SIMPLE ALGEBRAS THAT GENERALIZE THE JORDAN ALGEBRA M_3^8

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In this paper we discuss a generalization of the split exceptional Jordan algebra $M_3^8(\mathfrak{C})$ of the 3×3 hermitian matrices with elements in the split Cayley–Dickson algebra $\mathfrak{C}$ **(1)**. The generalization consists of replacing $\mathfrak{C}$ by the non-commutative Jordan algebra $\mathfrak{A} \equiv \mathfrak{A}(A, f, s, t)$ discussed in **(2; 3)** and forming the set of 3×3 hermitian matrices $M_3^m(\mathfrak{A}) \equiv M$ with elements in the m-dimensional algebra $\mathfrak{A}$. With the usual definition of multiplication $X \cdot Y = \frac{1}{2}(XY + YX)$, M becomes a commutative algebra and we have the following theorem, which shows how the structure of M is reflected by that of $\mathfrak{A}$.

THEOREM. *Let $\mathfrak{A}$ and M be as above, then:*

(1) *M is simple if and only if $\mathfrak{A}$ is simple;*

(2) *if $\mathfrak{A}$ is simple, then every element of M satisfies a generic minimum polynomial of degree three or M is power associative if and only if M is Jordan;*

(3) *the bilinear form $(X, Y) =$ trace $R(X \cdot Y)$ is an invariant form, which is non-degenerate if and only if $\mathfrak{A}$ is simple.*

In §1 we develop further relations for the algebra $\mathfrak{A}$, which are used in §2 to prove the simplicity of $M = M_3^m(\mathfrak{A})$. Now noting that if M is Jordan, then it is a power associative "cubic" algebra, we prove in §3 the converse statement given above in (2) by essentially showing that $M_3^m(\mathfrak{A}) \subset M_3^8(\mathfrak{C})$. Finally in §4 we prove statement (3) concerning the bilinear form (X, Y). We shall assume that the base field F is of characteristic zero since we want to consider trace; but it should be clear when this condition can be relaxed.

1. Some identities for $\mathfrak{A}(\mathfrak{A}, f, s, t)$. In this section we discuss briefly the properties of the algebra $\mathfrak{A} = \mathfrak{A}(A, f, s, t)$ necessary for this paper. The non-commutative Jordan algebras in **(2; 3)** are constructed as follows. Let $A \neq 0$ be an anti-commutative algebra with an invariant form $f(\alpha, \beta)$ (i.e. $f(\alpha\beta, \gamma) = f(\alpha, \beta\gamma)$), and let $\mathfrak{A} = \mathfrak{A}(A, f, s, t)$ denote the set of matrices

$$\begin{bmatrix} a & \alpha \\ \beta & b \end{bmatrix},$$

where $\alpha, \beta \in A$ and $a, b \in F$. For these matrices define the usual vector space operations co-ordinate-wise and define multiplication of two such matrices by

$$\begin{bmatrix} a & \alpha \\ \beta & b \end{bmatrix} \begin{bmatrix} c & \gamma \\ \delta & d \end{bmatrix} = \begin{bmatrix} ac + f(\alpha, \delta) & a\gamma + d\alpha + t\beta\delta \\ c\beta + b\delta + s\alpha\gamma & bd + f(\beta, \gamma) \end{bmatrix},$$

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where $f(\alpha, \beta)$ is the invariant form on A and $s, t \in F$. Thus letting $(x, y, z) = (xy)z - x(yz)$ denote the associator function, $\mathfrak{A}$ becomes an algebra with the following properties **(2)**:

(i) $(x, y, x) = 0$ for all $x, y \in \mathfrak{A}$, and $x^2 - (a + b)x + [ab - f(\alpha, \beta)]1 = 0$ for all

$$x = \begin{bmatrix} a & \alpha \\ \beta & b \end{bmatrix} \in \mathfrak{A},$$

that is, $\mathfrak{A}$ is a flexible quadratic algebra with identity element 1. Thus $(x^2, y, x) = 0$, so that $\mathfrak{A}$ is a non-commutative Jordan algebra.

(ii) $\mathfrak{A}$ is simple if and only if $f(\alpha, \beta)$ is non-degenerate on A; a proof of an analogous statement may be found in **(3)**.

