TWO PROBABILITY THEOREMS AND THEIR APPLICATION TO SOME FIRST PASSAGE PROBLEMS

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1. Introduction

Let X_i, $i = 1, 2, 3, \cdots$ be a sequence of independent and identically distributed random variables and write $S_n = X_1+X_2+\cdots+X_n$. If the mean of X_i is finite and positive, we have $Pr(S_n \leqq x) \to 0$ as $n \to \infty$ for all x, $-\infty < x < \infty$ using the weak law of large numbers. It is our purpose in this paper to study the rate of convergence of $Pr(S_n \leqq x)$ to zero. Necessary and sufficient conditions are established for the convergence of the two series

$$\sum_{n=1}^{\infty} n^k \, Pr(S_n \leqq x), \qquad -\infty < x < \infty$$

where k is a non-negative integer, and

$$\sum_{n=1}^{\infty} e^{rn} \, Pr(S_n \leqq x), \qquad -\infty < x < \infty$$

where $r > 0$. These conditions are applied to some first passage problems for sums of random variables. The former is also used in correcting a queueing Theorem of Finch [4].

2. Two Probability Theorems

Let X_i, $i = 1, 2, 3, \cdots$ be independent and identically distributed random variables. We write $S_n = X_1+X_2+\cdots+X_n$, $X_i^- = \min(0, X_i)$ and $X_i^+ = X_i+X_i^-$.

We shall establish the following two Theorems:

THEOREM A [1]. *Suppose* $E|X| < \infty$, $EX > 0$ *and let* k *be a non-negative*

[1] My attention has been drawn to a statement of Theorem A without proof in Smith, W. L. "On the elementary renewal theorem for non-identically distributed random variables" Univ. North Carolina Mimeographed Notes No. 352 (Feb. 1963). Professor Smith states that a proof of this result will appear in a paper entitled "On functions of characteristic functions and their applications to some renewal-theoretic random walk problems".

integer. A necessary and sufficient condition for the convergence of the series

$$\sum_{n=1}^{\infty} n^k \, Pr(S_n \leqq x), \qquad -\infty < x < \infty,$$

is that $E|X^-|^{k+2} < \infty$.

THEOREM B. *Suppose* $E|X| < \infty$ *and* $EX > 0$. *A necessary and sufficient condition for the convergence of the series*

$$\sum_{n=1}^{\infty} e^{rn} \, Pr(S_n \leqq x), \qquad -\infty < x < \infty,$$

for some $r > 0$ *is that* X^- *has an analytic characteristic function* [2].

(It is clear that analogous Theorems will hold in the case $EX < 0$). We defer the proofs of Theorems A and B until some Lemmas have been established.

LEMMA 1. *If* $E|X|^r < \infty$ *for some integer* $r \geqq 1$ *and* $EX > 0$, *then*

$$\sum n^{r-2} \, Pr(S_n \leqq x) < \infty, \qquad -\infty < x < \infty.$$

PROOF: Write $EX = \mu$. Using Katz [5], Theorem 1, we have

$$\sum n^{r-2} \, Pr\{|S_n - n\mu| \geqq n\varepsilon\} < \infty, \text{ every } \varepsilon > 0$$

from which we obtain, in particular

$$\sum n^{r-2} \, Pr(S_n \leqq (\mu - \varepsilon)n) < \infty, \text{ every } \varepsilon > 0. \tag{1}$$

Now we choose ε so small that $\varepsilon < \mu$. We then have, for n sufficiently large,

$$Pr\,(S_n \leqq x) \leqq Pr(S_n \leqq (\mu - \varepsilon)n)$$

and the result follows immediately from (1).

LEMMA 2. *Let* $E|X| < \infty$ *and* $EX > 0$ *or else* $E|X| = \infty$ *and, in either case,* $E|X^-|^r < \infty$ *for some integer* $r \geqq 1$. *Then*

$$\sum n^{r-2} \, Pr(S_n \leqq x) < \infty, \qquad -\infty < x < \infty.$$

PROOF. We define a new random variable Y as follows

$$\begin{aligned} Y &= X \text{ if } X < K \\ &= 0 \text{ otherwise,} \end{aligned}$$

where the constant $K (> 0)$ is chosen so that $EY > 0$. Then, $Y \leqq X$ and $E|Y|^r < \infty$. It follows from Lemma 1 that

[2] The term "analytic characteristic function" is used for a characteristic function which is analytic in a strip containing the origin as an interior point.

