BULL. AUSTRAL. MATH. SOC. 46L70, 46A55, 17C65
VOL. 35 (1987) 349-362

EXTREME POSITIVE LINEAR MAPS BETWEEN JORDAN BANACH ALGEBRAS

CHO-HO CHU AND NIGEL P.H. JEFFERIES

Let A and B be unital JB-algebras. We study the extremal structure of the convex set $S(A,B)$ of all identity preserving positive linear maps from A to B. We show that every unital Jordan homomorphism from A to B is an extreme point of $S(A,B)$. An extreme point of $S(A,B)$ need not be a homomorphism and we show that, given A, every extreme point of $S(A,B)$ is a homomorphism for any B if, and only if, $\dim A \leq 2$. We also determine when $S(A,B)$ is a simplex.

1. Introduction

Let A and B be unital JB-algebras. In this paper, we study the extreme points of the convex set $S(A,B)$ of all identity preserving positive linear maps from A to B.

Motivated by the results in C^*-algebras [2, 4, 10], we begin by showing that every unital Jordan homomorphism from A to B is an extreme point of $S(A,B)$. We then focus our attention on the natural question of the converse. We study conditions under which the extreme points of $S(A,B)$ are Jordan homomorphisms. If A and B are associative, it is known that the extreme points of $S(A,B)$ are exactly the unital

Received 19 May 1986.

homomorphisms from A to B . In the nonassociative case, however, our results indicate that only in very special situations can one expect that every extreme point of $S(A,B)$ is a homomorphism. For instance, given A , every extreme point of $S(A,B)$ is a homomorphism for any B if, and only if, $\dim A \leq 2$. Also, if B is the self-adjoint part of a finite-dimensional nonabelian C^*-algebra, then every extreme point of $S(A,B)$ is a homomorphism if, and only if, $\dim A \leq 2$.

When B is the real field $\mathbb{R}$, the set $S(A,\mathbb{R})$ is the state space of A and in this case, every extreme point of $S(A,\mathbb{R})$ is a homomorphism if any only if $S(A,\mathbb{R})$ is a (Choquet) simplex. It is natural to ask whether this is still true for any B . The answer is negative. In fact, we will show that $S(A,B)$ is a simplex if and only if either $A = \mathbb{R}$ or A is associative with $B = \mathbb{R}$.

2. JB-algebras and extreme maps

We will use [8] as our main reference for JB-algebras. In the sequel, by a JB-algebra we mean a real Jordan algebra A , with identity 1, which is also a Banach space where the Jordan product and the norm are related as follows

$$||a \circ b|| \leq ||a||\cdot||b||$$

$$||a||^2 = ||a^2|| \leq ||a^2 + b^2||$$

for $a, b \in A$. We note that A is partially ordered by the cone $A_+ = \{a^2 : a \in A\}$ and that A is an order-unit normed Banach space with order-unit 1. Moreover, the second dual A^{**} of A is a JBW-algebra and A embeds into A^{**} as a subalgebra. The self-adjoint part of a unital C^*-algebra is a JB-algebra with the usual Jordan product and the self-adjoint part of a von Neumann algebra is a JBW-algebra.

Let A and B be JB-algebras and let $L(A,B)$ be the real Banach space of bounded linear maps from A to B . A linear map $\phi : A \to B$ is positive if $\phi(A_+) \subset B_+$. Let $S(A,B)$ be the set of all positive linear maps $\phi : A \to B$ such that $\phi(1) = 1$. Then $S(A,B)$ is a convex subset of $L(A,B)$. An extreme point of $S(A,B)$ will be called an extreme map. We note that $S(A,B)$ always contains extreme points. Indeed $S(A,\mathbb{R}) = \{f \in A^* : f(1) = 1 = ||f||\}$ is the state space of A and it can be embedded as a convex subset of $S(A,B)$ via the map $f \in S(A,\mathbb{R}) \mapsto \phi_f \in S(A,B)$ where $\phi_f(a) = f(a)1_B$ for $a \in A$. Moreover, since the states of B separate points of B , by composing with the states of B , it is easy to see that if f is a pure state of A , then ϕ_f is an extreme point of $S(A,B)$. If A and B are the self-adjoint parts of C^*algebras $\underline{A}$ and $\underline{B}$ respectively, we let $\underline{L}(\underline{A},\underline{B})$ be the space of bounded (complex) linear maps from $\underline{A}$ to $\underline{B}$ and let $\underline{S}(\underline{A},\underline{B}) = \{\phi \in \underline{L}(\underline{A},\underline{B}) : \phi \geq 0 , \phi(1) = 1\}$ where each map ϕ in $\underline{S}(\underline{A},\underline{B})$ satisfies $\phi(a^*) = \phi(a)^*$ for $a \in \underline{A}$. It follows that the restriction map $\phi \in \underline{S}(\underline{A},\underline{B}) \mapsto \phi|_A \in S(A,B)$ is a real affine isomorphism and in particular, the two sets $\underline{S}(\underline{A},\underline{B})$ and $S(A,B)$ have the same extremal structure.

