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Certain non-algebras in harmonic analysis

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Given $q_o \in [1, 2)$, functions $f = f_{q_o} \in \bigcap_{p>1} A^p$ and $g_q = g_{q,q_o} \in A^q$ are constructed such that $fg_q \notin A^q$ for every $q \in [1, q_o]$. In particular, if $p \in (1, 2)$, A^p is not an algebra.

1. Introduction and preliminaries

We consider functions on the circle group T and write

$$A^p = C \cap Fl^p , \quad 1 \le p < \infty ,$$

where C denotes the set of continuous functions on T and

$$Fl^p = \{f \in L^1(T) : \hat{f} \in l^p(Z)\} .$$

In private correspondence with the author, Professor Yitzhak Katznelson suggested in outline a proof that A^p is not an algebra when $1 < p < 2$, and has since formulated *existential* proofs of more general results. Meanwhile the author has concentrated on a more *constructive* approach, the details of which are set out below. The author would like to thank Professor Katznelson for suggesting the use of the polynomials D_n and P^*_ν introduced below, and the form of Lemma 1.1.

It is known that A^p is a Banach space under the norm

$$N_p : h \mapsto \|h\|_\infty + \|\hat{h}\|_p = \|h\|_\infty + M_p(h) .$$

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We define e_ν to be the function $e^{it} \mapsto e^{i\nu t}$ on T and note that, for $h \in A^p$,

(1.1) $$N_p(e_\nu h) = N_p(h) \ ; \ M_p(e_\nu h) = M_p(h) \ .$$

The spectrum of $h \in L^1(T)$ is defined by

$$\operatorname{sp}(h) = \{n \in Z : \hat{h}(n) \neq 0\} \ .$$

Throughout the following we assume that $p \in (1, 2)$, $q \in [1, 2)$. $a_1, a_2, \ldots$ will denote positive absolute constants. If φ and ψ are positive functions on $\{1, 2, \ldots\}$, we write $\varphi \sim \psi$ iff $0 < \inf \varphi^{-1}\psi \leq \sup \varphi^{-1}\psi < \infty$.

If D_n denotes the Dirichlet kernel of degree n,

(1.2) $$\|D_n\|_\infty = 2n + 1 \sim n \ ; \ M_p(D_n) = (2n+1)^{\frac{1}{p}} \sim n^{\frac{1}{p}} \ ; \ M_p\left(D_n^2\right) \sim n^{1+\frac{1}{p}} \ .$$

In [1], p. 33, the Rudin-Shapiro polynomials P_m $(m = 0, 1, 2, \ldots)$ are defined by

$$P_m = \sum_{n=0}^{2^m-1} \varepsilon_m(n) e_n \ ,$$

where the $\varepsilon_m(n) \in \{-1, 1\}$ are chosen in such a way that

$$|P_m| \leq 2^{\frac{m+1}{2}} \ ; \ M_p(P_m) = 2^{\frac{m}{p}} \ , \quad m = 0, 1, 2, \ldots \ .$$

We shall use the polynomials P_ν^* $(\nu = 1, 2, \ldots)$, where $P_\nu^* = P_m$ for the unique m such that $2^m \leq \nu < 2^{m+1}$. Then

(1.3) $$\|P_\nu^*\|_\infty \leq 2^{\frac{m+1}{2}} = 2^{\frac{1}{2}} 2^{\frac{m}{2}} \leq 2^{\frac{1}{2}} \nu^{\frac{1}{2}}$$

and

(1.4) $$\nu^{\frac{1}{p}} \geq M_p(P_\nu^*) = 2^{\frac{m}{p}} > \left(\frac{\nu}{2}\right)^{\frac{1}{p}} = 2^{-\frac{1}{p}} \nu^{\frac{1}{p}} \ .$$

The following lemma will be needed.