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$$\sum n^{r-2} Pr(Y_1+Y_2+\cdots+Y_n \leqq x) < \infty, \qquad -\infty < x < \infty.$$

Also, if $X_1+X_2+\cdots+X_n \leqq x$, then $Y_1+Y_2+\cdots+Y_n \leqq x$ so that

$$Pr(Y_1+Y_2+\cdots+Y_n \leqq x) \geqq Pr(X_1+X_2+\cdots+X_n \leqq x) = Pr(S_n \leqq x)$$

and hence

$$\sum n^{r-2} Pr(S_n \leqq x) < \infty, \qquad -\infty < x < \infty.$$

This completes the proof.

LEMMA 3. *Let* $E|X| < \infty$, $EX = \mu > 0$ *and*

$$\sum n^k Pr(S_n \leqq x) < \infty, \qquad -\infty < x < \infty$$

for some non-negative integer k. *Then* $E|X^-|^{k+2} < \infty$.

Our proof relies heavily on techniques used by Erdös [3].

PROOF. Write $X_i^* = X_i - \mu$ and $Z_n = \sum_{i=1}^n X_i^*$. We then have

$$Pr(S_n \leqq x) = Pr(Z_n \leqq x - n\mu) \geqq Pr(Z_n \leqq -nc)$$

for $c > \mu$ and n sufficiently large.

Now from the fact that $E|X| < \infty$, it follows by a simple rearrangement that

$$\sum_{n=1}^{\infty} Pr(X_i^* < -(c+\varepsilon)n) < \infty,$$

for arbitrary $\varepsilon > 0$. Also, since the terms in this series are non-increasing, we have

$$nPr(X_i^* < -(c+\varepsilon)n) \to 0 \text{ as } n \to \infty. \tag{2}$$

Write $\{E\}$ for the event E and $\{\bar{E}\}$ for the complement of $\{E\}$. We have

$$\{Z_n \leqq -nc\} \supseteqq \bigcup_{i=1}^{n} [\{X_i^* < -n(c+\varepsilon)\} \cap \{|X_1^*+\cdots+X_{i-1}^*+X_{i+1}^*+\cdots +X_n^*| < (n-1)\varepsilon\}],$$

and for the sake of brevity, we put

$$A_i = \{X_i^* < -n(c+\varepsilon)\},$$
$$B_i = \{|X_1^*+\cdots+X_{i-1}^*+X_{i+1}^*+\cdots+X_n^*| < (n-1)\varepsilon\},$$

$i = 1, 2, 3, \cdots, n$. Thus,

$$
\begin{aligned}
Pr(Z_n \leqq -nc) &\geqq Pr[\bigcup_{i=1}^{n} (A_i \cap B_i)] \\
&= Pr[\bigcup_{i=1}^{n} \{(\overline{A_1 \cap B_1}) \cap (\overline{A_2 \cap B_2}) \cap \cdots \cap (\overline{A_{i-1} \cap B_{i-1}}) \cap (A_i \cap B_i)\}] \\
&= \sum_{i=1}^{n} Pr[(\overline{A_1 \cap B_1}) \cap \cdots \cap (\overline{A_{i-1} \cap B_{i-1}}) \cap (A_i \cap B_i)] \\
&\geqq \sum_{i=1}^{n} Pr(\bar{A}_1 \cap \bar{A}_2 \cap \cdots \cap \bar{A}_{i-1} \cap A_i \cap B_i) \qquad (3) \\
&\geqq \sum_{i=1}^{n} [Pr(A_i \cap B_i) - Pr\{(A_1 \cup A_2 \cup \cdots \cup A_{i-1}) \cap A_i\}] \\
&\geqq \sum_{i=1}^{n} [Pr(B_i) - (i-1) Pr(A_1)] Pr(A_i) \\
&\geqq \sum_{i=1}^{n} [Pr(B_i) - n\, Pr(A_1)] Pr(A_i).
\end{aligned}
$$

