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THE L_r CONVERGENCE AND WEAK LAWS OF LARGE NUMBERS FOR $\widetilde{\rho}$-MIXING RANDOM VARIABLES

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Abstract

The L_r convergence and a class of weak laws of large numbers are obtained for sequences of $\widetilde{\rho}$-mixing random variables under the uniform Cesàro-type condition. This is weaker than the pth-order Cesàro uniform integrability.

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1. Introduction

Let $\{X_n,\ n \in \mathbf{N}\}$ be a sequence of random variables on a probability space $(\Omega,\ \mathcal{M},\ P)$. For any $S \subset \mathbf{N}$, define $\mathcal{F}_S = \sigma\{X_k \mid k \in S\}$. Given σ-fields $\mathcal{F}, \mathcal{G} \subset \mathcal{M}$, let

$$\rho(\mathcal{F},\ \mathcal{G}) = \sup\{|\mathrm{corr}(f, g)| \in L_2(\mathcal{F}),\ g \in L_2(\mathcal{G})\}.$$

Similar to Bradley's work [1], we define the following coefficients of dependence:

$$\widetilde{\rho}(k) = \sup\{\rho(\mathcal{F}_S,\ \mathcal{F}_T)\},$$

where $k \ge 0$, and the supremum is taken over all pairs of nonempty finite sets $S, T \subset \mathbf{N}$ such that $\mathrm{dist}(S, T) \ge k$.

Definition 1.1. A sequence of random variables $\{X_n,\ n \in \mathbf{N}\}$ is said to be a $\widetilde{\rho}$*-mixing sequence* if

$$\lim_{k\to\infty} \widetilde{\rho}(k) < 1. \tag{1.1}$$

Since $0 \le \widetilde{\rho}(k) \le \widetilde{\rho}(k-1) \le \cdots \le \widetilde{\rho}(1) \le 1$, condition (1.1) is equivalent to

$$\widetilde{\rho}(k_0) < 1 \quad \text{for some } k_0 \ge 1. \tag{1.2}$$

Bradley [1, 2] and Miller [8] obtained several limit properties under the condition $\widetilde{\rho}(k) \to 0$. Bryc and Smolenski [3] and Peligrad [9, 10] pointed out the importance

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of condition (1.1). For the $\widetilde{\rho}$-mixing sequence, we refer to Bryc and Smolenski [3] for moment inequalities and almost sure convergence, to Yang [16] for moment inequalities and strong laws of large numbers (SLLNs), and to Peligrad and Gut [11] for almost sure results. Also, we refer to Utev and Peligrad [13] for maximal inequalities, Kuczmaszewska [5] for a Chung–Teicher type SLLN, Wu [15] for complete convergence and a weak law of large numbers (WLLN). In this paper, we will obtain L_r convergence and a class of WLLNs under the uniform Cesàro-type condition [7], which is weaker than the pth-order Cesàro uniform integrability [4].

Definition 1.2. A sequence of random variables $\{X_n,\ n \in \mathbf{N}\}$ is said to be of pth-order Cesàro uniform integrability if

$$\lim_{x\to\infty}\Big[\sup_{n\ge 1} n^{-1}\sum_{k=1}^{n} E|X_k|^p I_{\{|X_k|>x\}}\Big] = 0. \tag{1.3}$$

Remark 1.3. Note that

$$\begin{aligned} E|X_n|^p I_{\{|X_n|>x\}} &= \Big(\int_0^{x^p} + \int_{x^p}^{\infty}\Big) P(|X_n|^p I_{\{|X_n|>x\}} > t)\,dt \\ &= \int_0^{x^p} P(|X_n| > x)\,dt + \int_{x^p}^{\infty} P(|X_n|^p > t)\,dt. \end{aligned}$$

Note that (1.3) holds if and only if

$$\lim_{x\to\infty}\Big[\sup_{n\ge 1} n^{-1}\sum_{k=1}^{n} x^p P(|X_k| > x)\Big] = 0 \tag{1.4}$$

and

$$\lim_{x\to\infty}\Big[\sup_{n\ge 1} n^{-1}\sum_{k=1}^{n} \int_{x^p}^{\infty} P(|X_k|^p > t)\,dt\Big] = 0$$

both hold.