LEMMA 1.1 (Katznelson). *Let s be a positive integer, φ a trigonometric polynomial of degree less than s, and ψ any trigonometric polynomial. Write $\psi_{(2s)} : e^{it} \mapsto e^{i2st}$. Then*

$$M_p\left(\varphi\psi_{(2s)}\right) = M_p(\varphi)M_p(\psi) .$$

Proof. We can write

$$\theta = \varphi\psi_{(2s)} = \sum_{m\in Z} \hat{\psi}(m)e_{2sm}\varphi ,$$

a finite sum. Also, $e_{2sm}\varphi$ and $e_{2sm'}\varphi$ have disjoint spectra whenever $m \neq m'$. Thus, by (1.1),

$$\begin{aligned}\left(M_p(\theta)\right)^p &= \sum_{m\in Z} |\hat{\psi}(m)|^p\left[M_p\left(e_{2sm}\varphi\right)\right]^p \\ &= \sum_{m\in Z} |\hat{\psi}(m)|^p\left(M_p(\varphi)\right)^p \\ &= \left(M_p(\varphi)\right)^p\left(M_p(\psi)\right)^p .\end{aligned}$$

2. Construction

Let $f_{n,p} = D_n P^*_{m_1(2s_1)}$ and $g_{n,p,q} = D_n P^*_{m_2(2s_2)}$, where $m_1 = \left[n^{\frac{2(p-1)}{2-p}}\right]$, $m_2 = \left[n^{\frac{2(q-1)}{2-q}}\right]$ and the choice of s_1, s_2 will be specified below.

If $s_1 > n$, Lemma 1.1 and (1.2) - (1.4) show that

$$\text{(2.1)} \qquad \begin{aligned} N_p\left(f_{n,p}\right) &\le a_1\left(nm_1^{\frac{1}{2}} + n^{\frac{1}{p}}m_1^{\frac{1}{p}}\right) \\ &\le a_2 n^{\frac{1}{2-p}} .\end{aligned}$$

Similarly, if $s_2 > n$,

$$\text{(2.2)} \qquad N_q\left(g_{n,p,q}\right) \le a_3 n^{\frac{1}{2-q}} .$$

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Again, if $s_2 > 2n + 2s_1m_1$ and $s_1 > 2n$,

$$M_q(f_{n,p,q}) = M_q(D_n^2 P^*_{m_1}(2s_1) P^*_{m_2}(2s_2))$$
$$= M_q(D_n^2) M_q(P^*_{m_1}(2s_1)) M_q(P^*_{m_2}(2s_2)) .$$

Taking $s_1 = 4n$ and $s_2 = 4n + 16nm_1$, we have

(2.3) $$M_q(f_{n,p}g_{n,p,q}) \geq a_4 n^{\kappa} ,$$

where

$$\kappa = \frac{8q-4-2q^2+pq^2-5pq+4p}{q(2-p)(2-q)} .$$

From (2.1), (2.2) and (2.3),

(2.4) $$\frac{M_q(f_{n,p}g_{n,p,q})}{N_p(f_{n,p})N_q(g_{n,p,q})} \geq a_5 n^{\sigma} ,$$

where

$$\sigma = \frac{(p-1)(2-q)}{q(2-p)} > 0 .$$

We next estimate the size of $\mathrm{sp}(f_{n,p})$ and $\mathrm{sp}(g_{n,p,q})$:

$$\mathrm{sp}(f_{n,p}) \subseteq [-n,\ n+2s_1m_1] ,$$

$$\mathrm{sp}(g_{n,p,q}) \subseteq [-n,\ n+2s_2m_2] .$$

Thus, according to the choice of s_1 and s_2 specified above,

(2.6) $$\mathrm{sp}(f_{n,p}) \cup \mathrm{sp}(g_{n,p,q}) \subseteq \left[-n,\ n(1+8n^{\rho}+32n^{\tau})\right] ,$$

where

$$\rho = \frac{2(q-1)}{(2-q)} , \quad \tau = \frac{6(p+q)-4(pq+2)}{(2-q)(2-p)} .$$

