J. Aust. Math. Soc. **96** (2014), 289–302
doi:10.1017/S144678871400007X

CAYLEY SUM GRAPHS OF IDEALS OF A COMMUTATIVE RING

M. AFKHAMI, Z. BARATI, K. KHASHYARMANESH✉ and N. PAKNEJAD

(Received 30 January 2013; accepted 21 January 2014)

Communicated by B. Alspach

Abstract

Let R be a commutative ring, $I(R)$ be the set of all ideals of R and S be a subset of $I^*(R) = I(R) \setminus \{0\}$. We define a Cayley sum digraph of ideals of R, denoted by $\overrightarrow{\mathrm{Cay}}^+(I(R), S)$, as a directed graph whose vertex set is the set $I(R)$ and, for every two distinct vertices I and J, there is an arc from I to J, denoted by $I \longrightarrow J$, whenever $I + K = J$, for some ideal K in S. Also, the Cayley sum graph $\mathrm{Cay}^+(I(R), S)$ is an undirected graph whose vertex set is the set $I(R)$ and two distinct vertices I and J are adjacent whenever $I + K = J$ or $J + K = I$, for some ideal K in S. In this paper, we study some basic properties of the graphs $\overrightarrow{\mathrm{Cay}}^+(I(R), S)$ and $\mathrm{Cay}^+(I(R), S)$ such as connectivity, girth and clique number. Moreover, we investigate the planarity, outerplanarity and ring graph of $\mathrm{Cay}^+(I(R), S)$ and also we provide some characterization for rings R whose Cayley sum graphs have genus one.

2010 *Mathematics subject classification*: primary 05C10; secondary 05C69, 13A15.

Keywords and phrases: Cayley sum graph, planar graph, clique number, genus number.

1. Introduction

The investigation of graphs related to algebraic structures is a very large and growing area of research. One of the most important classes of graphs considered in this framework is that of Cayley graphs. These graphs have been considered for example in [3, 13, 17, 18]. Let us refer the reader to the survey article [21] for an extensive bibliography devoted to various applications of Cayley graphs. In particular, the Cayley graphs of semigroups are related to automata theory, as explained in [16] and the monograph [15]. Several other classes of graphs associated to algebraic structures have also been actively investigated. For example, power graphs and divisibility graphs have been considered in [19, 20]. Graphs associated with rings have been studied with respect to several ring constructions (see [12, 14]). The zero-divisor graphs of rings have been investigated in [4–8]. Since most properties of a ring are closely tied to the behaviour of its ideals, it is useful and interesting to associate graphs and digraphs to the ideals of a ring. To see some instances of these graphs, the reader is referred to [1, 2, 10, 22].

In this paper, we define a Cayley sum digraph of ideals of a commutative ring. Let R be a commutative ring, $I(R)$ be the set of all ideals of R and S be a subset of $I^*(R) = I(R) \setminus \{0\}$. We define the Cayley sum digraph of ideals of R, denoted by $\overrightarrow{\mathrm{Cay}}^+(I(R), S)$, as a directed graph whose vertex set is the set $I(R)$ and, for every two distinct vertices I and J, there is an arc from I to J, denoted by $I \longrightarrow J$, whenever $I + K = J$, for some ideal K in S. In fact, the Cayley sum digraph $\overrightarrow{\mathrm{Cay}}^+(I(R), S)$ is a Cayley digraph of semigroups. Also, the Cayley sum graph $\mathrm{Cay}^+(I(R), S)$ is an undirected graph whose vertex set is the set $I(R)$ and two distinct vertices I and J are adjacent, denoted by $I \sim J$, whenever $I + K = J$ or $J + K = I$, for some ideal K in S.

In Section 2, we study some basic properties of the graphs $\overrightarrow{\mathrm{Cay}}^+(I(R), S)$ and $\mathrm{Cay}^+(I(R), S)$ such as connectivity, girth and clique number. In Section 3, we characterize all rings whose Cayley sum graphs are planar, outerplanar and ring graphs. Finally, in Section 4, we study Cayley sum graphs with genus one.

Throughout this paper, all rings are assumed to be commutative with nonzero identity. By $I(R)$, $\mathrm{Nil}(R)$ and $\mathrm{Max}(R)$, we denote the set of all ideals, the set of all nilpotent elements and the set of all maximal ideals of R, respectively. Moreover, $I^*(R)$ is the set of all nonzero ideals of R. A nonzero ideal I of R is said to be *minimal* if there is no nontrivial ideal of R properly contained in I. We denote the set of all minimal ideals of R and the set of all prime ideals of R by $\mathrm{Min}(R)$ and $\mathrm{Spec}(R)$, respectively. Also, we denote the Jacobson radical of R by $J(R)$.

Now, we recall some definitions and notation on graphs. We use the standard terminology of graphs following [9]. Let $G = (V, E)$ be a graph, where V is the set of vertices and E is the set of edges. The graph $H = (V_0, E_0)$ is a *subgraph* of G if $V_0 \subseteq V$ and $E_0 \subseteq E$. Moreover, H is called a *spanning subgraph* of G if its vertex set is V. The *distance* between two distinct vertices a and b in G, denoted by $\mathrm{d}(a, b)$, is the length of the shortest path connecting a and b, if such a path exists; otherwise, we set $\mathrm{d}(a, b) := \infty$. The *diameter* of a graph G is $\mathrm{diam}(G) = \max\{\mathrm{d}(a, b) \colon a$ and b are distinct vertices of $G\}$. The *girth* of G is the length of the shortest cycle in G, denoted by $\mathrm{gr}(G)$ ($\mathrm{gr}(G) := \infty$ if G has no cycles). Also, for two distinct vertices a and b in G, the notation $a \sim b$ means that a and b are adjacent. A graph G is said to be *connected* if there exists a path between any two distinct vertices, and it is *complete* if it is connected with diameter one. We use K_n to denote the complete graph with n vertices. For a vertex x in G, the *degree* of x is the number of vertices adjacent to x and it is denoted by $\deg(x)$. A vertex x is an *isolated vertex* if $\deg(x) = 0$. A *clique* of a graph is a complete subgraph of it and the number of vertices in a largest clique of G is called the *clique number* of G and is denoted by $\omega(G)$. An *independent set* of G is a subset of the vertices of G such that no two vertices in the subset represent an edge of G. The *independence number* of G, denoted by $\alpha(G)$, is the cardinality of a largest independent set. For a positive integer r, an *r-partite graph* is one whose vertex set can be partitioned into r subsets, so that no edge has both ends in any one subset.

A *complete r-partite graph* is one in which each vertex is joined to every vertex that is not in the same subset. The complete bipartite graph (2-partite graph) with part sizes m and n is denoted by $K_{m,n}$. A graph is said to be *planar* if it can be drawn in the plane so that its edges intersect only at their ends. A *subdivision* of a graph is any graph that can be obtained from the original graph by replacing edges by paths. A remarkable simple characterization of planar graphs was given by Kuratowski in 1930. Kuratowski's theorem says that a graph is planar if and only if it contains no subdivision of K_5 or $K_{3,3}$ (cf. [9, page 153]).