Next we derive some new relations for $\mathfrak{A}$, which are similar to conjugation in the split Cayley–Dickson algebra. For

$$x = \begin{bmatrix} a & \alpha \\ \beta & b \end{bmatrix}, \qquad y = \begin{bmatrix} c & \gamma \\ \delta & d \end{bmatrix} \in \mathfrak{A},$$

define

$$\bar{x} = \begin{bmatrix} b & -\alpha \\ -\beta & a \end{bmatrix},$$

then a straightforward computation shows that $x \to \bar{x}$ is linear and

(1) $$x\bar{x} = \bar{x}x = n(x), \qquad \text{where } n(x) = (ab - f(\alpha, \beta))1,$$

(2) $$\overline{xy} = \bar{y}\,\bar{x},$$

so that $x \to \bar{x}$ is an involution. Next define the bilinear form on $\mathfrak{A}$,

$$n(x, y) = \tfrac{1}{2}[n(x + y) - n(x) - n(y)],\ x, y \in \mathfrak{A}.$$

Then

(3) $$n(x, y) = \tfrac{1}{2}(x\bar{y} + y\bar{x}) = \tfrac{1}{2}(\bar{x}y + \bar{y}x)$$

and $n(x, y)$ is non-degenerate if and only if $f(\alpha, \beta)$ is non-degenerate on A (which is equivalent to $\mathfrak{A}$ being simple **(2)**). For, using (1),

(4) $$\begin{aligned} n(x, y) &= \tfrac{1}{2}[(a + c)(b + d) - f(\alpha + \gamma, \beta + \delta) \\ &\qquad - (ab - f(\alpha, \beta)) - (cd - f(\gamma, \delta))]1 \\ &= \tfrac{1}{2}[cb + ad - f(\alpha, \delta) - f(\gamma, \beta)]1 \\ &= \tfrac{1}{2}(x\bar{y} + y\bar{x}), \end{aligned}$$

and from the second equation we see that if $n(x, y)$ is non-degenerate, so is $f(\alpha, \beta)$. Conversely, suppose $f(\alpha, \beta)$ is non-degenerate and $n(x, y) = 0$ for all $y \in \mathfrak{A}$. Then using the above equations with $c = 1$, $d = \delta = \gamma = 0$, we have $b = 0$; similarly $a = 0$. Choosing $\gamma = 0$ and δ arbitrary yields $\alpha = 0$; similarly $\beta = 0$ so that $x = 0$ and therefore $n(x, y)$ is non-degenerate. Next we have

(5) $$n(x\bar{y}, z) = n(\bar{z}x, y).$$

For, letting (x, y, z) denote the association function, we have, using (3),

$$2n(x\bar{y}, z) - 2n(\bar{z}x, y) = (y, \bar{x}, z) + \overline{(y, \bar{x}, z)} = 0,$$

since for any $x, y, z \in \mathfrak{A}$ we have

$$\begin{aligned} (x, y, z) &= -(z, y, x), && \text{since } \mathfrak{A} \text{ is flexible,} \\ &= (z, y, \bar{x}), && \text{since } x + \bar{x} = (a+b)1 \in 1F, \\ &= (\bar{z}, \bar{y}, \bar{x}) \\ &= \overline{(\bar{z}\bar{y})\bar{x} - \bar{z}(\bar{y}\bar{x})} \\ &= \overline{x(y, z) - (xy)z} \\ &= -\overline{(x, y, z)}. \end{aligned}$$

We shall need the following lemma.

LEMMA. *Let $A \neq 0$ be an anti-commutative algebra with a non-degenerate invariant form $f(\alpha, \beta)$ such that for all $\alpha, \beta, \gamma \in A$*

$$st\beta(\alpha\gamma) = f(\alpha, \beta)\gamma - f(\beta, \alpha)\alpha.$$

Then $\mathfrak{A} = \mathfrak{A}(A, f, s, t)$ is a split Cayley–Dickson algebra $\mathfrak{C}$ or a "split" quaternion associative algebra $\mathfrak{Q}$. In either case $M_3{}^8(\mathfrak{C})$ and $M_3{}^4(\mathfrak{Q})$ are Jordan algebras and if F is algebraically closed, we may consider $\mathfrak{C} \supset \mathfrak{Q}$ and therefore $M_3{}^8(\mathfrak{C}) \supset M_3{}^4(\mathfrak{Q})$ as Jordan algebras.