The following lemma has been proved in [12].

LEMMA 1. *Let* A *be a JB-algebra. Then an element* p *in* A *is an extreme point of the positive unit ball* $\{a \in A : 0 \leq a \leq 1\}$ *if and only if* p *is a projection, that is,* $p^2 = p$.

A linear map $\phi : A \to B$ is called a (Jordan) homomorphism if $\phi(a^2) = \phi(a)^2$ for all a in A . Plainly, every unital Jordan homomorphism is a positive linear map. In fact, it is even an extreme map.

THEOREM 2. *If* $\phi : A \to B$ *is a unital Jordan homomorphism, then* ϕ *is an extreme point of* $S(A,B)$.

Proof. As the second dual map $\phi^{**} : A^{**} \to B^{**}$ is weakly continuous, it is a Jordan homomorphism by density of A in A^{**} . Suppose that $\phi = \frac{1}{2}(\rho + \psi)$ with $\rho , \psi \in S(A,B)$, then $\phi^{**} = \frac{1}{2}(\rho^{**} + \psi^{**})$. Let e be a projection in A^{**} . Then $\phi^{**}(e)$ is a projection in B^{**} and hence an extreme point of the positive unit ball in B^{**} , by Lemma 1. Now $\phi^{**}(e) = \frac{1}{2}\rho^{**}(e) + \frac{1}{2}\psi^{**}(e)$ and $0 \leq \rho^{**}(e), \psi^{**}(e) \leq 1$ imply that

$\phi^{**}(e) = \rho^{**}(e) = \psi^{**}(e)$. Let a be any element in A^{**} and let $\varepsilon > 0$. By [8; 4. 2. 3], there exist projections $e_1, \ldots, e_n$ in A^{**} and real numbers $\lambda_1, \ldots, \lambda_n$ such that

$$||a - \sum_{j=1}^{n} \lambda_j e_j|| < \varepsilon .$$

So we have

$$\begin{aligned} ||\phi^{**}(a) - \rho^{**}(a)|| &= ||(\phi^{**} - \rho^{**})(a - \Sigma \lambda_j e_j) + (\phi^{**} - \rho^{**})(\Sigma \lambda_j e_j)|| \\ &= ||(\phi^{**} - \rho^{**})(a - \Sigma \lambda_j e_j)|| \\ &\leq \varepsilon \, ||\phi^{**} - \rho^{**}|| . \end{aligned}$$

This shows that $\phi^{**}(a) = \rho^{**}(a)$ for every a in A^{**} . Hence $\phi = \rho$ and ϕ is an extreme point of $S(A,B)$.

In general, not every extreme map is a homomorphism as the following lemma shows.

LEMMA 3. *Let* A *be a JB-algebra. The following conditions are equivalent:*

(i) A *is associative;*

(ii) A *is isometric isomorphic to the self-adjoint part of an abelian C*-algebra;*

(iii) the dual cone A^*_+ *of* A_+ *is a lattice;*

(iv) the state space $S(A,\mathbb{R})$ *is a (Choquet) simplex;*

(v) every extreme point of $S(A,\mathbb{R})$ *is a Jordan homomorphism.*

Proof. (i) ⇒ (ii) see [8; 3. 2. 2.].