2. Basic properties of the Cayley sum graph of ideals

Throughout this paper, S is a subset of $I^*(R)$. For simplicity, we call the Cayley sum digraph and the Cayley sum graph of ideals of R as the Cayley digraph and the Cayley graph of ideals and denote them by $\overrightarrow{\mathrm{Cay}}(I(R), S)$ and $\mathrm{Cay}(I(R), S)$, respectively. We also denote the zero ideal of R by 0.

In this section, we study some basic properties of the graphs $\overrightarrow{\mathrm{Cay}}(I(R), S)$ and $\mathrm{Cay}(I(R), S)$. First, we bring the following definition.

Definition 2.1. A set S is minimal with respect to addition, and we denote it by m.r.a., whenever we have $I \nsubseteq K_1 + K_2 + \cdots + K_t$, for every distinct ideals $I, K_1, \ldots, K_t \in S$, where $t \geqslant 1$.

Proposition 2.2. *Suppose that* $I(R) = (S)$. *Then* $\mathrm{Cay}(I(R), S)$ *is a connected graph.*

Proof. Let $I \in I(R)$. Since $I(R) = (S)$, there are ideals $I_1, I_2, \ldots, I_k \in S$ such that $I = I_1 + I_2 + \cdots + I_k$, where $k \geqslant 1$. Now, the directed path

$$0 \longrightarrow I_1 \longrightarrow I_1 + I_2 \longrightarrow \cdots \longrightarrow I_1 + I_2 + \cdots + I_k$$

shows that the zero ideal connects to every ideal in $I(R)$. Therefore, $\mathrm{Cay}(I(R), S)$ is connected. □

Proposition 2.3. *If* $\mathrm{Cay}(I(R), S)$ *is connected and* S *is m.r.a., then* $I(R) = (S)$.

Proof. Suppose that $I \in I(R)$. Since $\mathrm{Cay}(I(R), S)$ is connected, we have a path between the vertices 0 and I as follows:

$$0 \sim I_1 \sim \cdots \sim I_{k-1} \sim I.$$

Since 0 is adjacent to I_1, we have $I_1 \in S$. Now, if $I_2 \longrightarrow I_1$, then there exists an ideal K in S such that $K \subseteq I_1$. Thus, we have $I_1 = I_1 + K$, which is impossible. So, we have $I_1 \longrightarrow I_2$, and hence there is an ideal K_1 in S such that $I_2 = I_1 + K_1$. Therefore, we have $I_2 \in (S)$.

Now, let r be the minimum integer such that $I_{r+1} \longrightarrow I_r$, where $1 \leq r \leq k$, and so, for every integer i with $1 \leq i \leq r$, there exist ideals $K_1, K_2, \ldots, K_{i-1}$ in S such that $I_i = I_1 + K_1 + \cdots + K_{i-1}$. Therefore, there are ideals $K, K_1, \ldots, K_{i-1}$ in S such that $I_{r+1} + K = I_r = I_1 + K_1 + \cdots + K_{r-1}$. Thus, we have $K \subseteq I_1 + K_1 + \cdots + K_{r-1}$, which is a contradiction since S is m.r.a. Hence, for every integer r with $1 \leq r \leq k - 1$, we have $I_r \longrightarrow I_{r+1}$, and therefore $I \in (S)$. □

https://doi.org/10.1017/S144678871400007X Published online by Cambridge University Press

Remark 2.4. As seen in the proof of Propositions 2.2 and 2.3, these propositions can be formulated for every monoid.

Recall that R is called a *principal ring* if every ideal of R is principal, and R is called a *special ring* if $J(R) = \mathrm{Nil}(R)$.

In the following theorem, we provide a necessary and sufficient condition for the completeness of the Cayley graph $\mathrm{Cay}(\mathbb{I}(R), S)$.

Theorem 2.5. *The Cayley graph* $\mathrm{Cay}(\mathbb{I}(R), S)$ *is complete if and only if* $S = \mathbb{I}^*(R)$, *and* R *is either a local special principal ring or a local principal ideal domain.*

Proof. First, assume that $\mathrm{Cay}(\mathbb{I}(R), S)$ is complete. Then, for every ideal $I \in \mathbb{I}(R)$, we have $0 \sim I$. Thus, $I \in S$, which implies that $S = \mathbb{I}^*(R)$. Now, let $\mathfrak{m}$ and $\mathfrak{m}'$ be two distinct maximal ideals of R. Since $\mathrm{Cay}(\mathbb{I}(R), S)$ is complete, we have either $\mathfrak{m} \longrightarrow \mathfrak{m}'$ or $\mathfrak{m}' \longrightarrow \mathfrak{m}$. This means that $\mathfrak{m} \subseteq \mathfrak{m}'$ or $\mathfrak{m}' \subseteq \mathfrak{m}$, which is impossible. Hence, R is a local ring.

If x and y are two distinct elements in a minimal generating set of $\mathfrak{m}$, then one can easily see that $x = ry$ or $y = rx$, for some element $r \in R$, which is impossible. Thus, $\mathfrak{m}$ is a principal ideal. Also, by a similar argument, one can see that every ideal of R is principal.

Now, suppose that $\mathfrak{m} = (x)$. If $\mathrm{Ann}(x) = 0$, then R is an integral domain. So, in this situation, R is a principal ideal domain. Otherwise, $\mathrm{Ann}(x) \neq 0$. In this case there exists an element $ux^i \in \mathfrak{m}$ such that $ux^{i+1} = 0$, where $u \in U(R)$. Therefore, $x^{i+1} = 0$. So, we have that $x \in \mathrm{Nil}(R)$. Hence, $J(R) = \mathrm{Nil}(R)$, which means that R is a special principal ring.

Conversely, let $S = \mathbb{I}^*(R)$ and R be a local special principal ring or a local principal ideal domain. Assume that $\mathfrak{m} = (x)$ is a maximal ideal of R. Now, if $I = (ux^i)$ and $J = (vx^j)$ are two distinct ideals of R, where $j \leq i$, then we have $I + (J \setminus I) = J$. Therefore, the Cayley graph $\mathrm{Cay}(\mathbb{I}(R), S)$ is complete. □

Theorem 2.6. *The Cayley graph* $\mathrm{Cay}(\mathbb{I}(R), S)$ *is a star graph if and only if* R *is a field or* $S = \{R\}$.

Proof. Let $\mathrm{Cay}(\mathbb{I}(R), S)$ be a star graph with centre I. First, assume that $I = 0$. Hence, $S = \mathbb{I}^*(R)$. Now, if there is a proper ideal J in $\mathbb{I}(R)$, then one can find a triangle $0 \longrightarrow J \longrightarrow R \longleftarrow 0$ in $\mathrm{Cay}(\mathbb{I}(R), S)$, which is impossible. So, R is a field. Now, assume that $I \neq 0$. Then $I \in S$. Also, if $J \neq I$ is an ideal in S, then we have the triangle $0 \longrightarrow I \sim J \longleftarrow 0$, which is impossible. Thus, $S = \{I\}$. If $I \neq R$, then we have $0 \longrightarrow I \longrightarrow R$, and so $I + I = R$, which is a contradiction. Therefore, $S = \{R\}$.