Define

$$f_n^{\circ} = \frac{n^{-2}f_{k(n),p(n)}}{N_{p(n)}(f_{k(n),p(n)})} ,$$

$$g^{\circ}_{n,q} = \frac{\beta(n,q)g_{k(n),p(n),q}}{N_q\left(g_{k(n),p(n),q}\right)} ,$$

where

$$k(n) = 2^{n^2} , \quad p(n) = 1 + \frac{1}{2n} , \quad \beta(n, q) = 2^{-\frac{1}{2}\left(\frac{2-q}{2q}\right)n} .$$

Then (2.4) gives

(2.6) $$M_q\left(f^{\circ}_n g^{\circ}_{n,q}\right) \geq a_5 n^{-2} 2^{\psi} , \text{ where } \psi = \frac{(2-q)(2n+1)}{4q\left(2-\frac{1}{n}\right)}$$

$$\to \infty \quad \text{as} \quad n \to \infty$$

Since N_p is a decreasing function of p , the formula

$$N_p\left(f^{\circ}_n\right) = \frac{n^{-2}N_p\left(f_{k(n),p(n)}\right)}{N_{p(n)}\left(f_{k(n),p(n)}\right)}$$

shows that to any $p > 1$ corresponds $n_{\circ}(p)$ such that

(2.7) $$N_p\left(f^{\circ}_n\right) \leq n^{-2} \quad \text{for} \quad n \geq n_{\circ}(p) .$$

Also,

(2.8) $$N_q\left(g^{\circ}_{n,q}\right) = \beta(n, q) .$$

Finally, let $f_n = e_{\nu_n} f^{\circ}_n$, $g_{n,q} = e_{\nu_n} g^{\circ}_{n,q}$, where the integers $\nu_n = \nu_{n,q_{\circ}}$ will be chosen appropriately, and consider

(2.9) $$f = f_{q_{\circ}} = \sum_{n=1}^{\infty} f_n , \quad g_q = g_{q,q_{\circ}} = \sum_{n=1}^{\infty} g_{n,q} .$$

By (1.1) and (2.7),

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$$\begin{aligned} N_p(f) &\le \sum_{n=1}^{\infty} N_p(f_n) \\ &= \sum_{n=1}^{\infty} N_p\left(f_n^{\circ}\right) \\ &\le \sum_{n=1}^{n_{\circ}(p)} N_p\left(f_n^{\circ}\right) + \sum_{n>n_{\circ}(p)} n^{-2} \\ &< \infty \quad \text{for all } p > 1 , \end{aligned}$$

and so $f \in \bigcap_{p>1} A^p$. Similarly, by (1.1) and (2.8),

$$\begin{aligned} N_q(g_q) &\le \sum_{n=1}^{\infty} N_q(g_{n,q}) \\ &= \sum_{n=1}^{\infty} N_q\left(g_{n,q}^{\circ}\right) \\ &\le \sum_{n=1}^{\infty} \beta(n, q) \\ &< \infty , \end{aligned}$$

and so $g_q \in A^q$ for $1 \le q < 2$.

By (2.9),

$$\text{(2.10)} \qquad \begin{aligned} fg_q &= \sum_{r,s} f_r g_{s,q} \\ &= f_m g_{m,q} + \sum_{(r,s)\ne(m,m)} f_r g_{s,q} . \end{aligned}$$

Let

$$\text{(2.11)} \qquad \begin{aligned} F_{n,q_{\circ}} &= \cup\left\{\mathrm{sp}\left(f_n^{\circ}\right) \cup \mathrm{sp}\left(g_{n,q}^{\circ}\right) : q \le q_{\circ}\right\} \\ &= \cup\left\{\mathrm{sp}\left(f_{k(n),p(n)}\right) \cup \mathrm{sp}\left(g_{k(n),p(n),q}\right) : q \le q_{\circ}\right\} . \end{aligned}$$

