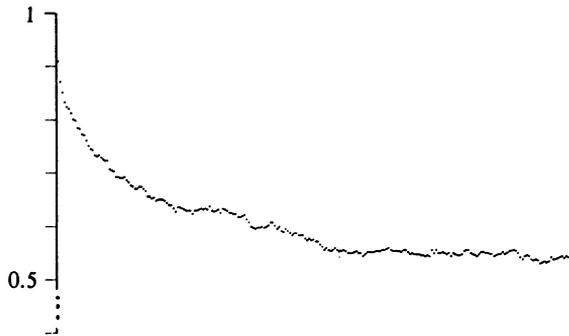


Figure 2. Sexual reproduction,  $\lambda = 3.8$

where  $T$  is even and tile the plane with translates of  $B$ : let  $B_{m,n} = (Km, Tn) + B$  for  $(m, n) \in \mathcal{L}'$ . We call  $(m, n) \in \mathcal{L}'$  open and set  $\eta'(m, n) = 1$  if a certain good event occurs in  $B_{m,n}$ . Let  $H_k = \{(i, j) : j \leq k\}$ . The good event for  $B_{m,n}$  will depend on the state of the sites  $(i, j) \in H_{nT}$  but will be defined in such a way that

if we condition on  $\mathcal{F}_{nT} =$  the  $\sigma$ -field generated by  $\{\eta(i, j) : (i, j) \in H_{nT}\}$ , then  
 (9)  $\eta'(m, n), m \in 2\mathbb{Z} - n$  are independent and are 1 with probability  $\geq \phi_T(p)$ .

We define points  $i_{m,n}, (m, n) \in \mathcal{L}'$  inductively, starting with  $i_{0,0} = 2$ . The definitions guarantee that

(10) if  $i_{m,n} < \infty$  then  $(i_{m,n}, nT)$  can be reached by a path from  $(2, 0)$ .

We begin with an informal description. When  $i_{m,n} < \infty$ , we set  $\eta'(m, n) = 1$  if  $I_{m+1,n+1}$  and  $I_{m-1,n+1}$  can be reached from  $(i_{m,n}, nT)$  inside  $B_{m,n}$ . When  $i_{m,n} = \infty$  our construction has failed to produce a path from  $(2, 0)$  to  $I_{m,n}$ . To have  $\eta'(m, n)$  defined in this case, we check to see if there are paths from  $(2, 0) + (Km, Tn)$  to  $I_{m+1,n+1}$  and to  $I_{m-1,n+1}$ .

To give formal definitions now, let  $i_{0,0} = 2$ . We proceed inductively assuming  $n \geq 0$  and that the  $i_{m,n}$  have been defined for  $(m, n) \in \mathcal{L}'$  with  $-n \leq m \leq n$ . (We only need  $\eta'(m, n)$  for  $-n \leq m \leq n$  to compute  $\mathcal{C}_0$ , so we do not worry about the other  $\eta'$ .)

Case 1.  $i_{m,n} < \infty$ . Let  $J_m = Km + J$  and let  $W_{m,n} = \{j : (j, (n+1)T) \text{ that can be reached from } (i_{m,n}, nT) \text{ inside } B_{m,n}\}$ . If  $W_{m,n} \cap J_{m+1} \neq \emptyset$  and  $W_{m,n} \cap J_{m-1} \neq \emptyset$  then we set  $\eta'(m, n) = 1$ ,

$$j_{m-1,n+1}^+ = \inf(W_{m,n} \cap J_{m-1}) \quad \text{and} \quad j_{m+1,n+1}^- = \inf(W_{m,n} \cap J_{m+1}).$$

Otherwise we set  $\eta'(m, n) = 0, j_{m-1,n+1}^+ = \infty$ , and  $j_{m+1,n+1}^- = \infty$ . Here  $j_{m-1,n+1}^+$  is the contribution to  $I_{m-1,n+1}$  from  $I_m$ , the  $+$  indicating that  $(m-1)+1 = m$ .

Case 2.  $i_{m,n} = \infty$ . We set  $\eta'(m, n) = 1$  if  $I_{m-1,n+1}$  and  $I_{m+1,n+1}$  can be reached from  $(2, 0) + (Km, Tn)$  inside  $B_{m,n}$ , and  $\eta'(m, n) = 0$  otherwise. In either case we set  $j_{m-1,n+1}^+ = \infty$ , and  $j_{m+1,n+1}^- = \infty$  to remind us that there is no guarantee that these points can be reached from  $(2, 0)$ .

Finally let  $j_{-n-1,n+1}^- = \infty$ , and  $j_{n+1,n+1}^+ = \infty$  (these variables are undefined) and set

$$i_{m,n} = \min\{j_{m,n+1}^+, j_{m,n+1}^-\}.$$

Since the  $B_{m,n}$  are disjoint, it is easy to see that (9) holds. The definitions guarantee (10), and Lemma 1 follows.

### 3. Continuous-time methodology

When dealing with a continuous-time process, we look at a corresponding discrete-time chain. For example, to simulate the contact process on  $I = \{0, 1, \dots, K - 1\}$  we use the following algorithm (which assumes  $\lambda > 1/2$ ):

- (i) Pick a site  $i \in I$  at random.
- (ii) If  $i$  is occupied we generate a random number  $r$  uniform on  $(0, 1)$  and kill the particle if  $r < 1/2\lambda$ .
- (iii) If  $i$  is vacant, we pick a neighbor ( $i - 1$  or  $i + 1$ ) at random. If the neighbor is occupied then  $i$  becomes occupied.