Proof. Since $\mathfrak{A}$ is flexible, we first show that $x^2y = x(xy)$ so that $\mathfrak{A}$ is alternative. Thus for $x, y \in \mathfrak{A}$ as in the first part of this section we have

$$x^2 = \begin{bmatrix} a^2 + f(\alpha, \beta) & (a+b)\alpha \\ (a+b)\beta & b^2 + f(\alpha, \beta) \end{bmatrix}$$

and

$$x^2y = \begin{bmatrix} c(a^2 + f(\alpha, \beta)) + (a+b)f(\alpha, \delta) & [a^2 + f(\alpha, \beta)]\gamma + (a+b)d\alpha \\ & \quad + s(a+b)\beta\delta \\ c(a+b)\beta + [b^2 + f(\alpha, \beta)]\delta & d(b^2 + f(\alpha, \beta)) + (a+b)f(\beta, \gamma) \\ \quad + t(a+b)\alpha\gamma & \end{bmatrix}$$

Also

$$x(xy) = \begin{bmatrix} a(ac + f(\alpha, \delta) & a(a\gamma + d\alpha + s\beta\delta) \\ \quad + f(\alpha, c\beta + b\delta + t\alpha\gamma) & \quad + (bd + f(\beta, \gamma))\alpha \\ & \quad + s\beta(c\beta + b\delta + t\alpha\gamma) \\ (ac + f(\alpha, \delta))\beta + b(c\beta + b\delta + t\alpha\gamma) & b(bd + f(\beta, \gamma)) \\ \quad + t\alpha(a\gamma + d\alpha + s\beta\delta) & \quad + f(\beta, a\gamma + d\alpha + s\beta\delta) \end{bmatrix}$$

and using the hypothesis we obtain the desired equality.

Now since $f(\alpha, \beta)$ is non-degenerate, $\mathfrak{A}$ is simple and therefore is the split Cayley–Dickson algebra $\mathfrak{C}$, or an associative algebra. In the latter case we let

$$z = \begin{bmatrix} e & \lambda \\ \mu & f \end{bmatrix} \in \mathfrak{A}$$

and compute the 2×2 matrix $(x, y, z) = 0$ in $\mathfrak{A}$. From the (1, 1) position in this matrix we obtain $tf(\beta\delta, \mu) - sf(\gamma\lambda, \alpha) = 0$. If $st \neq 0$, then since the elements in A in this expression are arbitrary, we have by choosing $\alpha = 0$ (or $\mu = 0$) that $f(\beta\delta, \mu) = 0$ (or $f(\gamma\lambda, \alpha) = 0$), which implies that $A^2 = 0$ by the non-degeneracy of $f(\alpha, \beta)$. But by hypothesis this yields $f(\alpha, \beta)\gamma = f(\beta, \gamma)\alpha$, and consequently the dimension of A is one; the same result holds if $st = 0$. Thus for $f(\beta, \beta) = b \neq 0$ we have $A = \beta F$, and $\mathfrak{Q} = \mathfrak{A}(\beta F, f, s, t)$ is associative. In both of these cases $M_3{}^8(\mathfrak{C})$ and $M_3{}^4(\mathfrak{Q})$ are Jordan algebras.

Next for $A = \beta F$ and F algebraically closed, we can find $\alpha \in A$ such that $f(\alpha, \alpha) = 1$; and consequently the map

$$\begin{bmatrix} a_{11} & a_{12}\,\alpha \\ a_{21}\,\alpha & a_{22} \end{bmatrix} \rightarrow \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

is an isomorphism of $\mathfrak{Q}$ onto the 2×2 matrix algebra over F, which may be regarded as the "split" quaternion algebra (**4**, pp. 135 and 162). Now we may regard $\mathfrak{C} \supset \mathfrak{Q}$ as follows. Since f is non-degenerate and symmetric, there exists $\alpha \in A$ (where $\mathfrak{C} = \mathfrak{A}(A, f, s, t)$) with $f(\alpha, \alpha) \neq 0$; assume that $f(\alpha, \alpha) = 1$. With this $\alpha \in A$, we see that $\mathfrak{Q}$ is isomorphic to $\mathfrak{A}(\alpha F, f, s, t)$ and therefore consider that $\mathfrak{C} \supset \mathfrak{Q}$ by this isomorphism; consequently $M_3{}^8(\mathfrak{C}) \supset M_3{}^4(\mathfrak{Q})$.

