Can. J. Math., Vol. XXVII, No. 2, 1975, pp. 469–474

A NOTE ON CERTAIN SPACES WITH BASES (mod K)

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In this note all spaces are assumed to be regular T_1 spaces and all undefined terms and notations may be found in [**8**]. In particular let $\mathrm{cl}(A)$ denote the closure of the set A and let Z^+ denote the set of natural numbers.

Definition 1. Let X be a topological space and $\mathscr{K}$ a covering of X by compact sets. An open covering $\mathscr{G}$ of X is said to be a *basis* (mod K) if whenever $x \in K_x \in \mathscr{K}$ and an open set V contains K_x, then there exists $G \in \mathscr{G}$ such that $x \in G \subset V$. In such a case X is written as the ordered triple $(X, \mathscr{K}, \mathscr{G})$.

A topological space $X = (X, \mathscr{K}, \mathscr{G})$ is *first-countable* (mod K) provided that X has an open covering $\mathscr{G}$ which satisfies the following condition: if $x \in K_x \in \mathscr{K}$, then there is a sequence $\{G_n: n \in Z^+\}$ in $\mathscr{G}$ such that if $K_x \subset V$, where V is an open set in X, then there is a natural number n such that $x \in G_n \subset V$.

The notions of various types of bases (mod K) was motivated by a result of Arhangel'skiĭ [**1**, Theorem 22] and a result of Michael and Lutzer [**12**].

If X is a compact space, then letting $\mathscr{K} = \{X\}$ and $\mathscr{G} = \{X\}$ it is apparent that $(X, \mathscr{K}, \mathscr{G})$ is first-countable (mod K). It is also apparent that each first-countable space is a first-countable (mod K) space.

Using the Tychonoff Product Theorem, it may be shown that the topological product of a countable family of first-countable (mod K) spaces is first-countable (mod K). The property of being first-countable (mod K) is preserved by open compact maps (i.e. an open continuous map with compact fibers) and is weakly hereditary, but not hereditary. If there is a perfect map (i.e. closed, continuous map with compact fibers) from a space X onto a first-countable (mod K) space, then X is first-countable (mod K). However first-countability (mod K) is not preserved by perfect maps. To see this, the following two definitions are needed.

Definition 1. A set A in a topological space X has *countable character* if there exists a sequence of open sets $\{U_n\}$ such that if $A \subset V$ where V is an open set, then there exists a natural number n such that $A \subset U_n \subset V$.

Definition 2. A topological space X is a *semi-stratifiable space* if, to each open set $U \subset X$ one can assign a sequence $\{U_n: n \in Z^+\}$ of closed subsets of X such that

(a) $\bigcup\{U_n: n \in Z^+\} = U$, and

(b) $U_n \subset V_n$ whenever $U \subset V$, where $\{V_n: n \in Z^+\}$ is the sequence assigned to V.

Received October 30, 1973 and in revised form, July 2, 1974.

Now let R denote any regular semi-metric space which has a compact subset A which is not of countable character. It was pointed out by C. Borges [**3**] that Example 9.2 of [**6**] has such a subset. Let φ be the identity map from R into R/A, the space obtained from R by identifying the points of A to a single point. The map φ is perfect and therefore, R/A is semi-stratifiable [**7**]. Since A is not of countable character, the space R/A is not first-countable. In [**7**] it is shown that compact semi-stratifiable spaces are metrizable. Thus, if R/A was first-countable (mod K), then R/A would be first-countable by Lemma 1 below. It follows that R/A is not first-countable (mod K).