Now take arbitrary ρ, $0 < \rho < 1$ and $\delta > 0$ such that $1-2\delta \geqq \rho$. It follows from the weak law of large numbers that we can find an integer N_1 such that

$$Pr(B_i) > 1-\delta \text{ for } n \geqq N_1.$$

Also, from (2), we can find an integer N_2 such that

$$n\, Pr(A_1) < \delta \text{ for } n \geqq N_2.$$

Thus, for $n \geqq \max (N_1, N_2)$, we have from (3)

$$Pr(Z_n \leqq -nc) \geqq n\rho\, Pr(X_i^* < -(c+\varepsilon)n), \tag{4}$$

and hence

$$\sum n^{k+1} Pr(X_i^* < -(c+\varepsilon)n) < \infty.$$

We now introduce the random variable Y defined by

$$
\begin{aligned}
Y &= X^* \text{ if } X^* < 0 \\
&= 0 \text{ otherwise,}
\end{aligned}
$$

and obtain

$$\sum n^{k+1} Pr(Y < -(c+\varepsilon)n) < \infty.$$

It follows from this, by a simple rearrangement, that $E|Y|^{k+2} < \infty$ and hence that $E|X^-|^{k+2} < \infty$. This completes the proof.

Proof of Theorem A. Theorem A follows immediately from Lemmas 2 and 3.

We now go on to give two Lemmas leading up to a proof of Theorem B.

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The development of the proof is similar to that used above in the proof of Theorem A.

LEMMA 4. *Suppose X^- has an analytic characteristic function and either $E|X| < \infty$ and $EX > 0$ or $E|X| = \infty$. There exists a constant $R > 0$ such that*

$$\sum e^{rn} Pr(S_n \leqq x) < \infty$$

for every x, $-\infty < x < \infty$ and every r, $0 < r < R$.

PROOF. Since X^- has an analytic characteristic function, there exists a constant $K > 0$ such that

$$\Phi(\theta) = E(e^{-\theta X}) < \infty$$

for all θ in $0 \leqq \theta < K$. Now for such a θ, a well-known Chebyshev type inequality gives

$$Pr(S_n \leqq x) \leqq e^{\theta x} E(e^{-\theta S_n}) = e^{\theta x}\{\Phi(\theta)\}^n.$$

Also, in view of our assumption that $E|X^-| < \infty$ and either $E|X| < \infty$ and $EX > 0$ or $E|X| = \infty$, we must have $\Phi(\theta) < 1$ for sufficiently small positive θ. We then choose R so small that $e^R\Phi(\theta) < 1$ and for all r, $0 < r < R$,

$$\sum e^{rn} Pr(S_n \leqq x) < \infty.$$

This completes the proof of the Lemma. I am indebted to the referee for this direct proof. My original proof was based on Baum, Katz and Read [1], Theorem 2, 190.

Lemma 4 is a generalization of the well-known result of Stein [9] which deals with the case $X^- = 0$. It should be noted that although Stein's result is correct, his proof is invalidated by a misinterpretation of the Markov chain property of the sequence $\{S_n\}$ of sums.

LEMMA 5. *Let $E|X| < \infty$. Suppose $EX = \mu > 0$ and*

$$\sum e^{rn} Pr(S_n \leqq x) < \infty$$

for all r, $0 < r < R$ and all x, $-\infty < x < \infty$. Then X^- has an analytic characteristic function.

PROOF. We retain the notation of Lemma 3. Proceeding precisely as in Lemma 3, we obtain (4)

$$Pr(Z_n \leqq -nc) \geqq n\rho\, Pr(X_i^* < -(c+\varepsilon)n),$$

so that certainly

$$\sum e^{rn} Pr(X_i^* < -(c+\varepsilon)n) < \infty.$$

We now introduce the random variable Y defined by

$$Y = X^* \text{ if } X^* < 0$$
$$= 0 \text{ otherwise,}$$

and obtain

$$\sum e^{rn} Pr(Y < -(c+\varepsilon)n) < \infty.$$

It follows immediately, using Lukacs [7], Theorem 7.2.1, 137, that Y and hence X^- has an analytic characteristic function. This completes the proof of the Lemma.

Proof of Theorem B: Theorem B follows immediately from Lemmas 4 and 5.

It is worth remarking that it is quite likely that in Theorems A and B the condition $E|X| < \infty$, $EX > 0$ can be replaced by the condition $E|X| < \infty$, $EX > 0$ or $E|X^-| < \infty$, $E|X| = \infty$.