2. Simplicity of M. Let

$$X = \begin{bmatrix} \alpha_1 & a_3 & \bar{a}_2 \\ \bar{a}_3 & \alpha_2 & a_1 \\ a_2 & \bar{a}_1 & \alpha_3 \end{bmatrix}, \qquad Y = \begin{bmatrix} \beta_1 & b_3 & \bar{b}_2 \\ \bar{b}_3 & \beta_2 & b_1 \\ b_2 & \bar{b}_1 & \beta_3 \end{bmatrix} \tag{6}$$

be 3×3 hermitian matrices in M, where $a_i, b_i \in \mathfrak{A}$, $\alpha_i, \beta_i \in F$, and where $x \rightarrow \bar{x}$ is the involution in A defined in §1. The commutative multiplication in M is given by $X \cdot Y = \frac{1}{2}(XY + YX)$ and the resulting 3×3 matrix is formally the same as obtained in $M_3{}^8(\mathfrak{C})$.

Next let $\{e_{ij}\}$ denote the usual matrix basis for the 3×3 matrices over F. Then $e_i = e_{ii}$, $i = 1, 2, 3$, are orthogonal idempotents in M. For $a \in \mathfrak{A}$ define for $i \neq j$

$$a_{ij} = (a)_{ij} = ae_{ij} + \bar{a}e_{ji}.$$

Then $\bar{a}_{ji} = a_{ij}$ and setting

$$M_{ij} = \{a_{ij} : a \in \mathfrak{A}\}$$

we have the Peirce decomposition relative to the e_i given by

$$M = e_1 F \oplus e_2 F \oplus e_3 F \oplus M_{12} \oplus M_{13} \oplus M_{23}.$$

From this we see that if the dimension of A is n (so that the dimension of $\mathfrak{A}$ is $m = 2(n + 1)$), then the dimension of M is

$$3 + 3[2(n + 1)] = 3[2(n + 1) + 1].$$

For $a, b \in \mathfrak{A}$ the multiplication of the basis elements of M is given by

$$\begin{aligned} e_i \cdot e_j &= \delta_{ij} e_i, \\ e_i \cdot a_{ij} &= \tfrac{1}{2} a_{ij} = a_{ij} \cdot e_j, \\ e_k \cdot a_{ij} &= 0, \qquad k \neq i,\ k \neq j, \\ a_{ij} \cdot b_{ij} &= n(a, b)(e_i + e_j), \\ 2a_{ij} \cdot b_{jk} &= (ab)_{ik}, \qquad i \neq j \neq k \neq i. \end{aligned}$$

Next we consider the simplicity of M. Assume that $f(\alpha, \beta)$ is non-degenerate and therefore $n(a, b)$ is non-degenerate on M. Suppose B is a non-zero ideal of M containing the non-zero element

$$X = \alpha_1 e_1 + \alpha_2 e_2 + \alpha_3 e_3 + a_{12} + \bar{b}_{13} + c_{23}.$$

Now $e_1 \cdot X = \alpha_1 e_1 + \frac{1}{2} a_{12} + \frac{1}{2} \bar{b}_{13} \in B$; therefore $(e_1 \cdot X) \cdot e_2 = \frac{1}{4} a_{12} \in B$ and $(e_1 \cdot X) \cdot e_3 = \frac{1}{4} \bar{b}_{13} \in B$. Thus since $e_1 \cdot X \in B$, $\alpha_1 e_1 \in B$. Similarly $\alpha_2 e_2$ and $\alpha_3 e_3$ are in B. Now suppose that some $\alpha_i \neq 0$, say $\alpha_1 \neq 0$. Then $e_1 \in B$ and therefore

$$\begin{aligned} M_{12} &= e_1 \cdot M_{12} \subset B, \\ M_{13} &= e_1 \cdot M_{13} \subset B, \\ M_{23} &= M_{21} \cdot M_{13} = M_{12} \cdot M_{13} \subset B. \end{aligned}$$

Next since $n(a, b)$ is non-degenerate on $\mathfrak{A}$, there exists $a \in \mathfrak{A}$ with $n(a) \neq 0$ and therefore $n(a)(e_2 + e_3) = a_{23}^2 \in M_{23} \subset B$. Thus $e_2 + e_3 \in B$; similarly $e_1 + e_2 \in B$. Since $e_1 \in B$, e_2 and e_3 are in B so that $B = M$.