Wu [14] obtained the following results.

Theorem 1.4. *Let $\{X_n, n \in \mathbf{N}\}$ be a sequence of $\widetilde{\rho}$-mixing random variables with zero mean. If condition (1.3) holds for $1 < p < 2$, then*

$$n^{-1/p}\sum_{k=1}^{n} X_k \xrightarrow{L_p} 0, \quad n \to \infty.$$

Theorem 1.5. *Let $\{X_n, n \in \mathbf{N}\}$ be a sequence of $\widetilde{\rho}$-mixing random variables with zero mean. If condition (1.4) holds for $1 < p < 2$, then*

$$n^{-1/p}\sum_{k=1}^{n} X_k \xrightarrow{p} 0, \quad n \to \infty.$$

2. Main results

THEOREM 2.1. *Let $\{X_n, n \in \mathbf{N}\}$ be a sequence of $\widetilde{\rho}$-mixing random variables with zero mean. If, for $1 < p < 2$, (1.4) holds and $\sup_{n\geq 1} n^{-1} \sum_{k=1}^{n} E|X_k|^p < \infty$, then for any $r \in (0, p)$ we have*

$$n^{-1/p} \sum_{k=1}^{n} X_k \xrightarrow{L_r} 0, \quad n \to \infty.$$

THEOREM 2.2. *Let $\{X_n, n \in \mathbf{N}\}$ be a sequence of $\widetilde{\rho}$-mixing random variables. Suppose that there exists a positive function $g(x)$ for $x \geq 0$ and $g(0) = 0$, such that $g(x)$ is strictly increasing with $g(x) \uparrow \infty$ and $g(x)/x$ is nondecreasing for $x > 0$. Also, assume that the uniform Cesàro-type condition*

$$\lim_{x\to\infty} \left[\sup_{n\geq 1} n^{-1} \sum_{k=1}^{n} xP\{|X_k|^p > g(x)\}\right] = 0 \tag{2.1}$$

holds for some $p \in (0, 2)$. Then we have

$$g^{-1/p}(n) \sum_{k=1}^{n} [X_k - E(X_k I_{\{|X_k|^p \leq g(n)\}})] \xrightarrow{p} 0, \quad n \to \infty.$$

Setting $g(x) = x$, we obtain the following result.

COROLLARY 2.3. *Let $\{X_n, n \in \mathbf{N}\}$ be a sequence of $\widetilde{\rho}$-mixing random variables. Suppose that the uniform Cesàro-type condition*

$$\lim_{x\to\infty} \sup_{n\geq 1} n^{-1} \sum_{k=1}^{n} xP\{|X_k|^p > x\} = 0 \tag{2.2}$$

holds for some $p \in (0, 2)$. Then

$$n^{-1/p} \sum_{k=1}^{n} [X_k - E(X_k I_{\{|X_k|^p \leq n\}})] \xrightarrow{p} 0, \quad n \to \infty.$$

REMARK 2.4. Observe that condition (2.2) is equivalent to (1.4). Our result is more general than Theorem 1.5, since $p \in (0, 2)$ and "$EX_n = 0$" is not required in Corollary 2.3.

REMARK 2.5. The uniform Cesàro-type condition (2.2) is weaker than the pth-order Cesàro uniform integrability,

$$\lim_{x\to\infty} \sup_{n\geq 1} n^{-1} \sum_{k=1}^{n} E|X_k|^p I_{\{|X_k|^p > x\}} = 0, \quad 0 < p < 2.$$

In the remainder of this paper, C stands for a positive finite constant whose value may differ from one place to another.