(iii) ⇒ (iv). $S(A,\mathbb{R})$ is a base of the lattice cone A^*_+ and hence is a simplex (see [3; p.138]).

(iv) ⇒ (v). Let f be an extreme point of $S(A,\mathbb{R})$. Then $\{f\}$ is a split face of $S(A,\mathbb{R})$ (see [3]) and so the kernel $f^{-1}(0)$ is a Jordan ideal in A (see [7; Theorem 2.3], [5; Corollary 3.4]). So f is a Jordan homomorphism.

https://doi.org/10.1017/S0004972700013332 Published online by Cambridge University Press

(v) ⇒ (i). This follows from the fact that the extreme points of $S(A,\mathbb{R})$ separate points in A and that for each extreme point f in $S(A,\mathbb{R})$, we have $f((a \circ b) \circ c) = f(a)\, f(b)\, f(c) = f(a \circ (b \circ c))$ for $a, b, c \in A$.

Now we study conditions under which the extreme maps are Jordan homomorphisms. As in [8], we define the centre Z_A of a JB-algebra A to be the set of all elements in A which operator commute with every other element in A where two elements a and b are said to operator commute if $a \circ (c \circ b) = (a \circ c) \circ b$ for all c in A. We note that Z_A is an associative JB-subalgebra of A. The following theorem is a straightforward extension of a result of Størmer in [10; Theorem 3.1]. We sketch a proof for the sake of completeness.

THEOREM 4. *Let* ϕ *be an extreme point of* $S(A,B)$. *If* $a \in Z_A$ *and* $\phi(a) \in Z_B$, *then* $\phi(a \circ b) = \phi(a) \circ \phi(b)$ *for all* $b \in A$.

Proof. We may assume $||a|| < \frac{1}{2}$, then $||\phi(a)|| < \frac{1}{2}$. By spectral theory, $\frac{1}{2}1 - a$ and $\frac{1}{2}1 - \phi(a)$ are positive and invertible in Z_A and Z_B respectively, with $\frac{1}{2}1 - \phi(a) \geq \lambda 1$ for some $\lambda > 0$. Define $\psi : A \to B$ by

$$\psi(b) = \phi(b \circ (\tfrac{1}{2}1 - a)) \circ (\tfrac{1}{2}1 - \phi(a))^{-1}$$

for $b \in A$. Then $\psi \in S(A,B)$ and $\lambda\psi \leq \phi$. As ϕ is extreme, we have $\psi = \phi$ which gives

$$\phi(b) = \psi(b) = \phi(b \circ (\tfrac{1}{2}1 - a)) \circ (\tfrac{1}{2}1 - \phi(a))^{-1}$$

and hence $\phi(a \circ b) = \phi(a) \circ \phi(b)$.

Since B is associative if and only if $B = Z_B$, the following result follows immediately from Theorem 2 and Theorem 4.

COROLLARY 5. *If* B *is associative and if* ϕ *is an extreme point of* $S(A,B)$, *then the restriction* $\phi|Z_A$ *is an extreme point of* $S(Z_A,B)$.

We note that the above result need not be true if B is not associative. We refer to [10; 4.14] for an example. Theorem 2 and Theorem 4 also imply the following corollary (see [1, 6, 9, 10]).

COROLLARY 6. *Let* A *and* B *be associative JB-algebras. Then the extreme points of* $S(A,B)$ *are exactly the unital Jordan homomorphisms from* A *to* B .