LEMMA 1. *Let X be a first-countable* (mod K) *space by virtue of a compact covering $\mathscr{K}$. If each member of $\mathscr{K}$ is a first-countable subspace of X, then X is first-countable.*

Proof. Let $x \in X$ and $K_x \in \mathscr{K}$ such that $x \in K_x$. Since each element of $\mathscr{K}$ is a first-countable subspace and since X is a regular T_1-space, a countable collection $\{H_n: n \in Z^+\}$ of open subsets of X may be found such that $\text{cl}(H_{n+1}) \subset H_n$ for each natural number n and, if V is any open set containing x, then there is a natural number n such that $x \in H_n \cap K_x \subset V \cap K_x$. Now, let V be any open set containing x. Choose a natural number n such that $x \in H_n \cap K_x \subset V \cap K_x$. Then it follows that

$$\text{cl}(H_{n+1} \cap K_x) \subset \text{cl}(H_{n+1}) \cap K_x \subset H_n \cap K_x \subset V \cap K_x.$$

Note that $K_x \cap (H_n - V) = \emptyset$. Thus, $K_x \cap (\text{cl}(H_{n+1}) - V) = \emptyset$. Let $\{G_m: m \in Z^+\}$ be a first-countable (mod K) base for X. There is a natural number m such that $x \in G_m \subset X - (\text{cl}(H_{n+1}) - V)$. It follows that $x \in H_{n+1} \cap G_m \subset V$ and that $\{H_n \cap G_m: n \in Z^+, m \in Z^+\}$ is a local base at x.

Topological spaces that are closed, continuous images of a metrizable space are called Lasnev spaces. In [**13**; **15**] Morita, Hanai, and Stone have shown that a Lasnev space is metrizable if and only if it is first-countable. Also, in [**15**], Stone has shown that a locally countably compact space which is a Lasnev space is metrizable. Using these facts and Lemma 1 the following theorem is established.

THEOREM 1. *If Y is a Lasnev space, then Y is metrizable if and only if Y is first-countable* (mod K).

Proof. Let f be a closed, continuous map from the metrizable space X onto $Y = (Y, \mathscr{K}, \mathscr{G})$, a first-countable (mod K) space. The map f restricted to $f^{-1}(K)$ is closed whenever $K \in \mathscr{K}$. Since $f^{-1}(K)$ is metrizable, K (as a subspace) must also be metrizable by Stone's Theorem. By Lemma 1, Y is first-countable. The metrizability of Y now follows by the Morita-Hanai-Stone Theorem.

For further information on Lasnev spaces, see [**14**].

In [**4**] Burke has shown that the following definitions for p-spaces and strict p-spaces are equivalent to the original definitions.

https://doi.org/10.4153/CJM-1975-055-5 Published online by Cambridge University Press

Definition 3. A completely regular space X is a p-space if there is a sequence $\mathscr{H} = \{\mathscr{H}_n : n \in Z^+\}$ of open coverings of X satisying: If $x \in X$ and $H_n \in \mathscr{H}_n$ such that $x \in H_n$, then

(a) $\bigcap\{\text{cl}(H_n) : n \in Z^+\}$ is compact, and

(b) if $x_n \in \bigcap\{\text{cl}(H_k) : k = 1, \ldots, n\}$ then $\{x_n : n \in Z^+\}$ has a cluster point.

Definition 4. A completely regular space X is a *strict-p-space* if there is a sequence $\mathscr{H} = \{\mathscr{H}_n : n \in Z^+\}$ of open coverings of X satisfying:

(a) $P_x = \bigcap\{\text{st}(x, \mathscr{H}_n) : n \in Z^+\}$ is a compact for each $x \in X$, and

(b) $\{\text{st}(x, \mathscr{H}_n) : n \in Z^+\}$ is a neighborhood base for P_x.

The following definition is familiar, but it is included for completeness.

Definition 5. A regular topological space X is a *Moore space* if there is a sequence of open covers $\{G_n : n \in Z^+\}$ such that $\{\text{st}(x, G_n) : n \in Z^+\}$ is a local base at x for each $x \in X$.

The following definition is a natural extension of the concept of a Moore space in the (mod K) setting.