*Note.* In the actual computer program one has to make sure that if  $i$  is made vacant in step (ii) then (iii) will be skipped.

Rules (i)–(iii) define a transition probability  $q$  for a discrete-time chain on the set of subsets of  $I$ . We compute the iterates of  $q$  and then recover the transition probability for the contact process by

$$p_t(\xi, \zeta) = \sum_{n=0}^{\infty} \exp(-2\lambda Kt) \frac{(2\lambda Kt)^n}{n!} q^n(\xi, \zeta).$$

For even moderate values of  $K$ , e.g.  $K = 12$ , the events of interest occur when the expected number of flips,  $2\lambda Kt$ , is fairly large, about 625 in the example cited. To compute  $\pi_k = e^{-625}(625)^k/k!$  we start by observing

$$\pi_{625} = \exp\left(\sum_{j=1}^{625} -1 + \log(625) - \log j\right)$$

and then use the recursion  $\pi_k = \pi_{k-1}(625)/k$  to compute the probabilities within five standard deviations of the mean.

### 4. Bounds for sexual reproduction

In this section we prove the bounds (6) and (7) given in the introduction.

*Proof of (6).* Let  $\xi_t$  denote the system starting from  $\xi_0 = \{0, -1, -2, \dots\}$ . Let  $r_t = \sup \xi_t$ . It follows from results in Durrett (1980) that  $\lim_{t \rightarrow \infty} r_t/t = \inf_{s \geq 0} Er_s/s$  almost surely, and if  $\alpha(\lambda) < 0$  then  $P_\lambda(\Omega_\infty) = 0$ . To get an upper bound on  $r_t$  we define a comparison process  $(s_t, \sigma_t)$  with state space  $\mathbb{Z} \times \{0, 1\}$  so that

(11) 
$$r_t \leq s_t,$$

and

(12) 
$$\text{if } r_t - 1 \in \xi_t \text{ then } \sigma_t = 1.$$

Intuitively  $(s_t, \sigma_t)$  is a version of  $\xi_t^x$  in which only the particles at  $r_t$  or  $r_t - 1$  are allowed to die, and  $\sigma_t = 1$  when there is a particle at  $r_t - 1$ .

To define the comparison formally we construct  $\xi_t$  in a special way. Let  $\{T_n^x, n \geq 1\}$ ,  $\{U_n^x, n \geq 1\}$ , and  $\{V_n^x, n \geq 1\}$  be independent Poisson processes with rates 1,  $\lambda$ , and  $\lambda$  respectively. At times  $T_n^x$  the particle at  $x$  is killed, at times  $U_n^x$  (or  $V_n^x$ ) a birth occurs at  $x$  if  $x - 2$  and  $x - 1$  (or  $x + 1$  and  $x + 2$ ) are occupied. The comparison process makes transitions as shown in Table 4.

TABLE 4

Time		$T_n^{x-1}$	$U_n^{x+1}$	$V_n^{x-1}$	$T_n^x$
$(y, 1)$	→	$(y, 0)$	$(y + 1, 1)$	$(y, 1)$	$(y - 1, 1)$
$(y, 0)$	→	$(y, 0)$	$(y, 0)$	$(y, 1)$	$(y - 2, 1)$

By examining the effect of each event it is easy to check that (11) and (12) hold.

Since  $\sigma_t$  changes  $1 \rightarrow 0$  at rate 1 and  $0 \rightarrow 1$  at rate  $\lambda + 1$ , the asymptotic fraction of the time  $\sigma_t = 1$  is  $(\lambda + 1)/(\lambda + 2)$ . The infinitesimal drift in the first component is  $\lambda - 1$  when  $\sigma_t = 1$  and  $-2$  when  $\sigma_t = 0$  so

$$s_t/t \rightarrow (\lambda - 1) \frac{\lambda + 1}{\lambda + 2} + (-2) \frac{1}{\lambda + 2} \quad \text{as } t \rightarrow \infty$$

which is  $< 0$  if  $\lambda < \sqrt{3}$ .

*Note.* By considering an approximation in which only particles at  $r_t, r_t - 1, \dots, r_t - k$  can die one can improve the last bound. Ziezold and Grillenberger (1988) have done this for the contact process.

*Proof of (7).* The proof is similar to the proof of (3) given on pp. 87–89 of Durrett (1988). Let  $\delta > 0$  and call a site in  $\mathcal{L} = \{(m, n) \in \mathbb{Z}^2 : m + n \text{ is even}\}$  open if

- (i) there are arrivals in  $T_k^{2m}$  or  $T_k^{2m+1}$  during  $((n - 1)\delta, (n + 1)\delta)$ ;
- (ii) there are births  $n\delta < U_j^{2m+2} < U_k^{2m+3} < (n + 1)\delta$ ;
- (iii) there are births  $n\delta < V_j^{2m-1} < V_k^{2m-2} < (n + 1)\delta$ .

To see the reason for the definition notice that when  $(0, 0)$  and  $(1, 1)$  are open,  $\xi_\delta^{(0,1)} \supset \{2, 3\}$ , and the states of different sites in  $\mathcal{L}$  are independent. By now familiar reasoning, if the probability sites are open is at least 0.819 then  $P(\xi^{(0,1)} \neq \emptyset \text{ for all } t) > 0$ . The probability a site is closed is smaller than

$$(1 - \exp(-4\delta)) + 2(1 + \lambda\delta)\exp(-\lambda\delta) < 0.18052$$

if  $\delta = 1/24$  and  $\lambda = 151$ .

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