The converse statement is clear. □

In the following theorem, we investigate the girth of the graphs $\mathrm{Cay}(\mathbb{I}(R), S)$ and $\overrightarrow{\mathrm{Cay}}(\mathbb{I}(R), S)$.

Proposition 2.7. *The following statements hold:*

(i) $\mathrm{gr}(\overrightarrow{\mathrm{Cay}}(\mathbb{I}(R), S)) = \infty$;
(ii) $\mathrm{gr}(\mathrm{Cay}(\mathbb{I}(R), S)) \in \{3, 4, \infty\}$.

https://doi.org/10.1017/S144678871400007X Published online by Cambridge University Press

PROOF. (i) If there exists a directed cycle $I_1 \longrightarrow I_2 \longrightarrow \cdots \longrightarrow I_n \longrightarrow I_1$ of minimal length with $n \geqslant 3$ in the Cayley digraph $\overrightarrow{\mathrm{Cay}}(I(R), S)$, then $I_1 \subset I_2 \subset \cdots \subset I_n \subset I_1$. So, we have $I_1 = I_2 = \cdots = I_n$, which is impossible. Thus, $\overrightarrow{\mathrm{Cay}}(I(R), S)$ contains no directed cycles.

(ii) First, assume that $S = \{I\}$. If we have the path $I_1 \longrightarrow I_2 \longrightarrow I_3$ in $\mathrm{Cay}(I(R), S)$, then $I_1 + I = I_2$ and $I_2 + I = I_3$, which implies that $I_2 = I_3$. If we have the path $I_1 \longleftarrow I_2 \longrightarrow I_3$ in $\mathrm{Cay}(I(R), S)$, then $I_1 = I + I_2 = I_3$. Therefore, any path of length two between I_1 and I_2 in $\mathrm{Cay}(I(R), S)$ is of the form $I_1 \longrightarrow I_2 \longleftarrow I_3$. Thus, $\mathrm{gr}(\mathrm{Cay}(I(R), S)) = \infty$.

Now, let $|S| \geq 2$ and I, J be two distinct elements in S. If $I \subset J$ or $J \subset I$, then we have the triangle $0 \sim I \sim J \sim 0$. Otherwise, one can find the cycle $0 \longrightarrow I \longrightarrow I + J \longleftarrow J \longleftarrow 0$. So, in this case, we have $\mathrm{gr}(\mathrm{Cay}(I(R), S)) \in \{3, 4\}$. □

For the rest of the paper, we assume that $S = I^*(R)$ and we denote the Cayley graph $\mathrm{Cay}(I(R), S)$ by $\mathrm{Cay}(I(R), I^*)$.

Recall that a graph on n vertices such that $n - 1$ of the vertices have valency one, and all of which are adjacent only to the remaining vertex a, is called a *star graph* with centre a. Also, a *refinement* of a graph H is a graph G such that the vertex sets of G and H are the same and every edge in H is an edge in G.

In the following corollary, we gather together some basic properties of the Cayley graph $\mathrm{Cay}(I(R), I^*)$, which can easily be gained by the definition of the graph.

COROLLARY 2.8. *The following statements hold:*

(a) $\mathrm{Cay}(I(R), I^*)$ *is a connected graph and* $\mathrm{diam}(\mathrm{Cay}(I(R), I^*)) \leq 2$*;*
(b) $\mathrm{Cay}(I(R), I^*)$ *contains a refinement of a star graph;*
(c) $\alpha(\mathrm{Cay}(I(R), I^*)) \geq \{|\mathrm{Max}(R)|, |\mathrm{Min}(R)|\}$.

3. Planar, outerplanar and ring graphs of $\mathrm{Cay}(I(R), I^*)$

In this section, first we study the planarity of the Cayley graph $\mathrm{Cay}(I(R), I^*)$. Then we characterize all rings R whose Cayley graphs $\mathrm{Cay}(I(R), I^*)$ are outerplanar graphs and ring graphs.

Recall that the *dimension* of R, denoted by $\dim(R)$, is the supremum of the lengths of prime ideals in R. It is a nonnegative integer or ∞.

THEOREM 3.1. *The clique number of the induced subgraph* $\mathrm{Cay}(I(R), I^*)$ *on* $\mathrm{Spec}(R)$ *is equal to* $\dim(R) + 1$.

PROOF. Let G be the induced subgraph of $\mathrm{Cay}(I(R), I^*)$ on $\mathrm{Spec}(R)$ and let ω be the clique number of G. At first, assume that $\dim(R) = \infty$. Since any chain of ideals with length n is a clique with n vertices, the clique number ω of G is infinite. Also, if $\dim(R) = n$, then there exists a chain $\mathfrak{p}_0 \subset \mathfrak{p}_1 \subset \cdots \subset \mathfrak{p}_n$, where $\mathfrak{p}_i \in \mathrm{Spec}(R)$. Since the elements of this chain form a complete subgraph in G, we have $n + 1 \leq \omega$.

Now, assume that $\{\mathfrak{q}_1, \mathfrak{q}_2, \ldots, \mathfrak{q}_\omega\} \subseteq \mathrm{Spect}(R)$ is the vertex set of a clique in the graph G. By using induction on ω, we show that the ideals $\mathfrak{q}_1, \mathfrak{q}_2, \ldots, \mathfrak{q}_\omega$ form a chain

https://doi.org/10.1017/S144678871400007X Published online by Cambridge University Press

of prime ideals. Clearly, for $\omega = 1$, there is nothing to prove. Now, suppose that the elements of $\Omega = \{\mathfrak{q}_1, \mathfrak{q}_2, \dots, \mathfrak{q}_k\}$, where $1 < k < \omega$, form a chain of prime ideals of R. We show that the elements of $\Omega' = \{\mathfrak{q}_1, \mathfrak{q}_2, \dots, \mathfrak{q}_k, \mathfrak{q}_{k+1}\}$ also form a chain. Let $\mathfrak{q}_{i_1} \subset \mathfrak{q}_{i_2} \subset \cdots \subset \mathfrak{q}_{i_k}$, where $i_j \in \{1, 2, \dots, k\}$, be the chain of elements in Ω. Now, if $\mathfrak{q}_{k+1} \subset \mathfrak{q}_{i_1}$, then there exists the chain $\mathfrak{q}_{k+1} \subset \mathfrak{q}_{i_1} \subset \mathfrak{q}_{i_2} \subset \cdots \subset \mathfrak{q}_{i_k}$ in Ω'. Otherwise, let $\mathfrak{q}_{i_t}$ be the maximal element of Ω which is a subset of $\mathfrak{q}_{k+1}$. Moreover, one can easily check that $\mathfrak{q}_{k+1} \subset \mathfrak{q}_{i_{t+1}}$. So, we have the following chain in Ω':

$$\mathfrak{q}_{i_1} \subset \cdots \subset \mathfrak{q}_{i_t} \subset \mathfrak{q}_{k+1} \subset \mathfrak{q}_{i_{t+1}} \subset \cdots \subset \mathfrak{q}_{i_k}.$$

Thus, $\omega \leq n + 1$. □

Theorem 3.2. *If the Cayley graph* $\mathrm{Cay}(I(R), I^*)$ *is planar, then* R *is an Artinian ring.*

Proof. It is clear that for every chain of ideals of R with length n, the Cayley graph $\mathrm{Cay}(I(R), I^*)$ has a complete subgraph isomorphic to K_{n+1}. Therefore, all chains have finite length, and so R is a Noetherian and also an Artinian ring. □

In the next theorem, we show that for a commutative ring with planar Cayley graph $\mathrm{Cay}(I(R), I^*)$, the set of maximal ideals has at most three elements.