We now show that there exists $X \in B$ with some $\alpha_i \neq 0$. Suppose $Y = a_{12} + \bar{b}_{13} + c_{23} \in B$ with, say, $a_{12} \neq 0$, the other cases being similar. Then $(e_1 \cdot Y) \cdot e_2 = \frac{1}{4} a_{12} \in B$. Now since $n(a, b)$ is non-degenerate on $\mathfrak{A}$, there exists $b \in \mathfrak{A}$ with $n(a, b) \neq 0$ and therefore

$$0 \neq n(a, b)(e_1 + e_2) = a_{12} \cdot b_{12} \in B.$$

Thus $X = e_1 + e_2 \in B$ is the desired element with $\alpha_1 \neq 0$. Thus we have shown that M is simple if $f(\alpha, \beta)$ is non-degenerate, which is equivalent to $\mathfrak{A}$ being simple.

Conversely, if $f(\alpha, \beta)$ is degenerate on A, set $C = \{\alpha \in A\ ; f(\alpha, A) = 0\}$; then

$$\mathfrak{N} = \left\{ \begin{bmatrix} 0 & \alpha \\ \beta & 0 \end{bmatrix} : \alpha, \beta \in C \right\} \tag{7}$$

is a proper ideal of $\mathfrak{A}$ and from (4) we have for $a \in \mathfrak{A}$, $b \in \mathfrak{N}$ that $n(a, b) = 0$. Next noting that $b \in \mathfrak{N}$ implies $\bar{b} \in \mathfrak{N}$, we see that $B = \mathfrak{N}_{12} + \mathfrak{N}_{13} + \mathfrak{N}_{23}$ is an ideal of M, where $\mathfrak{N}_{ij} = \{b_{ij} : b \in \mathfrak{N}\}$. For if $a \in \mathfrak{A}$, we have

$$\begin{aligned} e_i \cdot \mathfrak{N}_{ij} &= \mathfrak{N}_{ij} \cdot e_j = \tfrac{1}{2} \mathfrak{N}_{ij} \subset B, \\ e_k \cdot \mathfrak{N}_{ij} &= 0, \qquad k \neq i,\ k \neq j, \\ a_{ij} \cdot b_{ij} &= n(a, b)(e_i + e_j) = 0, \qquad \text{where } b \in \mathfrak{N}, \\ 2a_{ij} \cdot b_{jk} &= (ab)_{ik} \in B, \end{aligned}$$

since $ab \in \mathfrak{N}$. Thus B is a proper ideal of M, and this proves the first statement in the theorem.

3. Identities. In this section we prove the second statement of the main theorem. Let $X = \alpha_1 e_1 + \alpha_2 e_2 + \alpha_3 e_3 + a_{12} + \bar{b}_{13} + c_{23}$ be in M; then

$$X^2 = \begin{bmatrix} \alpha_1^2 + n(a,a) + n(b,b) & (\alpha_1 + \alpha_2)a + \bar{b}\,\bar{c} & (\alpha_1 + \alpha_3)\bar{b} + ac \\ (\alpha_1 + \alpha_2)\bar{a} + cb & \alpha_2^2 + n(a,a) + n(c,c) & (\alpha_2 + \alpha_3)c + \bar{a}\,\bar{b} \\ (\alpha_1 + \alpha_3)b + \bar{c}\,\bar{a} & (\alpha_2 + \alpha_3)\bar{c} + ba & \alpha_3^2 + n(b,b) + n(c,c) \end{bmatrix}.$$

Then computing $2X^3 = 2X \cdot X^2 = A_1 e_1 + A_2 e_2 + A_3 e_3 + f_{12} + \bar{g}_{13} + h_{23}$, we obtain

$$\text{(8)} \qquad \tfrac{1}{2}A_1 = \alpha_1^3 + (2\alpha_1 + \alpha_3)n(b,b) + (2\alpha_1 + \alpha_2)n(a,a) + n(b, \bar{c}\,\bar{a}) + n(a, \bar{b}\,\bar{c}),$$

$$\text{(9)} \qquad f_{12} = (\alpha_1 + \alpha_2)^2 a + (\alpha_1 + \alpha_2)\bar{b}\,\bar{c} + [\alpha_1^2 + \alpha_2^2 + 2n(a,a) + n(b,b) + n(c,c)]a + \bar{b}[\alpha_2 + \alpha_3)\bar{c} + ba] + [(\alpha_1 + \alpha_3)\bar{b} + ac]\bar{c}.$$