Let ℓ_n^∞ be the n-dimensional abelian C^*-algebra of (complex) finite sequences with the minimal projections $e_1 = (1, 0, \dots, 0)$, $e_2 = (0, 1, 0, \dots, 0), \dots, e_n = (0, \dots, 0, 1)$. We will denote by ℓ_n the self-adjoint part of ℓ_n^∞ . Let M_n be the C^*-algebra of $n \times n$ complex matrices and let H_n be its self-adjoint part consisting of (complex) hermitian matrices. Let $B(H)$ be the full operator algebra on a (complex) Hilbert space H . For any projections $p_1, \dots, p_n$ in $B(H)$, we say that they are weakly independent if their ranges $p_1(H), \dots, p_n(H)$ are weakly independent subspaces of H as defined in [2; p.165], this is equivalent to saying that for any $t_1, \dots, t_n \in B(H)$, $\sum_{j=1}^{n} p_j t_j p_j = 0$ implies $p_j t_j p_j = 0$ for $j = 1, \dots, n$ (see [11; p.102]). We note that if p is a minimal projection in $B(H)$, then for any $t \in B(H)$, $ptp = \lambda p$ for some complex number λ . Therefore, if $p_1, \dots, p_n$ are minimal projections, then they are weakly independent if and only if they are linearly independent. It has been shown in [2, 4, 11] that a map ϕ in $\underline{S}(\ell_n^\infty, M_m)$ is an extreme point if and only if the range projections ran $\phi(e_1), \dots,$ ran $\phi(e_n)$ are weakly independent in M_m .

PROPOSITION 7. *Let* B *be a JB-algebra. Then the extreme points of* $S(\ell_2, B)$ *are precisely the unital Jordan homomorphisms from* ℓ_2 *to* B .

Proof. Let ϕ be an extreme point of $S(\ell_2,B)$. To show that ϕ is a homomorphism, it suffices to show that $\phi(e_1)$ and $\phi(e_2)$ are projections in B since $1 = \phi(1) = \phi(e_1) + \phi(e_2)$. Equivalently, we show that $\phi(e_1)$ and $\phi(e_2)$ are extreme points of the positive unit ball of B . Suppose $\phi(e_1) = \frac{1}{2}b + \frac{1}{2}c$ with $0 \le b, c \le 1$ in B . Define two linear maps $\psi, \rho : \ell_2 \to B$ by $\psi(e_1) = b$, $\psi(e_2) = 1-b$; $\rho(e_1) = c$ and

https://doi.org/10.1017/S0004972700013332 Published online by Cambridge University Press

$\rho(e_2) = 1 - c$. Then clearly $\psi, \rho \in S(\ell_2, B)$ and $\phi = \frac{1}{2}(\psi + \rho)$. By extremality of ϕ , we have $\phi = \psi = \rho$ which gives $b = \psi(e_1) = \rho(e_1) = c$. This proves that $\phi(e_1)$ is an extreme point in the positive unit ball of B , so $\phi(e_1)$ is a projection. Likewise $\phi(e_2)$ is also a projection.

The above result is false for ℓ_n with $n \geq 3$.

Example 1. Define a unital positive (complex) linear map $\phi : \ell_3^\infty \to M_2$ by

$$\phi(e_1) = \begin{bmatrix} \frac{4}{9} & -\frac{2}{9} \\ -\frac{2}{9} & \frac{1}{9} \end{bmatrix}, \qquad \phi(e_2) = \begin{bmatrix} \frac{1}{9} & -\frac{2}{9} \\ -\frac{2}{9} & \frac{4}{9} \end{bmatrix},$$

$$\phi(e_3) = \begin{bmatrix} \frac{4}{9} & \frac{4}{9} \\ \frac{4}{9} & \frac{4}{9} \end{bmatrix}.$$

As $\phi(e_1), \phi(e_2), \phi(e_3)$ are linearly independent and also each $\phi(e_j)$ is a scalar multiple of a minimal projection in M_2 , it follows that the range projections ran $\phi(e_1)$, ran $\phi(e_2)$, ran $\phi(e_3)$ are linearly independent minimal projections which are therefore weakly independent. So ϕ is an extreme point of $\underline{S}(\ell_3^\infty, M_2)$ by the previous remark. But clearly ϕ is not a Jordan homomorphism.

We now consider $S(\ell_n, B)$ for $n \geq 3$. We note that a map $\phi \in S(\ell_n, B)$ is a Jordan homomorphism if and only if $\phi(e_j)$ is a projection in B for $j=1, \ldots, n$. If ϕ is an extreme point of $S(\ell_n, B)$, the following result shows when $\phi(e_j)$ is 'almost' a projection.

https://doi.org/10.1017/S0004972700013332 Published online by Cambridge University Press

PROPOSITION 8. *Let* ϕ *be an extreme point of* $S(\ell_n, B)$. *Then* $\phi(e_j)^2 \in \phi(\ell_n)$ *if and only if* $\phi(e_j)$ *is a scalar multiple of a projection in* B.