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3. Proof of main results

LEMMA 3.1 [13, Theorem 2.1]. *Let $\{X_n, n \in \mathbf{N}\}$ be a sequence of $\widetilde{\rho}$-mixing random variables, and assume that $EX_n = 0$, $E|X_n|^q < \infty$ for some $q \geq 2$ and for every $n \geq 1$. Then there exists a positive constant $C = C(q, k_0, \widetilde{\rho}(k_0))$ with k_0 and $\widetilde{\rho}(k_0)$ defined in (1.2), such that*

$$E \max_{1\leq j\leq n} \Big|\sum_{i=1}^{j} X_i\Big|^q \leq C\Big[\sum_{i=1}^{n} E|X_i|^q + \Big(\sum_{i=1}^{n} EX_i^2\Big)^{q/2}\Big], \quad n \geq 1.$$

In particular, if $q = 2$,

$$E \max_{1\leq j\leq n} \Big(\sum_{i=1}^{j} X_i\Big)^2 \leq C \sum_{i=1}^{n} EX_i^2.$$

By Lemma 3.1 and the Markov inequality, we get the Kolmogorov inequality [6] for $\widetilde{\rho}$-mixing random variables.

LEMMA 3.2. *Suppose that $\{X_n, n \in \mathbf{N}\}$ is a sequence of $\widetilde{\rho}$-mixing random variables with $EX_n = 0$ and $EX_n^2 < \infty$. Then, for any given $\epsilon > 0$, there exists a positive constant C such that*

$$P\Big(\max_{1\leq j\leq n} \Big|\sum_{i=1}^{j} X_i\Big| > \epsilon\Big) \leq \frac{C}{\epsilon^2} \sum_{i=1}^{n} EX_i^2.$$

LEMMA 3.3 [12, Lemma 3.2.3(ii)]. *Let $\{a_{ni}\}$ be a matrix of real numbers, and $\{x_i\}$ be a sequence of real numbers satisfying $x_i \to 0$, as $i \to \infty$. Then*

$$\sum_{i=1}^{\infty} |a_{ni}| \leq M < \infty, \quad \textit{for all } n \geq 1,$$

and

$$a_{ni} \to 0 \text{ as } n \to \infty, \quad \textit{for each } i \geq 1,$$

imply that

$$\sum_{i=1}^{\infty} a_{ni}x_i \to 0, \quad \textit{as } n \to \infty.$$

PROOF OF THEOREM 2.1. By Theorem 1.5, it is enough to show that $\{|n^{-1/p}S_n|^r, n \geq 1\}$ is uniformly integrable, where $S_n = \sum_{k=1}^{n} X_k$. Noting that $p/r > 1$, we need only prove that

$$\sup_{n\geq 1} E(|n^{-1/p}S_n|^r)^{p/r} < \infty. \tag{3.1}$$

Set $\alpha = \sup_{n\geq 1} n^{-1} \sum_{k=1}^{n} E|X_k|^p$. Note that

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$$
\begin{aligned}
E(|(2\alpha)^{-1}n^{-1/p}S_n|^r)^{p/r} &= n^{-1}\int_0^\infty P(|S_n| > 2\alpha s^{1/p})\,ds \\
&\le 1 + n^{-1}\int_n^\infty P(|S_n| > 2\alpha s^{1/p})\,ds \\
&\le 1 + n^{-1}\int_n^\infty \sum_{k=1}^n P(|X_k| > s^{1/p})\,ds \\
&\quad + n^{-1}\int_n^\infty P\Big(\Big|\sum_{k=1}^n X_k I_{\{|X_k|\le s^{1/p}\}}\Big| > 2\alpha s^{1/p}\Big)\,ds.
\end{aligned}
$$

Thus, to prove (3.1), it suffices to show that

$$
I_1 := \sup_{n\ge 1} n^{-1}\int_n^\infty \sum_{k=1}^n P(|X_k| > s^{1/p})\,ds < \infty
$$

and

$$
I_2 := \sup_{n\ge 1} n^{-1}\int_n^\infty P\Big(\Big|\sum_{k=1}^n X_k I_{\{|X_k|\le s^{1/p}\}}\Big| > 2\alpha s^{1/p}\Big)\,ds < \infty.
$$