3. Application to some first passage problems

Let X_i, $i = 1, 2, 3, \cdots$ be independent and identically distributed random variables and write $S_n = X_1+X_2+\cdots+X_n$. Consider a single boundary at $A\,(\geqq 0)$ so that if

$$F_0(x) = \begin{cases} 1 & x \geqq 0 \\ 0 & x < 0, \end{cases}$$
$$F_1(x) = Pr(S_1 \leqq x),$$
$$F_n(x) = Pr(S_n \leqq x;\ \max_{1 \leqq k \leqq n-1} S_k \leqq A),\ n > 1,$$

the probability p_n that the first passage time out of the interval $(-\infty, A]$ for the process S_n is n is given by

$$p_n = F_{n-1}(A) - F_n(A), \qquad n \geqq 1.$$

This passage problem in the case $A = 0$ arises, for example, in the busy period distribution of the queue $GI/G/1$ which has been considered by various authors such as Finch [4].

We introduce the probability generating function $P(\lambda) = \sum_{r=1}^{\infty} \lambda^r p_r$ for the first passage time distribution (henceforth abbreviated F.P.T.D.) $Pr(N = n) = p_n$. We have formally

$$P'(1) = E(N) = 1 + \sum_{r=1}^{\infty} F_r(A)$$
$$P''(1) = \sum_{r=2}^{\infty} r(r-1)p_r = E(N^2) - E(N) = 2\sum_{r=1}^{\infty} rF_r(A),$$

and in general for $k > 1$,

$$P^{(k)}(1) = (\alpha)_k \text{ (the } k\text{-th factorial moment of } N)$$
$$= k \sum_{r=k-1}^{\infty} (r)_k F_r(A) = \sum_{r=0}^{k} s(k, r) E(N^r)$$

where $(r)_k = r(r-1)(r-2) \cdots (r-k+1)$ and $s(k, r)$ are the Stirling numbers of the first kind. It is thus clear that $E(N^r) < \infty$ for some positive integer r if and only if $\sum n^{r-1} F_n(A) < \infty$. Also, the random variable N has an analytic characteristic function if and only if the radius of convergence of $P(\lambda)$ is greater than unity or equivalently if $\sum e^{rn} F_n(A) < \infty$ for some $r > 0$.

Now we write

$$q_n = Pr\,(\max_{1 \leq k \leq n} S_k \leqq 0), \qquad n \geqq 1; \qquad q_0 = 1.$$

Spitzer [8], 332, shows that

$$\sum_{n=0}^{\infty} q_n t^n = \exp\left\{\sum_{n=1}^{\infty} \frac{t^n}{n} Pr(S_n \leqq 0)\right\}, \tag{6}$$

a result originally due, in a slightly different form, to E. Sparre Andersen. From this we obtain

$$q_n \geqq \frac{1}{n} Pr(S_n \leqq 0).$$

Thus,

$$Pr(S_n \leqq A) \geqq Pr\,(\max_{1 \leq k \leq n} S_k \leqq A) = F_n(A) \geqq q_n \geqq \frac{1}{n} Pr(S_n \leqq 0),$$

and we readily obtain from Theorem B:

THEOREM 1. *The F.P.T.D. generated by the random variable X with $E|X| < \infty$ and $EX > 0$ has an analytic characteristic function if and only if X^- has an analytic characteristic function.*

Further, we obtain immediately from Theorem A:

THEOREM 2. *Let $r > 1$ be a positive integer. Consider the F.P.T.D. generated by the random variable X with $E|X| < \infty$ and $EX > 0$. If the F.P.T.D. has a finite r-th moment, then $E|X^-|^r < \infty$. If, on the other hand, $E|X^-|^r < \infty$, then the F.P.T.D. has finite moments at least up to the $(r-1)$th.*

In the particular case where $A = 0$, we can improve this Theorem by virtue of the relation (6). In fact, formally differentiating (6) $(r-1)$ times, we see that

$$\sum n^{r-2} Pr(S_n \leqq 0) < \infty \text{ if and only if } \sum n^{r-1} q_n < \infty.$$

Thus, in view of our comments above, the r-th moment of the F.P.T.D. exists if and only if

$$\sum n^{r-2} Pr(S_n \leqq 0) < \infty.$$

We therefore obtain immediately from Theorem A:

THEOREM 3. *Let $r > 1$ be a positive integer. The zero-barrier F.P.T.D. generated by the random variable X with $E|X| < \infty$ and $EX > 0$ has a finite r-th moment if and only if X^- has a finite r-th moment.*

Before ending this section, it is worth remarking that Derman and Robbins [2] show that it is possible to have $E|X^+| = \infty$, $E|X^-| = \infty$ and $Pr(S_n > 0 \text{ i.o.}) = 1$, $Pr(S_n \leqq 0 \text{ i.o.}) = 0$ and hence, following Kemperman [6], Theorem 15.2, 81, $\sum 1/n\, Pr(S_n > 0) = \infty$, $\sum 1/n\, Pr(S_n \leqq 0) < \infty$. This provides us with a limitation on eventual improvements of the Theorems given above.