Definition 6. A topological space $X = (X, \mathscr{K}, \mathscr{G})$ is *developable* (mod K) if $\mathscr{G} = \bigcup\{\mathscr{G}_i : i \in Z^+\}$ where $\mathscr{G}_i$ is an open covering of X for each natural number i and for each $x \in X$, if $x \in K \in \mathscr{K}$ and K is contained in an open set V, then there is a natural number $n(x)$ such that $\text{st}(x, \mathscr{G}_{n(x)}) \subset V$. A regular developable (mod K) space is called a *Moore* (mod K) *space* and $\mathscr{G}$ is called a *development* (mod K) for X.

The next several theorems relate Moore (mod K) spaces with p-spaces and strict p-spaces.

THEOREM 2. *Let* $X = (X, \mathscr{K}, \mathscr{G})$ *be a completely regular Moore* (mod K) *space; then,* X *is a* p*-space.*

Proof. Let $\mathscr{G} = \{\mathscr{G}_n : n \in Z^+\}$ be a development (mod K) for X. Appealing to Definition 3, let $\mathscr{H}_n = \mathscr{G}_n$ for each natural number n. Let $x \in K \in \mathscr{K}$ and, for each natural number n, let $x \in H_n \in \mathscr{H}_n$. It follows that

$$\bigcap\{\text{cl}(H_n) : n \in Z^+\} \subseteq K.$$

For if $y \in \bigcap\{\text{cl}(H_n) : n \in Z^+\}$ and $y \notin K$, then there is an open set U such that $K \subset U \subset \text{cl}(U) \subset X - \{y\}$. Choose a natural number n such that $\text{st}(x, \mathscr{H}_n) \subset U$. It follows that $\text{cl}(\text{st}(x, \mathscr{H}_n)) \subset \text{cl}(U)$ and, in particular, $y \notin \text{cl}(H_n)$. This is a contradiction and it follows that $\bigcap\{\text{cl}(H_n) : n \in Z^+\}$ is a subset of K and, thus, is compact.

Now suppose that $x_n \in \bigcap\{\text{cl}(H_i) : 1 \leqq i \leqq n\}$ for each natural number n. If $\{x_n : n \in Z^+\}$ does not have a cluster point then there is a natural number n such that $S = \{x_{n+i} : i \in Z^+\}$ is closed and disjoint from K. Since X is regular, a natural number $m \geqq n$ can be chosen such that $\text{cl}(\text{st}(x, \mathscr{H}_m)) \cap S = \emptyset$. It follows that $\text{cl}(H_m) \cap S = \emptyset$ and, in particular, $x_m \notin \bigcap\{\text{cl}(H_i) : 1 \leqq i \leqq m\}$

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which is a contradiction. Thus $\{x_n: n \in Z^+\}$ has a cluster point and X is a p-space.

THEOREM 3. *If* $X = (X, \mathscr{K}, \mathscr{G})$ *is a completely regular Moore* (mod K) *space such that* $\bigcap\{\mathrm{st}(x, \mathscr{G}_n): n \in Z^+\} = \bigcap\{\mathrm{cl}(\mathrm{st}(x, \mathscr{G}_n)): n \in Z^+\}$, *then* X *is a strict* p*-space.*

Proof. Let $\mathscr{G} = \{\mathscr{G}_n: n \in Z^+\}$ be a development (mod K) for X such that $\bigcap\{\mathrm{st}(x, \mathscr{G}_n): n \in Z^+\} = \bigcap\{\mathrm{cl}(\mathrm{st}(x, \mathscr{G}_n)): n \in Z^+\}$ for each $x \in X$. For each natural number n, let

$$\mathscr{H}_n = \{g_1 \cap \ldots \cap g_n: 1 \leqq i \leqq n, g_i \in \mathscr{G}_i\}.$$

Observe that $\bigcap\{\mathrm{st}(x, \mathscr{H}_n): n \in Z^+\} = \bigcap\{\mathrm{cl}(\mathrm{st}(x, \mathscr{H}_n)): n \in Z^+\}$ for each $x \in X$.