Theorem 3.3. *If the Cayley graph* $\mathrm{Cay}(I(R), I^*)$ *is planar, then* $|\mathrm{Max}(R)| \leq 3$.

Proof. Assume to the contrary that $|\mathrm{Max}(R)| > 3$ and that $\mathfrak{m}_1, \mathfrak{m}_2, \mathfrak{m}_3$ are distinct maximal ideals in $\mathrm{Max}(R)$. Then $\mathfrak{m}_1 \cap \mathfrak{m}_2 \cap \mathfrak{m}_3 \neq 0$ and we can find the following chain in R:

$$0 \subset (\mathfrak{m}_1 \cap \mathfrak{m}_2 \cap \mathfrak{m}_3) \subset (\mathfrak{m}_1 \cap \mathfrak{m}_2) \subset \mathfrak{m}_1 \subset R.$$

Therefore, the Cayley graph $\mathrm{Cay}(I(R), I^*)$ contains a subgraph isomorphic to K_5, which is a contradiction. □

The following remark is needed in the rest of the paper.

Remark 3.4. Suppose that $\mathfrak{m}$ and $\mathfrak{m}'$ are two distinct maximal ideals of R. If there exists $i \geq 2$ such that $\mathfrak{m}^i = 0$, then, for every $x \in \mathfrak{m} \setminus \mathfrak{m}'$, we have that $x^i = 0$, and so $x \in \mathfrak{m}'$, which is a contradiction. This means that if R has at least two maximal ideals, then, for all $i \geq 2$, $\mathfrak{m}^2 \neq 0$, for every maximal ideal $\mathfrak{m}$ of R.

In the sequel of this section, we determine the family of commutative rings whose Cayley graphs are planar.

Theorem 3.5. *The Cayley graph* $\mathrm{Cay}(I(R), I^*)$ *is a planar graph if and only if* $R \cong F_1 \times F_2$, *where the* F_i *are fields, or* $(R, \mathfrak{m})$ *is a local ring which satisfies in one of the following conditions:*

(i) $\dim_{\frac{R}{\mathfrak{m}}}(\frac{\mathfrak{m}}{\mathfrak{m}^2}) = 2$ *and* $I(R) = \{0, (x), (y), (x, y), R\}$, *where* $x, y \in \mathfrak{m}$;

(ii) $\dim_{\frac{R}{\mathfrak{m}}}(\frac{\mathfrak{m}}{\mathfrak{m}^2}) = 1$, $\mathfrak{m}^2 \neq 0$ *and* $I(R) = \{0, (x^2), (x), R\}$, *where* $x \in \mathfrak{m}$;

(iii) $\dim_{\frac{R}{\mathfrak{m}}}(\frac{\mathfrak{m}}{\mathfrak{m}^2}) = 1$, $\mathfrak{m}^2 = 0$ *and* $I(R) = \{0, (x), R\}$, *where* $x \in \mathfrak{m}$.

https://doi.org/10.1017/S144678871400007X Published online by Cambridge University Press

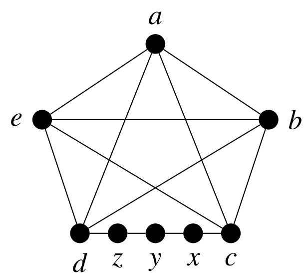

FIGURE 1. A subdivision of K_5.

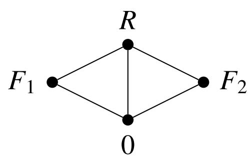

FIGURE 2.

PROOF. First, suppose that $\mathrm{Cay}(I(R), I^*)$ is planar. Hence, by Theorem 3.3, R has at most three maximal ideals. We have the following cases.

Case 1: $|\mathrm{Max}(R)| = 3$. Suppose that $\mathrm{Max}(R) = \{\mathfrak{m}_1, \mathfrak{m}_2, \mathfrak{m}_3\}$. If $J(R) \neq 0$, then we have the following chain that induces a subgraph of $\mathrm{Cay}(I(R), I^*)$ isomorphic to K_5, which is impossible:

$$0 \subset J(R) \subset \mathfrak{m}_1 \cap \mathfrak{m}_2 \subset \mathfrak{m}_1 \subset R.$$

If $J(R) = 0$, then $R \cong F_1 \times F_2 \times F_3$, where F_1, F_2, F_3 are fields. Therefore, we can find a subdivision of K_5 with the following vertices (see Figure 1):

$$\begin{aligned} &a = 0, && b = F_1 \times F_2 \times 0, && c = F_1 \times 0 \times 0, && d = 0 \times F_2 \times 0,\\ &e = F_1 \times F_2 \times F_3, && x = F_1 \times 0 \times F_3, && y = 0 \times 0 \times F_3, && z = 0 \times F_2 \times F_3. \end{aligned}$$

Thus, in this case $\mathrm{Cay}(I(R), I^*)$ is not planar.

Case 2: $|\mathrm{Max}(R)| = 2$. If $J(R) = 0$, then $R \cong F_1 \times F_2$, where the F_i are fields. Thus, $I(R) = \{0, F_1 \times 0, 0 \times F_2, R\}$ and, by Figure 2, $\mathrm{Cay}(I(R), I^*)$ is planar. Let $J(R) \neq 0$, $\mathrm{Max}(R) = \{\mathfrak{m}_1, \mathfrak{m}_2\}$ and I be a nonzero ideal with $I \subset J(R)$. Then we have a subgraph isomorphic to $K_{3,3}$ with the vertex set $\{0, I, J(R)\} \cup \{R, \mathfrak{m}_1, \mathfrak{m}_2\}$, which is impossible. Now, since $\mathrm{Cay}(I(R), I^*)$ is planar, for each nonzero ideal I with $I \neq J(R)$, we have $I \not\subseteq J(R)$. Therefore, $J(R)^2 = J(R)$ or $J(R)^2 = 0$. If $J(R)^2 = J(R)$, then, by Nakayama's lemma, $J(R) = 0$, which is a contradiction. Thus, $J(R)^2 = \mathfrak{m}_1^2\mathfrak{m}_2^2 = 0$, and so there exists i with $1 \leq i \leq 2$ such that $\mathfrak{m}_i^2 \neq \mathfrak{m}_i$. Without loss of generality, we may assume that $\mathfrak{m}_1^2 \neq \mathfrak{m}_1$. Now, if also $\mathfrak{m}_2^2 \neq \mathfrak{m}_2$, then $\mathfrak{m}_1^2 \cap \mathfrak{m}_2 = \mathfrak{m}_1^2 \cap \mathfrak{m}_2^2 = 0$. Since $\mathfrak{m}_1$ and $\mathfrak{m}_2$ are minimal prime ideals associated to 0, by the second uniqueness decomposition theorem, we get a contradiction. Therefore, $\mathfrak{m}_2 = \mathfrak{m}_2^2$ and $J(R)^2 = \mathfrak{m}_1^2 \cap \mathfrak{m}_2 = 0$.