Proof. It suffices to prove the necessity. Suppose $\phi(e_j)^2 \in \phi(\ell_n)$, then $\phi(e_j)^2 = \sum_{k=1}^{n} \lambda_k \phi(e_k)$ where $\lambda_k \in \mathbb{R}$. Without loss of generality we may assume $j = 1$. Define $\psi : \ell_n \to B$ by

$$\psi(a) = a_1(\lambda_1 1_B - \phi(e_1)) \circ \phi(e_1) + \sum_{k=2}^{n} a_k \lambda_k \phi(e_k)$$

where $a = (a_1, \ldots, a_n) \in \ell_n$. Then $\psi(1) = 0$ and we have $-\mu\phi \leq \psi \leq \mu\phi$ where $\mu = \max\{||\lambda_1 - \phi(e_1)||, |\lambda_2|, \ldots, |\lambda_n|\}$. Choose $t > 0$ such that $t\mu \leq 1$. Let $\phi_1 = \phi - t\psi$ and $\phi_2 = \phi + t\psi$. Then we have ϕ_1, $\phi_2 \in S(\ell_n, B)$ and also $\phi = \frac{1}{2}\phi_1 + \frac{1}{2}\phi_2$. As ϕ is an extreme map, we have $\phi = \phi_1$ which gives $(\lambda_1 - \phi(e_1)) \circ \phi(e_1) = 0$, that is, $\phi(e_1)^2 = \lambda_1 \phi(e_1)$. So $\phi(e_1)$ is a scalar multiple of a projection.

From the above result we see that if ϕ is an extreme map in $S(\ell_n, B)$ and if each $\phi(e_j)^2$ is in $\phi(\ell_n)$ with $||\phi(e_j)|| = 1$ (or 0), then ϕ is a homomorphism. One might conjecture that if an extreme map $\phi : \ell_n \to B$ is such that $\phi(\ell_n)$ is a Jordan algebra, then ϕ is a homomorphism. This is false as the following example shows.

<u>Example 2</u>. Define a positive linear map $\phi : \ell_4 \to H_2$ by

$$\phi(e_1) = \frac{1}{5+4\sqrt{2}} \begin{bmatrix} 4 & -2 \\ -2 & 1 \end{bmatrix}, \quad \phi(e_2) = \frac{1}{5+4\sqrt{2}} \begin{bmatrix} 1 & -2 \\ -2 & 4 \end{bmatrix},$$

$$\phi(e_3) = \frac{2}{5+4\sqrt{2}} \begin{bmatrix} \sqrt{2} & 1+i \\ 1-i & \sqrt{2} \end{bmatrix}, \quad \phi(e_4) = \frac{2}{5+4\sqrt{2}} \begin{bmatrix} \sqrt{2} & 1-i \\ 1-i & \sqrt{2} \end{bmatrix}.$$

Then each $\phi(e_j)$ is a scalar multiple of a minimal projection and as in Example 1, ϕ is an extreme point of $S(\ell_4, H_2)$. Moreover $\phi(\ell_4) = H_2$ is a Jordan algebra but ϕ is not a Jordan homomorphism.

Actually, if A is a 'nontrivial' JB-algebra, then there is always an extreme map $\phi : A \to H_2$ which is not a homomorphism. We have the following result.

THEOREM 9. *Let A be a JB-algebra. The following conditions are equivalent:*

(i) For any JB-algebra B, every extreme point of $S(A,B)$ is a Jordan homomorphism;

(ii) Every extreme point of $S(A, H_2)$ is a Jordan homomorphism;

(iii) $\dim A \le 2$, that is, $A = \mathbb{R}$ or ℓ_2.