Note that

$$
I_1 \le \sup_{n\ge 1} n^{-1}\sum_{k=1}^n \int_0^\infty P(|X_k| > s^{1/p})\,ds = \sup_{n\ge 1} n^{-1}\sum_{k=1}^n E|X_k|^p < \infty.
$$

Since $EX_n = 0$ and $n \in \mathbf{N}$, we have

$$
\begin{aligned}
\sup_{s\ge n} s^{-1/p}\Big|E\sum_{k=1}^n X_k I_{\{|X_k|\le s^{1/p}\}}\Big| &\le \sup_{s\ge n} s^{-1/p}\sum_{k=1}^n E|X_k| I_{\{|X_k|> s^{1/p}\}} \\
&\le n^{-1/p}\sum_{k=1}^n E|X_k| I_{\{|X_k|> n^{1/p}\}} \\
&\le n^{-1}\sum_{k=1}^n E|X_k|^p \le \alpha. \qquad (3.2)
\end{aligned}
$$

Also, since $\{X_k I_{\{|X_k|\le s^{1/p}\}} - EX_k I_{\{|X_k|\le s^{1/p}\}}, k \in \mathbf{N}\}$ is a sequence of $\widetilde{\rho}$-mixing random variables with finite second moment and zero mean, by using (3.2) and Lemma 3.2 we obtain

$$
\begin{aligned}
I_2 &\le \sup_{n\ge 1} n^{-1}\int_n^\infty P\Big(\Big|\sum_{k=1}^n [X_k I_{\{|X_k|\le s^{1/p}\}} - EX_k I_{\{|X_k|\le s^{1/p}\}}]\Big| > \alpha s^{1/p}\Big)\,ds \\
&\le C\sup_{n\ge 1} n^{-1}\int_n^\infty s^{-2/p}\sum_{k=1}^n EX_k^2 I_{\{|X_k|\le s^{1/p}\}}\,ds \\
&= C\sup_{n\ge 1} n^{-1}\sum_{k=1}^n \int_n^\infty s^{-2/p} EX_k^2 I_{\{|X_k|\le s^{1/p}\}}\,ds.
\end{aligned}
$$

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Moreover,

$$\begin{aligned}
\int_n^{\infty} s^{-2/p} EX_k^2 I_{\{|X_k|\le s^{1/p}\}}\,ds &\le \sum_{m=n}^{\infty}\int_m^{m+1} s^{-2/p} EX_k^2 I_{\{|X_k|\le s^{1/p}\}}\,ds \\
&\le \sum_{m=n}^{\infty} m^{-2/p} EX_k^2 I_{\{|X_k|\le (m+1)^{1/p}\}} \\
&\le \sum_{m=1}^{\infty} m^{-2/p} EX_k^2 I_{\{|X_k|\le (m+1)^{1/p}\}} \\
&= \sum_{m=1}^{\infty} m^{-2/p} \sum_{i=1}^{m} EX_k^2 I_{\{i<|X_k|^p\le i+1\}} + \sum_{m=1}^{\infty} m^{-2/p} EX_k^2 I_{\{|X_k|^p\le 1\}} \\
&\le \sum_{m=1}^{\infty} m^{-2/p} \sum_{i=1}^{m} EX_k^2 I_{\{i<|X_k|^p\le i+1\}} + C \\
&= \sum_{i=1}^{\infty} EX_k^2 I_{\{i<|X_k|^p\le i+1\}} \sum_{m=i}^{\infty} m^{-2/p} + C \\
&\le C\sum_{i=1}^{\infty} i^{1-2/p} EX_k^2 I_{\{i<|X_k|^p\le i+1\}} + C \le CE|X_k|^p + C,
\end{aligned}$$