Now if X is to satisfy a generic minimum cubic polynomial $m_X(\lambda)$, we see, by comparing the elements in the $(1, 1)$ position of 1, X, X^2, and X^3, that we must have

$$m_X(\lambda) = \lambda^3 - (\alpha_1 + \alpha_2 + \alpha_3)\lambda^2 + (\alpha_1\alpha_2 + \alpha_1\alpha_3 + \alpha_2\alpha_3 - a\bar{a} - b\bar{b} - c\bar{c})\lambda - (\alpha_1\alpha_2\alpha_3 + s(a,b,c) - \alpha_1 c\bar{c} - \alpha_2 b\bar{b} - \alpha_3 a\bar{a})1,$$

where $s(a, b, c) = n(b, \bar{c}\,\bar{a}) + n(a, \bar{b}\,\bar{c})$. Next since X must satisfy $m_X(\lambda)$, we compare the elements in the $(1, 2)$ position of 1, X, X^2, and X^3 to obtain

$$\begin{aligned} 0 &= \tfrac{1}{2}f_{12} - (\alpha_1 + \alpha_2 + \alpha_3)[(\alpha_1 + \alpha_2)a + \bar{b}\,\bar{c}] + (\alpha_1\alpha_2 + \alpha_1\alpha_3 + \alpha_2\alpha_3 - a\bar{a} - b\bar{b} - c\bar{c})a \\ &= \bar{b}(ba) - (\bar{b}b)a + (ac)\bar{c} - a(c\bar{c}) \end{aligned}$$

for all $a, b, c \in \mathfrak{A}$. This equation is satisfied if and only if $\bar{x}(xy) = (\bar{x}x)y$ for all $x, y \in \mathfrak{A}$; but a straightforward computation shows that the above equation holds if and only if

$$st\beta(\alpha\gamma) = f(\alpha, \beta)\gamma - f(\beta, \gamma)\alpha$$

for all $\alpha, \beta, \gamma \in A$. Thus by the lemma of §1 M is a Jordan algebra.

Finally we consider the power associativity of M. Computing

$$X^2 \cdot X^2 = B_1 e_1 + B_2 e_2 + B_3 e_3 + \dots,$$

we obtain

$$\begin{aligned} B_1 &= [\alpha_1^2 + n(a,a) + n(b,b)]^2 \\ &\quad + n[(\alpha_1 + \alpha_2)a + \bar{b}\,\bar{c},\ (\alpha_1 + \alpha_2)a + \bar{b}\,\bar{c}] \\ &\quad + n[(\alpha_1 + \alpha_3)b + \bar{c}\,\bar{a},\ (\alpha_1 + \alpha_3)b + \bar{c}\,\bar{a}] \\ &= \alpha_1^4 + n(a,a)^2 + n(b,b)^2 + 2\alpha_1^2 n(a,a) \\ &\quad + 2\alpha_1^2 n(b,b) + 2n(a,a)n(b,b) \\ &\quad + (\alpha_1 + \alpha_2)^2 n(a,a) + n(\bar{b}\,\bar{c}, \bar{b}\,\bar{c}) \\ &\quad + (\alpha_1 + \alpha_3)^2 n(b,b) + n(\bar{c}\,\bar{a}, \bar{c}\,\bar{a}) \\ &\quad + 2(2\alpha_1 + \alpha_2 + \alpha_3)n(a, \bar{b}\,\bar{c}), \end{aligned}$$

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using (5) to obtain this last term. Next computing $X \cdot X^3 = C_1 e_1 + C_2 e_2 + C_3 e_3 + \ldots$, we obtain

$$C_1 = \alpha_1(\tfrac{1}{2}A_1) + n(a, \tfrac{1}{2}f_{12}) + n(b, \tfrac{1}{2}g_{13}),$$

where A_1 and f_{12} are given by (8) and (9) and

$$\begin{aligned} g_{13} = {} & (\alpha_1 + \alpha_3)^2 b + (\alpha_1 + \alpha_3)\bar{c}\,\bar{a} \\ & + [\alpha_1^2 + \alpha_3^2 + n(a, a) + 2n(b, b) + n(c, c)]b \\ & + [(\alpha_2 + \alpha_3)\bar{c} + ba]\bar{a} + \bar{c}[(\alpha_1 + \alpha_2)\bar{a} + cb]. \end{aligned}$$