https://doi.org/10.1017/S144678871400007X Published online by Cambridge University Press

Hence, $R \cong R/\mathfrak{m}_1^2 \times R/\mathfrak{m}_2$, and so $J(R) \cong J(R/\mathfrak{m}_1^2) \times J(R/\mathfrak{m}_2)$. Thus, $\mathfrak{m}_1\mathfrak{m}_2 \cong \mathfrak{m}_1 \times 0$, which is impossible. Therefore, in this case $\mathrm{Cay}(I(R), I^*)$ is planar if and only if R is the direct product of two fields.

Case 3: $|\mathrm{Max}(R)| = 1$. So, suppose that $(R, \mathfrak{m})$ is a local ring and $t = \dim_{\frac{R}{\mathfrak{m}}}(\frac{\mathfrak{m}}{\mathfrak{m}^2})$. It is easy to see that if $t \geq 3$, then the Cayley graph $\mathrm{Cay}(I(R), I^*)$ has a subgraph isomorphic to K_5. Hence, suppose that $t < 3$. If $t = 2$, then there exist distinct elements x and y in a minimal generating set of $\mathfrak{m}$ such that $\mathfrak{m} = (x, y)$. Since $\mathrm{Cay}(I(R), I^*)$ is planar and there exists the chain $0 \subset (xy) \subset (x) \subset (x, y) \subset R$, we have $(xy) = (x)$ or $(xy) = 0$. Clearly, $(xy) \neq (x)$, and so $(xy) = 0$. Now, suppose that $I \in I(R) \setminus \{0, (x), (y), (x, y)\}$. Then we have a subgraph of $\mathrm{Cay}(I(R), I^*)$ isomorphic to $K_{3,3}$ with vertex set $\{0, (x, y), R\} \cup \{(x), (y), I\}$. Therefore, $I(R) = \{0, (x), (x, y), R\}$. Now, assume that $t = 1$. Then R is a principal ring and, by using a proof similar to that we used in the proof of Theorem 2.5, $\mathrm{Cay}(I(R), I^*)$ is complete. So, $|I(R)| \leq 4$. If $\mathfrak{m}^2 = 0$, then $I(R) = \{0, \mathfrak{m}, R\}$ and the graph $\mathrm{Cay}(I(R), I^*)$ is isomorphic to K_3. Otherwise, $\mathfrak{m}^2 \neq 0$. In this situation $I(R) = \{0, \mathfrak{m}, \mathfrak{m}^2, R\}$, and therefore the Cayley graph $\mathrm{Cay}(I(R), I^*)$ is isomorphic to K_4.

The converse statement is clear. □

Let G be a graph with n vertices and q edges. We recall that a *chord* is any edge of G joining two nonadjacent vertices in a cycle of G. Let C be a cycle of G. We say that C is a *primitive cycle* if it has no chords. Also, a graph G has the *primitive cycle property* (PCP) if any two primitive cycles intersect in at most one edge.

The number frank(G) is called the *free rank* of G and it is the number of primitive cycles of G. Also, the number $\mathrm{rank(G)} = q - n + r$ is called the *cycle rank* of G, where r is the number of connected components of G. The cycle rank of G can be expressed as the dimension of the cycle space of G. By [11, Proposition 2.2], we have $\mathrm{rank(G)} \leq \mathrm{frank(G)}$. A graph G is called a *ring graph* if it satisfies one of the following equivalent conditions (see [11]):

(i) rank(G) = frank(G);
(ii) G satisfies the PCP and G does not contain a subdivision of K_4 as a subgraph.

Also, an undirected graph is an *outerplanar* graph if it can be drawn in the plane without crossings in such a way that all of the vertices belong to the unbounded face of the drawing. There is a characterization for outerplanar graphs that says that a graph is outerplanar if and only if it does not contain a subdivision of K_4 or $K_{2,3}$.
Clearly, every outerplanar graph is a ring graph and every ring graph is a planar graph.

In the next two propositions, we characterize all rings R with ring graph and outerplanar Cayley graph.

Proposition 3.6. *The Cayley graph* $\mathrm{Cay}(I(R), I^*)$ *is a ring graph if and only if* R *satisfies one of the following conditions:*

(i) $R \cong F_1 \times F_2$, *where the* F_i *are fields;*
(ii) $(R, \mathfrak{m})$ *is a local ring,* $\dim_{\frac{R}{\mathfrak{m}}}(\mathfrak{m}) = 1$ *and* $I(R) = \{0, \mathfrak{m}, R\}$.

https://doi.org/10.1017/S144678871400007X Published online by Cambridge University Press

PROOF. Let $\mathrm{Cay}(I(R), I^*)$ be a ring graph. Since every ring graph is planar, it is enough to consider planar Cayley graphs. Thus, by Theorem 3.5, we have the following cases.

Case 1: $R \cong F_1 \times F_2$. Then one can easily check that the Cayley graph $\mathrm{Cay}(I(R), I^*)$ is a ring graph.

Case 2: $(R, \mathfrak{m})$ *is a local ring and* $I(R) = \{0, (x), (y), (x, y), R\}$. Then the Cayley graph has a subgraph isomorphic to K_4 with vertices $0, (x), (x, y), R$ and therefore the Cayley graph $\mathrm{Cay}(I(R), I^*)$ is not a ring graph.

Case 3: $(R, \mathfrak{m})$ *is a local ring and* $I(R) = \{0, \mathfrak{m}^2, \mathfrak{m}, R\}$. So, the Cayley graph is isomorphic to K_4. Thus, in this case, the Cayley graph cannot be a ring graph.

Case 4: $(R, \mathfrak{m})$ *is a local ring and* $I(R) = \{0, \mathfrak{m}, R\}$. Then the Cayley graph $\mathrm{Cay}(I(R), I^*)$ is a triangle, and hence it is a ring graph. □

PROPOSITION 3.7. *The Cayley graph* $\mathrm{Cay}(I(R), I^*)$ *is an outerplanar graph if and only if the Cayley graph* $\mathrm{Cay}(I(R), I^*)$ *is a ring graph.*

PROOF. Assume that the Cayley graph $\mathrm{Cay}(I(R), I^*)$ is a ring graph. So, $\mathrm{Cay}(I(R), I^*)$ is either a triangle or $R \cong F_1 \times F_2$, where the F_i are fields. It is clear that in both cases the graph is outerplanar.