which implies $I_2 < \infty$. Thus, the proof of Theorem 2.1 is complete. □

Proof of Theorem 2.2. For $n \ge 1$, set $Y_k = X_k I_{\{|X_k|^p\le g(n)\}}$, $1 \le k \le n$, and $T_n = \sum_{k=1}^{n} Y_k$. By (2.1), for any given $\epsilon > 0$, we have

$$\begin{aligned}
P\Big(\Big|g^{-1/p}(n)\sum_{k=1}^{n} X_k - g^{-1/p}(n)\sum_{k=1}^{n} Y_k\Big| > \epsilon\Big) &\le P\Big(\bigcup_{k=1}^{n}\{|X_k|^p > g(n)\}\Big) \le \sum_{k=1}^{n} P(|X_k|^p > g(n)) \\
&= n^{-1}\sum_{k=1}^{n} nP(|X_k|^p > g(n)) \to 0, \quad \text{as } n \to \infty.
\end{aligned}$$

So it is sufficient to prove that

$$g^{-1/p}(n)\sum_{k=1}^{n}(Y_k - EY_k) \xrightarrow{P} 0, \quad \text{as } n \to \infty. \tag{3.3}$$

Since $\{(Y_k - EY_k)/g^{1/p}(n),\ k \ge 1\}$ is a sequence of $\widetilde{\rho}$-mixing random variables with finite second moment and zero mean, by Lemma 3.1 we get

$$\begin{aligned}
&g^{-2/p}(n)E\Big|\sum_{k=1}^{n}(Y_k - EY_k)\Big|^2 \\
&\quad\le Cg^{-2/p}(n)\sum_{k=1}^{n} EY_k^2 = Cg^{-2/p}(n)\sum_{k=1}^{n} EX_k^2 I_{\{|X_k|^p\le g(n)\}}
\end{aligned}$$

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$$
\begin{aligned}
&= Cg^{-2/p}(n)\sum_{k=1}^{n}\sum_{j=1}^{n}\int_{\{g(j-1)<|X_k|^p\le g(j)\}} X_k^2\,dP\\
&\le Cg^{-2/p}(n)\sum_{k=1}^{n}\sum_{j=1}^{n} g^{2/p}(j)\{P(|X_k|^p > g(j-1)) - P(|X_k|^p > g(j))\}\\
&= Cg^{-2/p}(n)\sum_{k=1}^{n}\Big[g^{2/p}(1)P(|X_k|^p > g(0)) - g^{2/p}(n)P(|X_k|^p > g(n))\\
&\qquad + \sum_{j=1}^{n-1}\{g^{2/p}(j+1) - g^{2/p}(j)\}P(|X_k|^p > g(j))\Big]\\
&\le Cg^{2/p}(1)ng^{-2/p}(n) + Cg^{-2/p}(n)\\
&\qquad \times \sum_{k=1}^{n}\sum_{j=1}^{n-1}\{g^{2/p}(j+1) - g^{2/p}(j)\}P(|X_k|^p > g(j))\\
&\le Cg^{2/p}(1)ng^{-2/p}(n)\\
&\qquad + Cng^{-2/p}(n)\sum_{j=1}^{n-1}\frac{g^{2/p}(j+1) - g^{2/p}(j)}{j}\sup_{n\ge1} n^{-1}\sum_{k=1}^{n} jP(|X_k|^p > g(j))\\
&=: I_3 + I_4. \qquad (3.4)
\end{aligned}
$$

Note that $g(n)/n$ is nondecreasing and $g(n) \uparrow \infty$, and we have

$$
I_3 = C\frac{g^{2/p}(1)n}{g(n)}\frac{1}{g^{2/p-1}(n)} \le C\frac{g^{2/p}(1)}{g(1)}\frac{1}{g^{2/p-1}(n)} \to 0, \quad \text{as } n \to \infty. \qquad (3.5)
$$