Expanding the formula for C_1, we obtain

$$\begin{aligned} C_1 = \alpha_1[\alpha_1^3 & + (2\alpha_1 + \alpha_2)n(a, a) + (2\alpha_1 + \alpha_3)n(b, b) + n(b, \bar{c}\,\bar{a}) + n(a, \bar{b}\,\bar{c})] \\ & + \tfrac{1}{2}n(a, (\alpha_1 + \alpha_2)^2 a + (\alpha_1 + \alpha_2)\bar{b}\,\bar{c} \\ & + [\alpha_1^2 + \alpha_2^2 + 2n(a, a) + n(b, b) + n(c, c)]a \\ & + \bar{b}[(\alpha_2 + \alpha_3)\bar{c} + ba] + [(\alpha_1 + \alpha_3)\bar{b} + ac]\bar{c}) \\ & + \tfrac{1}{2}n(b, (\alpha_1 + \alpha_3)^2 b + (\alpha_1 + \alpha_3)\bar{c}\,\bar{a} \\ & + [\alpha_1^2 + \alpha_3^2 + n(a, a) + 2n(b, b) + n(c, c)]b \\ & + [(\alpha_2 + \alpha_3)\bar{c} + ba]\bar{a} + \bar{c}[(\alpha_1 + \alpha_2)\bar{a} + cb]). \end{aligned}$$

Now if M is power associative, we must have $B_1 = C_1$; and using (5) on C_1, this yields

$$\begin{aligned} & n(a, a)n(b, b) + n(\bar{c}\,\bar{a}, \bar{c}\,\bar{a}) + n(\bar{b}\,\bar{c}, \bar{b}\,\bar{c}) \\ & \quad = \tfrac{1}{2}n(c, c)n(a, a) + \tfrac{1}{2}n(c, c)n(b, b) \\ & \qquad + \tfrac{1}{2}n(a, (ac)\bar{c}) + \tfrac{1}{2}n(b, \bar{c}(cb)) + n(a, \bar{b}(ba)). \end{aligned}$$

for all $a, b, c \in A$. Thus setting $c = 0$ and using (5),

$$n(a, a)\, n(b, b) = n(a, \bar{b}\overline{(\bar{a}\,\bar{b})}) = n(\bar{a}\,\bar{b}, \bar{a}\,\bar{b}) = n(ab, ab).$$

But using (1) this equation yields

$$f(\beta, f(\gamma, \delta)\alpha - f(\alpha, \delta)\gamma + st\delta(\alpha\gamma)) = 0$$

for all $\alpha, \beta, \gamma, \delta \in A$; and since $f(\alpha, \beta)$ is non-degenerate, we have by the lemma of §1 that M is Jordan.

4. Concerning invariant forms. Let $R(X)$ denote the mapping $Y \to Y \cdot X$ and let $(X, Y) = \operatorname{trace} R(X \cdot Y)$; then we shall show in this section that (X, Y) is an invariant form (i.e. $(X \cdot Y, Z) = (X, Y \cdot Z)$), which is non-degenerate if and only if $f(\alpha, \beta)$ is non-degenerate. Let $z_1, \ldots, z_{2(n+1)}$ be a basis for $\mathfrak{A}$; then $e_1, e_2, e_3, (z_j)_{12}, (z_j)_{13}, (z_j)_{23}, j = 1, \ldots, 2(n+1)$ is a basis for M. Let

$$X = \alpha_1 e_1 + \alpha_2 e_2 + \alpha_3 e_3 + (a_3)_{12} + (\bar{a}_2)_{13} + (a_1)_{23}$$

be in M, where $a_k = \sum a_{kj} z_j \in \mathfrak{A}$ with $a_{kj} \in F$. Then to compute trace $R(X)$ we have

$$\begin{aligned}
e_1 R(X) &= \alpha_1 e_1 + \ldots, \\
e_2 R(X) &= \alpha_2 e_2 + \ldots, \\
e_3 R(X) &= \alpha_3 e_3 + \ldots, \\
(z_j)_{12} R(X) &= \tfrac{1}{2}(\alpha_1 + \alpha_2)(z_j)_{12} + \ldots, \\
(z_j)_{13} R(X) &= \tfrac{1}{2}(\alpha_1 + \alpha_3)(z_j)_{13} + \ldots, \\
(z_j)_{23} R(X) &= \tfrac{1}{2}(\alpha_2 + \alpha_3)(z_j)_{23} + \ldots,
\end{aligned}$$