4. Correction to a theorem of Finch [4]

Let η be the difference between the inter-arrival and service time in a $GI/G/1$ queue. We refrain from stating the usual queueing assumptions for the sake of brevity. Let Π_n be the probability that n customers are served in a busy period. Then, as is well known,

$$\Pi_1 = Pr(\eta_1 > 0)$$

$$\Pi_n = Pr\,(\max_{1 \leqq k \leqq n-1} \eta_1 + \eta_2 + \cdots + \eta_k \leqq 0, \eta_n > 0), \qquad n > 1,$$

so that $Pr(T = n) = \Pi_n$ is a zero-barrier F.P.T.D.

Finch [4] gives the following Theorem (his Theorem 2, 223).

THEOREM *Suppose that $E|\eta| < \infty$. Write $\Pi = \sum_{j=1}^{\infty} \Pi_j$, $N = \sum_{j=1}^{\infty} j\Pi_j$, and $a_n = Pr(\eta_1 + \eta_2 + \cdots + \eta_n > 0)$. Then,*

$$\Pi = \begin{cases} 1 & \text{if } E\eta \geqq 0 \\ 1 - \exp\{-\sum_{n=1}^{\infty} n^{-1} a_n\} & \text{if } E\eta < 0 \end{cases}$$

$$N = \begin{cases} \exp\{\sum_{n=1}^{\infty} n^{-1}(1 - a_n)\} & \text{if } E\eta > 0 \\ \infty & \text{if } E\eta = 0 \\ \sum_{n=1}^{\infty} a_n \exp\{-\sum_{n=1}^{\infty} n^{-1} a_n\} & \text{if } E\eta < 0. \end{cases}$$

It is the final part of the statement of this Theorem that is incorrect, namely that

$$N = \sum_{n=1}^{\infty} a_n \exp\{-\sum_{n=1}^{\infty} n^{-1} a_n\} < \infty \text{ if } E\eta < 0.$$

In fact, under the condition $E\eta < 0$, we see from a negative mean analogue of Theorem A that $\sum_{n=1}^{\infty} a_n < \infty$ if and only if $E|\eta^+|^2 < \infty$. Thus, $N = \infty$ if $E\eta < 0$ and $E|\eta^+|^2 < \infty$. Finch's error arises from an invalid application of the Borel zero-one criterion which yields $Pr(S_n > 0 \text{ i.o.}) = 0$ or 1 according as $\sum a_n < \infty$ or $= \infty$. Actually, using Kemperman [6], Theorem 15.2, 81, $Pr(S_n > 0 \text{ i.o.}) = 0$ or 1 according as $\sum n^{-1} a_n < \infty$ or $= \infty$. Finch's Theorem and his proof of it can easily be repaired in terms of these comments. A correct statement of the Theorem is as follows:

THEOREM. *Suppose that* $E|\eta| < \infty$. *Write* $\Pi = \sum_{j=1}^{\infty} \Pi_j$, $N = \sum_{j=1}^{\infty} j\Pi_j$, *and* $a_n = Pr(\eta_1+\eta_2+\cdots+\eta_n) > 0$. *Then*

$$\Pi = \begin{cases} 1 & \text{if } E\eta \geqq 0 \\ 1-\exp\{-\sum_{n=1}^{\infty} n^{-1}a_n\} & \text{if } E\eta < 0 \end{cases}$$

$$N = \begin{cases} \exp\{\sum_{n=1}^{\infty} n^{-1}(1-a_n)\} & \text{if } E\eta > 0 \\ \infty & \text{if } E\eta = 0 \text{ or } E\eta < 0 \text{ and } E|\eta^+|^2 = \infty \\ \sum_{n=1}^{\infty} a_n \exp\{-\sum_{n=1}^{\infty} n^{-1}a_n\} & \text{if } E\eta < 0 \text{ and } E|\eta^+|^2 < \infty. \end{cases}$$

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