Proof. (ii) $\Rightarrow$ (iii). Let f be a pure state of A. Then, as remarked before, the map $\phi_f : a \to f(a)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is an extreme map in $S(A, H_2)$ and is therefore a Jordan homomorphism. It follows that f is a homomorphism on A. Thus, by Lemma 3, A is associative and we may assume that A is the self-adjoint part of the C^*-algebra $C(X)$ of continuous functions on a compact Hausdorff space X. If $\dim A \ge 3$, then X contains three distinct points x, y, z say. Define a (complex) linear map $\phi : C(X) \to M_2$ by

$$\phi(a) = a(x)\begin{bmatrix} \frac{4}{9} & -\frac{2}{9} \\ -\frac{2}{9} & \frac{1}{9} \end{bmatrix} + a(y)\begin{bmatrix} \frac{1}{9} & -\frac{2}{9} \\ -\frac{2}{9} & \frac{4}{9} \end{bmatrix} + a(z)\begin{bmatrix} \frac{4}{9} & \frac{4}{9} \\ \frac{4}{9} & \frac{4}{9} \end{bmatrix}$$

for $a \in C(X)$. Then ϕ is an extreme point of $\underline{S}(C(X), M_2)$ by weak independence as in Example 1 and by Arveson's theorem in [2; 1. 4. 10]. Now the restriction of ϕ to the self-adjoint part A of $C(X)$ is an extreme point of $S(A,H_2)$ and it is not a homomorphism. So $\dim A \le 2$.

(iii) ⇒ (i). If $\dim A \le 2$, then A is associative and so $A = \mathbb{R}$ or ℓ_2. If $A = \mathbb{R}$, then $S(A,B)$ is a singleton $\{\Phi\}$ and Φ is a homomorphism. If $A = \ell_2$, then Proposition 7 concludes the proof.

Remark. We see in the above proof that if every extreme point of $S(A,B)$ is a homomorphism, then A is associative. Therefore, if B is associative, then every extreme point of $S(A,B)$ is a homomorphism if and only if A is associative.

To prove our next theorem, we will use the following lemma of which the proof is routine and is omitted.

LEMMA 10. *Let A, B and C be JB-algebras. Suppose ϕ is an extreme map in $S(A,B)$ and ψ an extreme map in $S(A,C)$. Let $\Phi : A \to B \oplus C$ be defined by $\Phi(a) = \phi(a) \oplus \psi(a)$ for $a \in A$. Then Φ is an extreme map in $S(A, B \oplus C)$.*

We recall that a type I factor is isomorphic to the full operator algebra $B(H)$ on some Hilbert space H . Let $B(H)_{sa}$ be the self-adjoint part of $B(H)$. We will consider JB-algebras with a direct summand of $B(H)_{sa}$. We note that an atomic von Neumann algebra is a direct sum of type I factors and a finite-dimensional C^*-algebra is a finite direct sum of matrix algebras. Moreover, if a von Neumann algebra has a pure normal state, then it contains a direct summand of a type I factor (see [8; 7. 5. 13]) .

THEOREM 11. *Let A and B be JB-algebras. Suppose B contains a direct summand of $B(H)_{sa}$ with $\dim H \ge 2$. Then every extreme point of $S(A,B)$ is a Jordan homomorphism if and only if $\dim A \le 2$.*

Proof. The sufficiency follows from Theorem 9. We prove the necessity. So suppose every extreme point in $S(A,B)$ is a homomorphism. By the Remark following Theorem 9, A is associative and we may assume it is the self-adjoint part of some $C(X)$. We need to show $\dim A \le 2$. If $\dim A \ge 3$, then X contains three distinct points x, y and z say. We deduce a contradiction. By assumption, B contains a direct summand of $B(H)_{sa}$ with $\dim H \ge 2$. So we have $B = B(H)_{sa} \oplus C$ for some JB-algebra