Since every outerplanar graph is a ring graph, the converse statement is clear. □

4. Cayley graphs with genus one

We denote by S_g the surface formed by a connected sum of g tori. The number g is called the *genus* of the surface S_g. Recall that a simple graph which can be embedded in S_g but not in S_{g-1} is called a graph of genus g. The notation $\gamma(G)$ is denoted for the genus of a graph G. It is easy to see that $\gamma(H) \leq \gamma(G)$ for all subgraphs H of G. Also, a graph G is called *toroidal* if $\gamma(G) = 1$. Clearly, a graph G is planar if $\gamma(G) = 0$.

Recall that for a rational number q, $\lceil q \rceil$ is the first integer number greater than or equal to q. In the following two lemmas, we bring some well-known formulas for the genus of a graph (see [23, 24]).

LEMMA 4.1. *The following statements hold:*

(a) *for* $n \geq 3$, *we have* $\gamma(\mathrm{K}_n) = \lceil \frac{1}{12}(n-3)(n-4) \rceil$;
(b) *for* $m, n \geq 2$, *we have* $\gamma(\mathrm{K}_{m,n}) = \lceil \frac{1}{4}(m-2)(n-2) \rceil$.

According to Lemma 4.1, we have $\gamma(\mathrm{K}_n) = 0$, for $n = 3, 4$, $\gamma(\mathrm{K}_n) = 1$, for $n = 5, 6, 7$, and, for other values of n, $\gamma(\mathrm{K}_n) \geq 2$. Moreover, $\gamma(K_{4,4}) = \gamma(K_{3,m}) = 1$, if $m = 3, 4, 5, 6$.

LEMMA 4.2. *If* G *is a finite and connected graph with* n *vertices,* m *edges and genus* g, *then*

$$n - m + r = 2 - 2g,$$

where r *is the number of regions created when* G *is minimally embedded on a surface of genus* g.

Notation 4.3. We use $F_{i_1 i_2 \ldots i_t}$ instead of the ideal $F_1 \times \cdots \times F_s$ when all components are zero except for $F_{i_1}, F_{i_2}, \ldots, F_{i_t}$. For example, we use F_{134} for the set $F_1 \times 0 \times F_3 \times F_4 \times 0 \times 0$.

Proposition 4.4. *If the genus of the Cayley graph* $\mathrm{Cay}(I(R), I^*)$ *is finite, then* R *is an Artinian ring.*

Proof. Clearly, with every chain of length n, we can construct a complete graph isomorphic to K_{n+1}. So, if $\gamma(\mathrm{Cay}(I(R), I^*))$ is finite, then, by Lemma 4.1, the length of every chain is finite. Thus, every chain is stationary. □

In the following theorem, we provide a relation between the genus of the Cayley graph $\mathrm{Cay}(I(R), I^*)$ and the number of maximal ideals of R.

Proposition 4.5. *If* $|\mathrm{Max}(R)| = t$ *and* $\gamma(\mathrm{Cay}(I(R), I^*)) = g$ *is finite, then*

$$2t < 5(1 + \sqrt{2 + 2g}).$$

Proof. Let $\mathfrak{m}_1, \mathfrak{m}_2, \ldots, \mathfrak{m}_t$ be distinct maximal ideals of R. So, we have the following chain:

$$(\mathfrak{m}_1 \cap \mathfrak{m}_2 \cap \cdots \cap \mathfrak{m}_t) \subset (\mathfrak{m}_1 \cap \mathfrak{m}_2 \cap \cdots \cap \mathfrak{m}_{t-1}) \subset \cdots \subset (\mathfrak{m}_1 \cap \mathfrak{m}_2) \subset \mathfrak{m}_1 \subset R.$$

Hence, the Cayley graph $\mathrm{Cay}(I(R), I^*)$ has a subgraph isomorphic to K_{t+1}. So, by Lemma 4.1,

$$g \geq \gamma(K_{t+1}) = \lceil \tfrac{1}{12}((t+1) - 3)((t+1) - 4) \rceil.$$

Now clearly we have $12g > t^2 - 5t - 6$ and, with a simple calculation, we obtain $2t < 5(1 + \sqrt{2 + 2g})$. □

Theorem 4.6. *Let* R *be a ring with at least three maximal ideals. Then* $\gamma(\mathrm{Cay}(I(R), I^*)) = 1$ *if and only if* $R \cong F_1 \times F_2 \times F_3$, *where the* F_i *are fields.*

Proof. First, assume that the Cayley graph $\mathrm{Cay}(I(R), I^*)$ is toroidal. Then, by Proposition 4.5, one can easily check that $|\mathrm{Max}(R)| \leq 6$. Suppose that $|\mathrm{Max}(R)| = t$. Then we have the following cases.

Case 1: $t = 6$. Suppose that $\mathrm{Max}(R) = \{\mathfrak{m}_1, \mathfrak{m}_2, \ldots, \mathfrak{m}_6\}$ and $J(R) \neq 0$. Then, in view of Figure 3, we have a subgraph of $\mathrm{Cay}(I(R), I^*)$ isomorphic to K_8, with the following vertices:

$u_1 = 0$, $u_2 = \mathfrak{m}_1 \cap \mathfrak{m}_2 \cap \mathfrak{m}_3 \cap \mathfrak{m}_4 \cap \mathfrak{m}_6$, $u_3 = \mathfrak{m}_1$, $u_4 = \mathfrak{m}_1 \cap \mathfrak{m}_2 \cap \mathfrak{m}_3$,
$u_5 = J(R)$, $u_6 = \mathfrak{m}_1 \cap \mathfrak{m}_2 \cap \mathfrak{m}_3 \cap \mathfrak{m}_4$, $u_7 = R$, $u_8 = \mathfrak{m}_1 \cap \mathfrak{m}_2$.

Now, by Lemma 4.1, it is impossible because of the fact that the Cayley graph is toroidal.

Now, suppose that $J(R) = 0$. So, $R \cong F_1 \times F_2 \times F_3 \times F_4 \times F_5 \times F_6$, where the F_i are fields. In this situation, by considering the following vertices, we have a subdivision of K_8 in the Cayley graph $\mathrm{Cay}(I(R), I^*)$ (see Figure 4):

$a = 0$, $b = F_1$, $c = F_{12}$, $d = F_{123}$, $e = F_{1234}$,
$f = F_{1235}$, $g = F_{12345}$, $h = R$, $k = F_{13}$.

Again, by Lemma 4.1, it is impossible.

https://doi.org/10.1017/S144678871400007X Published online by Cambridge University Press

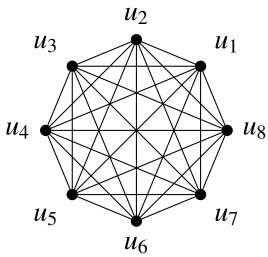

FIGURE 3. The complete graph K_8.