where ... denotes elements that make no contribution to the diagonal of the matrix of $R(X)$. Thus if I denotes the $2(n+1) \times 2(n+1)$ identity matrix, we have

$$\begin{aligned}
\operatorname{trace} R(X) &= \operatorname{trace} \begin{bmatrix} \alpha_1 & & & & & * \\ & \alpha_2 & & & & \\ & & \alpha_3 & & & \\ & & & \tfrac{1}{2}(\alpha_1+\alpha_2)I & & \\ & & & & \tfrac{1}{2}(\alpha_1+\alpha_3)I & \\ & & * & & & \tfrac{1}{2}(\alpha_2+\alpha_3)I \end{bmatrix} \\
&= [2(n+1)+1](\alpha_1+\alpha_2+\alpha_3).
\end{aligned}$$

Next for X, Y as in (6) we can show that

$$\begin{aligned}
X \cdot Y = (\alpha_1 \beta_1 + n(a_2, b_2) + n(a_3, b_3))e_1 \\
+ (\alpha_2 \beta_2 + n(a_1, b_1) + n(a_3, b_3))e_2 \\
+ (\alpha_3 \beta_3 + n(a_2, b_2) + n(a_1, b_1))e_3 + \ldots
\end{aligned}$$

so that

$$\begin{aligned}
(X, Y) &= \operatorname{trace} R(X \cdot Y) \\
&= [2(n+1)+1][\alpha_1\beta_1 + \alpha_2\beta_2 + \alpha_3\beta_3 + 2n(a_1, b_1) + 2n(a_2, b_2) \\
&\qquad + 2n(a_3, b_3)].
\end{aligned}$$

From this equation we see that if $f(\alpha, \beta)$ is non-degenerate, so is (X, Y). For suppose that $(X, Y) = 0$ for all $Y \in M$; then for $\beta_1 = 1$ and the rest zero we obtain $\alpha_1 = 0$; similarly $\alpha_2 = \alpha_3 = 0$. Next for b_1 arbitrary and $b_2 = b_3 = 0$ we obtain $n(a_1, b_1) = 0$, and since $n(a, b)$ is non-degenerate when $f(\alpha, \beta)$ is non-degenerate, then $a_1 = 0$; similarly $a_2 = a_3 = 0$. Thus $X = 0$. Conversely, if $f(\alpha, \beta)$ is degenerate, then for $b_1, b_2, b_3 \in \mathfrak{N}$, the ideal given in (7), and for $\beta_1 = \beta_2 = \beta_3 = 0$ we see from the above formula that the element Y is such that $(X, Y) = 0$ for all $X \in M$.

Next we shall show that $(X \cdot Y, Z) = (X, Y \cdot Z)$, that is

$$\operatorname{trace} R[(X \cdot Y)Z - X \cdot (Y \cdot Z)] = 0.$$

For

$$Z = \gamma_1 e_1 + \gamma_2 e_2 + \gamma_3 e_3 + (c_3)_{12} + (\bar{c}_2)_{13} + (c_1)_{23},$$

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a lengthy computation yields

$$\begin{aligned}&\operatorname{trace} R[(X\cdot Y)\cdot Z] - \operatorname{trace} R[X\cdot(Y\cdot Z)]\\ &\quad = 2(2(n+1)+1)[n(\bar{a}_3\bar{b}_2 + \bar{b}_3\bar{a}_2, c_1)\\ &\qquad + n(\bar{a}_1\bar{b}_3 + \bar{b}_1\bar{a}_3, c_2) + n(\bar{a}_2\bar{b}_1 + \bar{b}_2\bar{a}_1, c_3)\\ &\qquad - n(\bar{c}_3\bar{b}_2 + \bar{b}_3\bar{c}_2, a_1) - n(\bar{c}_1\bar{b}_3 + \bar{b}_1\bar{c}_3, a_2)\\ &\qquad - n(\bar{c}_2\bar{b}_1 + \bar{b}_2\bar{c}_1, a_3)]\\ &\qquad = 0,\end{aligned}$$

using (5) in the form $n(\bar{x}y, z) = n(y\bar{z}, x)$. Thus (X, Y) is an invariant form on M.

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