Let $x \in X$ and $K \in \mathscr{K}$ such that $x \in K$. If $y \in X - K$, choose a natural number n such that $y \notin \mathrm{st}(x, \mathscr{H}_n)$. Thus $y \notin \bigcap\{\mathrm{st}(x, \mathscr{H}_n): n \in Z^+\} = P_x$. It follows that $P_x \subset K$ and P_x is a compact subset of X.

Let P_x be contained in an open set U. If $C = K - U$, then since C is compact and $\mathrm{cl}(\mathrm{st}(x, \mathscr{H}_{n+1})) \subseteq \mathrm{cl}(\mathrm{st}(x, \mathscr{H}_n))$ for each natural number n, there is a natural number i such that $C \cap \mathrm{cl}(\mathrm{st}(x, \mathscr{H}_i)) = \emptyset$. It follows that

$$U \cup (X - \mathrm{cl}(\mathrm{st}(x, \mathscr{H}_i))$$

is an open set that contains K. Choose a natural number $j > i$ such that

$$\mathrm{st}(x, \mathscr{H}_j) \subset U \cup (X - \mathrm{cl}(\mathrm{st}(x, \mathscr{H}_i)).$$

For this natural number j, $\mathrm{st}(x, \mathscr{H}_j) \subset U$. Hence, $\{\mathrm{st}(x, \mathscr{H}_n): n \in Z^+\}$ is a neighborhood base for P_x and, therefore, X is a strict p-space.

The notion of a θ-refinable space was introduced by Worrell and Wicke [**16**] and studied in [**2**].

Definition 7. A space X is said to be θ-refinable if given any open covering $\mathscr{G}$ of X, there is a sequence $\{\mathscr{G}_i: i \in Z^+\}$ such that

(i) $\mathscr{G}_i$ is an open covering of X which refines $\mathscr{G}$ for each natural number i, and

(ii) if $x \in X$, then there is a natural number $n(x)$ such that x is in only finitely many members of $\mathscr{G}_{n(x)}$.

COROLLARY 1. *A completely regular, Moore* (mod K) *space* $X = (X, \mathscr{K}, \mathscr{G})$ *is a strict* p*-space if it is* θ*-refinable.*

Proof. Following the technique used by Burke in [**4**], a sequence

$$\{\mathscr{H}_{(i,j)}: i \in Z^+, j \in Z^+\}$$

of open covers can be constructed such that there is a cofinal subsequence $\{(n(i), m(i)): i \in Z^+\}$ that satisfies

$$\bigcap\{\mathrm{st}(x, \mathscr{H}_{(n(i), m(i))}): i \in Z^+\} = \bigcap\{\mathrm{cl}(\mathrm{st}(x, \mathscr{H}_{(n(i), m(i))})): i \in Z^+\}.$$

Thus the space is a strict p-space.

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Definition 8. A system $G = \{g(n, x): x \in X, n \in Z^+\}$ is a *graded system of open covers* if

(i) $x \in g(n, x)$ and $g(n, x)$ is open for each $x \in X$ and each natural number n,
(ii) $g(n + 1, x) \subseteq g(n, x)$ for all natural numbers n and each $x \in X$, and
(iii) $\{x\} = \bigcap\{g(n, x): n \in Z^+\}$ for each $x \in X$.

A graded system of open covers $\{g(n, x): n \in Z^+, x \in X\}$ is a *c-semi-stratification* for X provided that $A = \bigcap\{g(n, A): n \in Z^+\}$ for each closed compact set A where $g(n, A) = \bigcup\{g(n, x): x \in A\}$. A space is *c-semi-stratifiable* if it has a c-semi-stratification.

The notion of a c-semi-stratifiable space is a generalization of a semi-stratifiable space and is studied in [**11**].