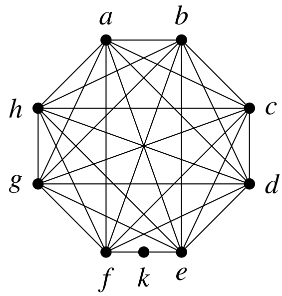

FIGURE 4. A subdivision of K_8.

Case 2: $t = 5$. Suppose that $J(R) \neq 0$. If $\mathrm{Max}(R) = \{\mathfrak{m}_1, \mathfrak{m}_2, \ldots, \mathfrak{m}_5\}$, then, by Figure 4, we have a subdivision of K_8 in the Cayley graph $\mathrm{Cay}(I(R), I^*)$ with the following vertices, which is impossible:

$$a = R, \quad b = \mathfrak{m}_1, \quad c = \mathfrak{m}_2\mathfrak{m}_1, \quad d = \mathfrak{m}_3\mathfrak{m}_2\mathfrak{m}_1, \quad e = \mathfrak{m}_4\mathfrak{m}_3\mathfrak{m}_2\mathfrak{m}_1,$$
$$k = \mathfrak{m}_1\mathfrak{m}_3, \quad f = \mathfrak{m}_5\mathfrak{m}_3\mathfrak{m}_2\mathfrak{m}_1, \quad g = \mathfrak{m}_5\mathfrak{m}_4\mathfrak{m}_3\mathfrak{m}_2\mathfrak{m}_1, \quad h = 0.$$

If $J(R) = 0$, then there exist fields $F_1, \ldots, F_5$ such that $R \cong F_1 \times \cdots \times F_5$. Hence, we have a subgraph isomorphic to $K_{5,5}$ with vertex set $\{a, b, c, d, e\} \cup \{a', b', c', d', e'\}$, where

$$a = F_1, \quad b = F_2, \quad c = F_3, \quad d = F_{12}, \quad e = F_{13},$$
$$a' = F_{123}, \quad b' = F_{1234}, \quad c' = F_{1235}, \quad d' = 0, \quad e' = R.$$

So, in this situation, the genus of the graph is at least two.

Case 3: $t = 4$. Suppose that $\mathrm{Max}(R) = \{\mathfrak{m}_1, \mathfrak{m}_2, \mathfrak{m}_3, \mathfrak{m}_4\}$. First, assume that $J(R) \neq 0$. So, we find a subdivision of $K_{4,6}$, which is pictured in Figure 5 with the following vertices, which is impossible:

$$a = \mathfrak{m}_1, \quad b = \mathfrak{m}_2, \quad c = \mathfrak{m}_3, \quad d = \mathfrak{m}_1\mathfrak{m}_2, \quad e = \mathfrak{m}_1\mathfrak{m}_3, \quad f = \mathfrak{m}_4,$$
$$a' = R, \quad b' = 0, \quad c' = J(R), \quad d' = \mathfrak{m}_1\mathfrak{m}_2\mathfrak{m}_3, \quad u = \mathfrak{m}_2\mathfrak{m}_3, \quad v = \mathfrak{m}_2\mathfrak{m}_3\mathfrak{m}_4.$$

https://doi.org/10.1017/S144678871400007X Published online by Cambridge University Press

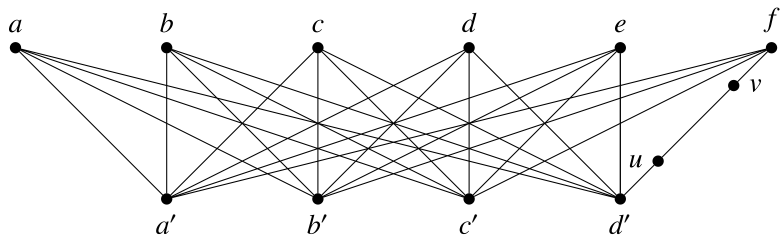

FIGURE 5. A subdivision of $K_{4,6}$.

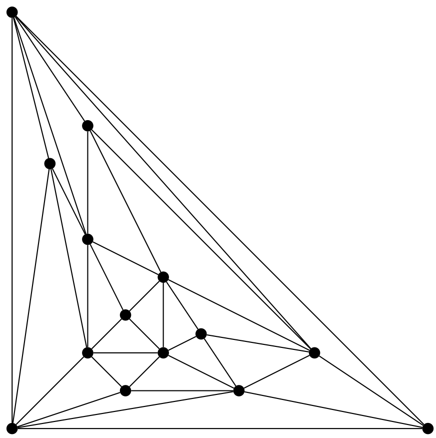

FIGURE 6.

Now, let $J(R) = 0$, and so there exist fields F_1, F_2, F_3, F_4 such that $R \cong F_1 \times F_2 \times F_3 \times F_4$. Therefore, the Cayley graph $\mathrm{Cay}(I(R), I^*)$ has 16 vertices. If we consider the induced subgraph G with vertex set $I(R) \setminus \{R\}$, then G has 15 vertices, 50 edges and without the vertex 0, as is shown in Figure 6, it has 23 faces. Now, by counting the triangles in G which contain the zero ideal as a vertex, we find that G has at least 59 faces and, by Lemma 4.2, we have $15 - 50 + 59 \neq 0$. So, G does not have genus one. Since G has a subgraph isomorphic to $K_{3,3}$, we have $\gamma(G) \geq 2$, and hence $\gamma(\mathrm{Cay}(I(R), I^*)) \geq 2$, which is impossible.

Case 4: $t = 3$. Suppose that $J(R) = 0$. Then $R \cong F_1 \times F_2 \times F_3$, and therefore $I(R) = \{0, F_1, F_2, F_3, F_{12}, F_{13}, F_{23}, R\}$. Therefore, by Figure 7, we have that $\mathrm{Cay}(I(R), I^*)$ is toroidal.