C . We first show that there is an extreme point ϕ in $S(A,B(H)_{sa})$ which is not a homomorphism. If $\dim H = 2$, such an extreme map exists by Theorem 9. Suppose $\dim H \geq 3$. Define the following positive operators in $B(H)$:

$$T_1 = \left[\begin{array}{cc|c} \frac{4}{9} & -\frac{2}{9} & \\ -\frac{2}{9} & \frac{1}{9} & 0 \\ \hline \multicolumn{2}{c|}{0} & I \end{array}\right], \quad T_2 = \left[\begin{array}{cc|c} \frac{1}{9} & -\frac{2}{9} & \\ -\frac{2}{9} & \frac{4}{9} & 0 \\ \hline \multicolumn{2}{c|}{0} & 0 \end{array}\right],$$

$$T_3 = \left[\begin{array}{cc|c} \frac{4}{9} & \frac{4}{9} & \\ \frac{4}{9} & \frac{4}{9} & 0 \\ \hline \multicolumn{2}{c|}{0} & 0 \end{array}\right],$$

where I is the identity operator on a subspace of H . Then $T_1 + T_2 + T_3$ is the identity in $B(H)$ and using Example 1, it can be verified that the range projections $\operatorname{ran} T_1$, $\operatorname{ran} T_2$, $\operatorname{ran} T_3$ are weakly independent in $B(H)$. Therefore the linear map $\Phi : C(X) \to B(H)$ defined by

$$\Phi(a) = a(x)T_1 + a(y)T_2 + a(z)T_3 \qquad (a \in C(X))$$

is an extreme point of $\underline{S}(C(X), B(H))$ by Arveson's theorem and its proof in [2; 1. 4. 10]. Evidently Φ is not a Jordan homomorphism. Hence the restriction of Φ to the self-adjoint part A of $C(X)$ gives an extreme map ϕ in $S(A,B(H)_{sa})$ which is not a homomorphism. Now let ρ be any

https://doi.org/10.1017/S0004972700013332 Published online by Cambridge University Press

extreme point of $S(A,C)$. Then by Lemma 10, the map $\psi(.) = \phi(.) \oplus \rho(.)$ is an extreme point of $S(A,B(H)_{sa} \oplus C) = S(A,B)$ and also ψ is not a homomorphism. This is a contradiction. So $\dim A \leq 2$. The proof is complete.

Remark. We do not know if the above theorem is true for every non-associative JB-algebra B .

3. Simplexes

We recall that a (non-empty) convex set S in a vector space E is a simplex if for $x \in E$ and $\alpha > 0$, the intersection $S \cap (x + \alpha S)$ is either empty or of the form $y + \beta S$ for some $y \in E$ and $\beta \geq 0$. It is well-known that if S is a base of a cone K , then S is a linearly compact simplex if and only if K is a lattice (see [3; p.138]).

Trivially $S(\mathbb{R},B)$ is a simplex for any JB-algebra B since it reduces to a singleton. On the other hand, Lemma 3 shows that $S(A,\mathbb{R})$ is a simplex if and only if A is an associative JB-algebra.

THEOREM 12. *Let A and B be JB-algebras. The following conditions are equivalent:*

(i) $S(A,B)$ is a simplex;

(ii) Either $A = \mathbb{R}$ or A is associative with $B = \mathbb{R}$.

Proof. We only need to prove (i) $\Rightarrow$ (ii). We first show that A is associative. Let $K = \bigcup_{\lambda \geq 0} \lambda S(A,B)$ be the cone generated by $S(A,B)$. Then K is a lattice. We show that the dual cone A^*_+ of A_+ is a lattice. Let $f, g \in A^*_+$. Define $\phi_f, \phi_g : A \to B$ by $\phi_f(a) = f(a)1_B$ and $\phi_g(a) = g(a)1_B$ for $a \in A$. Then $\phi_f, \phi_g \in K$. So the supremum $\phi = \phi_f \vee \phi_g$ exists in K . Let h be a state of B . We show that $h \circ \phi$ is the lattice supremum of f and g . Indeed, for $a \in A_+$, we have $(h \circ \phi)(a) = h((\phi_f \vee \phi_g)(a)) \geq h(\phi_f(a)), h(\phi_g(a))$ where $h(\phi_f(a)) = f(a)$ and $h(\phi_g(a)) = g(a)$. So $h \circ \phi \geq f, g$. Let $k \in A^*_+$ be such that

https://doi.org/10.1017/S0004972700013332 Published online by Cambridge University Press

$k \geq f, g$. Let $\phi_k : A \to B$ be the map $\phi_k(\cdot) = k(\cdot)1_B$. Then we have $\phi_k \geq \phi_f, \phi_g$ which implies $\phi_k \geq \phi_f \vee \phi_g$. This in turn implies $k \geq h \circ \phi$. So the supremum $f \vee g$ exists in A^*_+ . Hence A^*_+ is a lattice and A is associative by Lemma 3.