Then, for $q \le q_{\circ}$,

$$\mathrm{sp}(f_n) \cup \mathrm{sp}(g_{n,q}) \subseteq \nu_{n,q_{\circ}} + F_{n,q_{\circ}} ,$$

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and

$$\operatorname{sp}\left(f_r g_{s,q}\right) \subseteq \nu_{r,q_0} + F_{r,q_0} + \nu_{s,q_0} + F_{s,q_0} .$$

Moreover, (2.5) and (2.11) show that F_{n,q_0} is finite. Supposing the ν_{n,q_0} to be chosen to satisfy

$$(2.12) \quad \begin{cases} \left(\nu_{m,q_0}+F_{m,q_0}+\nu_{m,q_0}+F_{m,q_0}\right) \cap \left(\nu_{r,q_0}+F_{r,q_0}+\nu_{s,q_0}+F_{s,q_0}\right) = \emptyset \\ \qquad\qquad \text{whenever } (r, s) \neq (m, m) , \end{cases}$$

then, for every $q \le q_0$, $\left(fg_q\right)^\wedge$ and $\left(f_m g_{m,q}\right)^\wedge$ will agree on the support of the latter, and (2.10) will show that

$$\left[M_q\left(fg_q\right)\right]^q \ge \left[M_q\left(f_m g_{m,q}\right)\right]^q = \left[M_q\left(f_m^\circ g_{m,q}^\circ\right)\right]^q ,$$

the last step by (1.1). Hence, by (2.6),

$$M_q\left(fg_q\right) = \infty \quad \text{for} \quad 1 \le q \le q_0 .$$

Reverting to (2.12) it is simple to check that (omitting explicit reference to q_0) it suffices to choose $\nu_1 \in Z$ freely, and to make a choice by recurrence to satisfy

$$\nu_{n+1} \in Z \backslash F_n'\left(\nu_1, \ldots, \nu_n; q_0\right) , \quad 2\nu_{n+1} \in Z \backslash F_n''\left(\nu_1, \ldots, \nu_n; q_0\right) ,$$

where

$$F_n'\left(u_1, \ldots, u_n; q_0\right) = \bigcup_{i \le n} \left(u_i+F_{i,q_0}+F_{n+1,q_0}-F_{n+1,q_0}-F_{n+1,q_0}\right) \cup \bigcup_{\substack{i \le n \\ j \le n}} \left(2u_i-u_j+F_{n+1,q_0}+F_{n+1,q_0}-F_{i,q_0}-F_{j,q_0}\right) ,$$

$$F_n''\left(u_1, \ldots, u_n; q_0\right) = \bigcup_{i \le j \le n} \left(u_i+u_j+F_{i,q_0}+F_{j,q_0}-F_{n+1,q_0}-F_{n+1,q_0}\right) ,$$

for every $n \in N$ and every $\left(u_1, \ldots, u_n\right) \in Z^n$.

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REMARK. The preceding simple construction encounters difficulties if one tries to handle all $q < 2$ in one move. This is because the sets $F_n = \cup\left\{\mathrm{sp}\left(f_n^{\circ}\right) \cup \mathrm{sp}\left(g_{n,q}^{\circ}\right) : q < 2\right\}$ are infinite and it is no longer clear that integers ν_n can be chosen so that the analogue of (2.12) is satisfied. On the other hand, in one of the stronger existential results mentioned in §1, Professor Katznelson indicates that the *existence* of $f \in \bigcap_{p>1} A^p$ and $g_p \in Fl^p$ $(1 \le q < 2)$ satisfying $fg_q \notin Fl^q$ $(1 \le q < 2)$ follows on combining (2.4) with convexity and category arguments.

Reference

[1] Yitzhak Katznelson, *An introduction to harmonic analysis* (John Wiley, New York, London, Sydney, Toronto, 1968).

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