THEOREM 4. *A regular space X is a Moore space if and only if X is a c-semi-stratifiable space and a Moore* (mod K) *space.*

Proof. Let X be a c-semi-stratifiable space and let $X = (X, \mathscr{K}, \mathscr{G})$ where $\mathscr{G} = \{\mathscr{G}_n: n \in Z^+\}$ is a development (mod K). No generality is lost if it is assumed that the closures of elements of $\mathscr{G}_{n+1}$ refine $\mathscr{G}_n$ for each natural number n. Let $\{g(n, x): x \in X, n \in Z^+\}$ be a c-semi-stratification for X such that for each natural number n, $\{g(n, x): x \in X\}$ refines $\mathscr{G}_n$ and $\mathrm{cl}(g(n + 1, x)) \subseteq g(n, x)$. Since regular, compact c-semi-stratifiable spaces are metrizable [**11**], each member of $\mathscr{K}$ is first-countable. Thus X is first-countable by Lemma 1. No generality is lost then if it is assumed that $\{g(n, x): n \in Z^+\}$ is a local base at x.

Let K_x be an arbitrary member of $\mathscr{K}$ such that $x \in K_x$ and let $x \in g(n, x_n)$ for each natural number n. Suppose that $\{x_n: n \in z^+\}$ does not converge to x. Then there exists an open neighborhood V of x and subsequence $\{y_n: n \in Z^+\}$ of $\{x_n: n \in Z^+\}$ such that no y_n is in V. If $\{y_n: n \in Z^+\}$ is frequently in K_x, then because X is first-countable there exists $z \in K_x - V$ and subsequence $\{z_n: n \in Z^+\}$ of $\{y_n: n \in Z^+\}$ such that $\{z_n: n \in Z^+\}$ converges to z. Let $S = \{z\} \cup \{z_n: n \in Z^+\}$. The set S is compact and $x \notin S$. Thus there is a natural number m such that $x \notin g(m, S)$. Choose $n \geqq m$ such that $x_n \in S$. Then $x \notin g(m, x_n) \supseteq g(n, x_n)$ which is a contradiction. Thus $\{y_n: n \in Z^+\}$ is not frequently in K_x. Without loss of generality it may be assumed that $y_n \notin K_x$ for each natural number n. Let $D = \{y_n \in Z^+\}$. If D is not closed, then again there exists $y \in X$ such that $y \notin V$ and a subsequence $\{z_n: n \in Z^+\}$ of $\{y_n: n \in Z^+\}$ such that $\{z_n: n \in Z^+\}$ converges to y. As before a contradiction may be derived. If D is closed, then, since D is disjoint from K_x, a natural number m may be found such that $\mathrm{st}(x, \mathscr{G}_m) \cap D = \emptyset$. Since $\{g(m, x): x \in X\}$ refines $\mathscr{G}_m$, it follows that $x \notin g(m, D)$ which again leads to contradiction. Thus it follows that $\{x_n: n \in Z^+\}$ converges to x. By a result of Heath [**9**], X is semi-metrizable. Since X has a development (mod K), X is a p-space and Burke [**4**] has shown that a semi-metrizable p-space is developable.

Definition 9. A topological space X is said to be *quasi-metrizable* provided

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there is a real valued function $d:X \times X \to R$ (where R denotes the reals) satisfying the following conditions:

(i) $d(x, y) \geqq 0$,

(ii) $d(x, y) = 0$ if, and only if, $x = y$,

(iii) $d(x, z) \leqq d(x, y) + d(y, z)$ for all $x, y, z \in X$,

(iv) The collection $\{S_d(x, \epsilon): x \in X, \epsilon > 0\}$ forms a base for the topology on X where $S_d(x, \epsilon) = \{y: d(x, y) < \epsilon\}$.

As an example of a c-semi-stratifiable space that is not a Moore (mod K) space consider the example of D. K. Burke given in [**5**]. It is easily shown that Burke's example is a locally compact, quasi-metric space. It is shown [**11**] that every quasi-metric space is c-semi-stratifiable. The example is not a Moore space and, hence, by the preceding theorem, it is not a Moore (mod K) space.

The lexicographic ordering of the unit square L [**10**, p. 23] is a Moore (mod K) space by letting $X = L$, $\mathscr{K} = \{L\}$, $\mathscr{G} = \{L\}$. Since L is not a Moore space it is not c-semi-stratifiable.

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