We may now assume that A is the algebra of real continuous functions on some compact Hausdorff space X . Suppose $B \neq \mathbb{R}$. Then there exists $b \in B$ such that $0 \leq b \leq 1$ and b is not a scalar multiple of the identity 1. We show that $A = \mathbb{R}$. Suppose, for contradiction, that $A \neq \mathbb{R}$, then there are two distinct points x and y in X . The unit masses ε_x and ε_y are pure states of A and can be identified as extreme points of $S(A,B)$ as before. Define $\phi, \psi \in S(A,B)$ by

$$\phi(\cdot) = \varepsilon_x(\cdot)b + \varepsilon_y(\cdot)(1 - b)$$

$$\psi(\cdot) = \varepsilon_y(\cdot)b + \varepsilon_x(\cdot)(1 - b) .$$

The we have $\tfrac{1}{2}\phi + \tfrac{1}{2}\psi = \tfrac{1}{2}\varepsilon_x + \tfrac{1}{2}\varepsilon_y$.

Since $S(A,B)$ is a linearly compact simplex, $\{\varepsilon_x\}$ and $\{\varepsilon_y\}$ are split faces of $S(A,B)$ (see [3; 8. 1]) and so the convex hull $co\ \{\varepsilon_x, \varepsilon_y\}$ is a (split) face of $S(A,B)$. Now $\tfrac{1}{2}\phi + \tfrac{1}{2}\psi = \tfrac{1}{2}\,\varepsilon_x + \tfrac{1}{2}\,\varepsilon_y \in co\ \{\varepsilon_x, \varepsilon_y\}$ but $\phi \notin co\ \{\varepsilon_x, \varepsilon_y\}$ since $b \in \phi(A) \neq \mathbb{R}1_B$. This contradicts the fact that $co\ \{\varepsilon_x, \varepsilon_y\}$ is a face. Hence $A = \mathbb{R}$. The proof is complete.

Remark. The above arguments clearly extend to order-unit normed Banach spaces.

Thus, for example, $S(\ell_2, \ell_2)$ is not a simplex while every extreme point of $S(\ell_2, \ell_2)$ is a Jordan homomorphism.

References

[1] R.F. Arens and J.L. Kelly, "Characterizations of the space of continuous functions over a compact Hausdorff space", *Trans. Amer. Math. Soc.* 62 (1947) 499-508.

[2] W.B. Arveson, "Subalgebras of C*-algebras", *Acta Math.* 123 (1969) 141-224.

[3] L. Asimow and A.J. Ellis, *Convexity theory and its applications in functional analysis*, (Academic Press, 1980).

[4] M.D. Choi, *Positive linear maps on C*-algebras*, (Thesis, University of Toronto, 1972).

[5] C.M. Edwards, "Ideal theory in JB-algebras", *J. London Math. Soc.* 16 (1977) 507-513.

[6] A.J. Ellis, "Extreme positive operators", *Quart. J. Math.* 15 (1964) 342-344.

[7] H. Hanche-Olsen, "Split faces and ideal structure of operator algebras", *Math. Scand.* 48 (1981) 137-144.

[8] H. Hanche-Olsen and E. Størmer, *Jordan operator algebras*, (Pitman, 1984).

[9] A. Ionescu Tulcea and C. Ionescu Tulcea, "On the lifting property", *J. Math. Anal. Appl.* 3 (1961) 537-546.

[10] E. Størmer, "Positive linear maps of operator algebras", *Acta Math.* 110 (1963) 233-278.

[11] E. Størmer, *Positive linear maps of C*-algebras*, Lecture Notes in Physics Vol. 29 (Springer-Verlag, 1974).

[12] J.D.M. Wright and M.A. Youngson, "On isometrics of Jordan algebras", *J. London Math. Soc.* 17 (1978) 339-344.

Department of Mathematical Sciences
University of London Goldsmiths' College
London SE14
United Kingdom.