In order to estimate I_4, for every $n \ge 1$ and $j \ge 1$, denote

$$
\alpha_{nj} = n^{-1}\sum_{k=1}^{n} jP\{|X_k|^p > g(j)\}.
$$

Then, by equation (2.1), $\sup_{n\ge1}\alpha_{nj} = o(1)$ as $j \to \infty$. Define an array $\{\beta_{nj},\ 1 \le j < \infty,\ n \ge 1\}$ by

$$
\beta_{nj} = \begin{cases} \dfrac{n}{g^{2/p}(n)}\dfrac{g^{2/p}(j+1) - g^{2/p}(j)}{j}, & 1 \le j \le n-1,\\ 0, & j \ge n. \end{cases}
$$

We show that $\{\beta_{nj},\ 1 \le j < \infty,\ n \ge 1\}$ is a Toeplitz array, that is,

$$
\sum_{j=1}^{\infty}|\beta_{nj}| = O(1), \qquad (3.6)
$$

and

$$
\beta_{nj} \to 0 \quad \text{as } n \to \infty, \text{ for each } j \ge 1. \qquad (3.7)
$$

Clearly (3.7) holds, since $n/g^{2/p}(n) \to 0$, as $n \to \infty$. Noting that

$$\sum_{j=1}^{\infty} |\beta_{nj}| = \frac{n}{g^{2/p}(n)} \sum_{j=1}^{n-1} \frac{g^{2/p}(j+1) - g^{2/p}(j)}{j},$$

condition (3.6) follows, if $\sum_{j=1}^{n-1} [\{g^{2/p}(j+1) - g^{2/p}(j)\}/j] = O(g^{2/p}(n)/n)$. It suffices to show that for $r = 2/p > 1$,

$$\sum_{j=1}^{n-1} \frac{g^r(j+1) - g^r(j)}{j} = O\Big(\frac{g^r(n)}{n}\Big). \tag{3.8}$$

Note that

$$\begin{aligned}\sum_{j=1}^{n-1} \frac{g^r(j+1) - g^r(j)}{j} &= \sum_{j=1}^{n-1} \Big[\frac{g^r(j+1)}{j+1} + \frac{g^r(j+1)}{j(j+1)} - \frac{g^r(j)}{j}\Big] \\ &\le \frac{g^r(n)}{n} + \sum_{j=1}^{n-1} \frac{g^r(j+1)}{j(j+1)} \\ &\le \frac{g^r(n)}{n} + 2\sum_{j=1}^{n} \frac{g^r(j)}{j^2} \\ &\le \frac{g^r(n)}{n} + 2\frac{g^r(n)}{n^r} \sum_{j=1}^{n} \frac{1}{j^{2-r}}.\end{aligned}$$

Moreover, since

$$\sum_{j=1}^{n} \frac{1}{j^{2-r}} \le \int_0^{n+1} \frac{1}{x^{2-r}} dx = \frac{1}{r-1}(n+1)^{r-1} \le \frac{1}{r-1}(2n)^{r-1}$$

$r > 1$, we obtain

$$\frac{g^r(n)}{n} + 2\frac{g^r(n)}{n^r} \sum_{j=1}^{n} \frac{1}{j^{2-r}} \le \frac{g^r(n)}{n} + 2\frac{g^r(n)}{n^r} \frac{1}{r-1}(2n)^{r-1} = \Big(1 + \frac{2^r}{r-1}\Big)\frac{g^r(n)}{n}.$$

Thus (3.8) holds, and consequently condition (3.6) is satisfied. We have shown that $\{\beta_{nj}, 1 \le j < \infty, n \ge 1\}$ is a Toeplitz array, so by Lemma 3.3 we have

$$I_4 \to 0 \quad \text{as } n \to \infty. \tag{3.9}$$

Hence (3.3) follows from (3.4), (3.5) and (3.9). This completes the proof of Theorem 2.2. □

4. Conclusion

The L_r convergence and weak laws of large numbers for $\widetilde{\rho}$-mixing random variables under a condition weaker than the pth-order Cesàro uniform integrability are obtained. In a future work, the goal is to study strong convergence for $\widetilde{\rho}$-mixing random variables under a condition which is a little stronger than the pth-order Cesàro uniform integrability.

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