Now, suppose that $J(R) \neq 0$. Let I be a nonzero ideal of R such that $I \subseteq J(R)$ and $\mathfrak{m}_1, \mathfrak{m}_2, \mathfrak{m}_3$ be distinct maximal ideals of R. If $I \neq J(R)$, then we find a subgraph

https://doi.org/10.1017/S144678871400007X Published online by Cambridge University Press

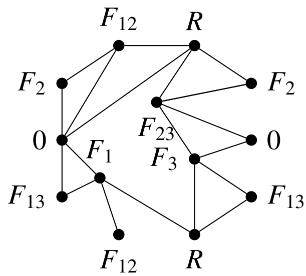

FIGURE 7.

isomorphic to $K_{4,6}$ with the following vertices, which is impossible:

$$v_1 = 0, \quad v_2 = J(R), \quad v_3 = I, \quad v_4 = R, \quad u_1 = \mathfrak{m}_1,$$
$$u_2 = \mathfrak{m}_2, \quad u_3 = \mathfrak{m}_3, \quad u_4 = \mathfrak{m}_1\mathfrak{m}_2, \quad u_5 = \mathfrak{m}_1\mathfrak{m}_3, \quad u_6 = \mathfrak{m}_2\mathfrak{m}_3.$$

Thus, we have $I = J(R)$. In particular, $J(R)^2 = J(R)$ or $J(R)^2 = 0$. If $J(R)^2 = J(R)$, then, by Nakayama's lemma, we have $J(R) = 0$, which is impossible. So, $J(R)^2 = \mathfrak{m}_1^2\mathfrak{m}_2^2\mathfrak{m}_3^2 = 0$. Now, if, for every $i = 1, 2, 3$, we have $\mathfrak{m}_i^2 \neq \mathfrak{m}_i$, then $\mathfrak{m}_i^2\mathfrak{m}_j^2\mathfrak{m}_k = J(R)$ or 0. Since R is an Artinian ring, the set of all maximal ideals of R is exactly the set of all minimal ideals of R. Therefore, by the second uniqueness decomposition theorem, we get a contradiction. Therefore, at least one of the maximal ideals is equal to its square, say $\mathfrak{m}_3^2 = \mathfrak{m}_3$. Now, by a similar method, we find that $\mathfrak{m}_i^2 = \mathfrak{m}_i$, for some $i = 1$ or 2. So, we may assume that $\mathfrak{m}_2^2 = \mathfrak{m}_2$. Since $J(R)^2 \neq J(R)$, we have $0 = \mathfrak{m}_1^2 \cap \mathfrak{m}_2 \cap \mathfrak{m}_3$. Thus, $R \cong R/\mathfrak{m}_1^2 \times R/\mathfrak{m}_2 \times R/\mathfrak{m}_3$, and so $J(R) \cong J(R/\mathfrak{m}_1^2) \times J(R/\mathfrak{m}_2) \times J(R/\mathfrak{m}_3)$. Hence, we have $\mathfrak{m}_1\mathfrak{m}_2\mathfrak{m}_3 \cong \mathfrak{m}_1 \times 0 \times 0$, which is impossible. So, if R is a ring with three maximal ideals and nonzero Jacobson radical, then $\mathrm{Cay}(I(R), I^*)$ is not toroidal.

The converse statement is clear. □

Acknowledgement

The authors would like to thank the referee for interest in the subject and careful reading of the article.

References

[1] M. Afkhami, M. R. Ahmadi, R. Jahani-Nezhad and K. Khashyarmanesh, 'Cayley graphs of ideals in a commutative ring', *Bull. Malays. Math. Sci. Soc.*, to appear.
[2] M. Afkhami, M. Karimi and K. Khashyarmanesh, 'On the regular digraph of ideals of commutative rings', *Bull. Aust. Math. Soc.* **88** (2013), 177–189.
[3] M. Afkhami, K. Khashyarmanesh and Kh. Nafar, 'Generalized Cayley graphs associated to commutative rings', *Linear Algebra Appl.* **437** (2012), 1040–1049.
[4] S. Akbari, H. R. Maimani and S. Yassemi, 'When a zero-divisor graph is planar or a complete r-partite graph', *J. Algebra* **270** (2003), 169–180.

https://doi.org/10.1017/S144678871400007X Published online by Cambridge University Press

[5] D. D. Anderson and M. Naseer, ‘Beck’s coloring of a commutative ring’, *J. Algebra* **159** (1993), 500–514.
[6] D. F. Anderson and A. Badawi, ‘On the zero-divisor graph of a ring’, *Comm. Algebra* **36** (2008), 3073–3092.
[7] D. F. Anderson and P. S. Livingston, ‘The zero-divisor graph of a commutative ring’, *J. Algebra* **217** (1999), 434–447.
[8] I. Beck, ‘Coloring of commutative rings’, *J. Algebra* **116** (1998), 208–226.
[9] J. A. Bondy and U. S. R. Murty, *Graph Theory with Applications* (American Elsevier, New York, 1976).
[10] I. Chakrabarty, S. Ghosh, T. K. Mukherjee and M. K. Sen, ‘Intersection graphs of ideals of rings’, *Discrete Math.* **309** (2009), 5381–5392.
[11] I. Gitler, E. Reyes and R. H. Villarreal, ‘Ring graphs and complete intersection toric ideals’, *Discrete Math.* **310** (2010), 430–441.
[12] A. V. Kelarev, ‘Directed graphs and nilpotent rings’, *J. Aust. Math. Soc.* **65** (1998), 326–332.
[13] A. V. Kelarev, ‘On undirected Cayley graphs’, *Australas. J. Combin.* **25** (2002), 73–78.
[14] A. V. Kelarev, *Ring Constructions and Applications* (World Scientific, River Edge, NJ, 2002).
[15] A. V. Kelarev, *Graph Algebras and Automata* (Marcel Dekker, New York, 2003).
[16] A. V. Kelarev, ‘Labelled Cayley graphs and minimal automata’, *Australas. J. Combin.* **30** (2004), 95–101.
[17] A. V. Kelarev, ‘On Cayley graphs of inverse semigroups’, *Semigroup Forum* **72** (2006), 411–418.
[18] A. V. Kelarev and C. E. Praeger, ‘On transitive Cayley graphs of groups and semigroups’, *European J. Combin.* **24** (2003), 59–72.
[19] A. V. Kelarev and S. J. Quinn, ‘A combinatorial property and power graphs of groups’, *Contrib. Gen. Algebra* **12** (2000), 229–235.
[20] A. V. Kelarev and S. J. Quinn, ‘Directed graphs and combinatorial properties of semigroups’, *J. Algebra* **251** (2002), 16–26.
[21] A. V. Kelarev, J. Ryan and J. Yearwood, ‘Cayley graphs as classifiers for data mining: the influence of asymmetries’, *Discrete Math.* **309** (2009), 5360–5369.
[22] M. J. Nikmehr and F. Shaveisi, ‘The regular digraph of ideals of a commutative ring’, *Acta Math. Hungar.* **134** (2012), 516–528.
[23] A. T. White, ‘The genus of the Cartesian product of two graphs’, *J. Combin. Theory Ser.* B **11** (1971), 89–94.
[24] A. T. White, *Graphs, Groups and Surfaces* (North-Holland, Amsterdam, 1973).

M. AFKHAMI, Department of Mathematics, University of Neyshabur,
PO Box 91136-899, Neyshabur, Iran
e-mail: mojgan.afkhami@yahoo.com

Z. BARATI, Department of Mathematics, Kosar University of Bojnord,
PO Box 9415615458, Bojnord, Iran
e-mail: za.barati87@gmail.com

K. KHASHYARMANESH, Department of Pure Mathematics,
Ferdowsi University of Mashhad, PO Box 1159-91775, Mashhad, Iran
e-mail: khashyar@ipm.ir

N. PAKNEJAD, Department of Pure Mathematics,
Ferdowsi University of Mashhad, PO Box 1159-91775, Mashhad, Iran
e-mail: paknejad.n@gmail.com