Canad. Math. Bull. Vol. **44** (1), 2001 pp. 93–96

Some Semigroup Laws in Groups

Dedicated to H. S. M. Coxeter in old friendship

B. H. Neumann

Abstract. A challenge by R. Padmanabhan to prove by group theory the commutativity of cancellative semigroups satisfying a particular law has led to the proof of more general semigroup laws being equivalent to quite simple ones.

1 Introduction

In 1998, on a visit to the University of Manitoba, I was challenged by Professor Padmanabhan to prove by group-theoretical methods a result on cancellative semigroups, namely that the law

$$x^2 \cdot y^2 \cdot x = y \cdot x^3 \cdot y \tag{1}$$

implies commutativity. This is indeed possible, and not too difficult. I here investigate some generalisations of Padmanabhan's law, and find the varieties of groups defined by them. The simplest example is the law

$$x^3 \cdot y^2 \cdot x = y \cdot x^4 \cdot y, \tag{2}$$

which implies $x^2 \cdot y = y \cdot x^2$. The variety defined by this law contains that generated by the quaternion group, or, equivalently, by the dihedral group of order 8, and indeed it contains all hamiltonian groups. It turns out that these two examples are typical of the general law I consider, namely

$$x^{s+t} \cdot y^2 \cdot x^t = y \cdot x^{s+2t} \cdot y \tag{3}$$

Padmanabhan's law is (3) with $s = t = 1$, and my slight generalisation is (3) with $s = 2, t = 1$.

I use standard notation: e is the unit element, and

$$[x, y] := x^{-1} \cdot y^{-1} \cdot x \cdot y, \quad x^y := y^{-1} \cdot x \cdot y;$$

and standard identities like

$$[x \cdot y, z] = [x, z]^y \cdot [y, z], \quad [x^{-1}, y] = [y, x]^{x^{-1}}$$

are used without explicit reference. The final results require lengthy calculations.

Received by the editors January 22, 1999; revised April 6, 2000.
I thank Professors C. Kanta Gupta and Narain D. Gupta, FsRSC, for arranging this visit to the University of Manitoba and for their generous hospitality. I am also grateful to Professor J. McCool for spotting a faulty argument in an earlier version of this paper, and thus giving me the opportunity to prove and improve the results.
AMS subject classification: Primary: 20E10; secondary: 20M07.

https://doi.org/10.4153/CMB-2001-011-9 Published online by Cambridge University Press

2 The Calculations

I put

$$u := x^s, \quad v := x^t,$$

so that (3) now becomes

(4) $$u \cdot v \cdot y^2 \cdot v = y \cdot v \cdot u \cdot v \cdot y$$

or

(5) $$[u \cdot v \cdot y, y \cdot v] = e.$$

Putting $y = e$ in this shows that

(6) $$u \cdot v = v \cdot u,$$

that is to say, u and v commute; as indeed they should from their definition as powers of the same element; however, in (4) they appear just as apparently independent variables. The commutativity (6) will be used frequently. Expanding (5),

$$e = [u \cdot v \cdot y, v] \cdot [u \cdot v \cdot y, y]^v = [u \cdot v, v]^y \cdot [y, v] \cdot [u \cdot v \cdot y, y]^v$$

and, using (6) and further expanding:

$$e = [y, v] \cdot [u \cdot v, y]^{y \cdot v} \cdot [y, y]^v,$$

giving

(7) $$[u \cdot v, y]^{y \cdot v} = [v, y].$$

Next I introduce $z := v \cdot y$, so that (5) becomes

$$e = [u \cdot z, z^v] = z^{-1} \cdot u^{-1} \cdot v^{-1} \cdot z^{-1} \cdot v \cdot u \cdot z \cdot v^{-1} \cdot z \cdot v,$$

$$e = [v \cdot u, z]^z \cdot [z, v],$$

giving, again using (6)

$$[u \cdot v, z]^z = [v, z].$$

Replacing z by y leads to

(8) $$[u \cdot v, y]^y = [v, y].$$

Comparison of (7) and (8) yields, first,

$$[v, y]^v = [v, y]$$

or

$$[v, y, v] = e,$$

that is to say:

Lemma 9 *The variable v commutes with its commutators and its conjugates.*

https://doi.org/10.4153/CMB-2001-011-9 Published online by Cambridge University Press

Next, expanding (8):

$$[u,y]^{v\cdot y}\cdot[v,y]^{y}=[v,y],$$

$$[u,y]^{v\cdot y}=[v,y]\cdot[y\cdot v]^{y},$$

$$[u,y]^{v}=[v,y]^{y^{-1}}\cdot[y,v]=[y^{-1},v]\cdot[y,v].$$

Now $[u,y]^{v}$ is seen to be a product of v-commutators, thus commutes with v, so

$$[u,y]=[y^{-1},v]\cdot[y,v]. \tag{10}$$

Replacing y by y^{-1}, then

$$[u,y^{-1}]=[y,v]\cdot[y^{-1},v],$$

and, using the fact that v-commutators commute with each other [being products of conjugates of v and its inverse],

$$[y,u]^{y^{-1}}=[y^{-1},v]\cdot[y,v]=[u,y].$$

Thus

$$[u,y]^{y}=[y,u]=[u,y]^{-1},$$

$$[u,y^{2}]=[u,y]^{y}\cdot[u,y]=e:$$

Lemma 11 *The commutator $[u,y]$ is inverted by transformation by y, and u commutes with all squares.*

I apply this fact to (10):

$$[u,y]^{y}=[u,y]^{-1}=[v,y]\cdot[v,y^{-1}]=[v,y]\cdot[y,v]^{y^{-1}}.$$

Transforming by y again, this gives

$$[u,y]=[v,y]^{y}\cdot[y,v],$$

and comparison with (10) yields

$$[v,y]^{y}=[y^{-1},v]=[v,y]^{y^{-1}},$$

and

$$[v,y]^{y^{2}}=[v,y].$$

Lemma 12 *The commutator $[v,y]$ commutes with y^{2}.*

https://doi.org/10.4153/CMB-2001-011-9 Published online by Cambridge University Press

3 Consequences

As v commutes with its commutators and with the u commutators [as noted above, $[u, y]$ is a product of conjugates of v], and as u commutes with all squares, and thus with all commutators, a routine argument shows that the element $u^k \cdot v^l$, for integers k and l, commutes with all commutators. Now k and l can be so chosen that

$$u^k \cdot v^l = x^r,$$

where $r := (s, t)$. This gives the result:

Theorem 13 *The law*

$$x^{s+t} \cdot y^2 \cdot x^t = y \cdot x^{s+2t} \cdot y \tag{3}$$

implies the law

$$[x^r, y, x^r] = e, \tag{14}$$

where r is the greatest common divisor of s and t. Thus r-th powers commute with all their conjugates.

The most interesting case is that of coprime s and t, that is, $r = 1$. Then (14) becomes

$$[x, y, x] = e, \tag{15}$$

which is the Engel law and implies that all elements commute with their conjugates,

In particular, as $x^t \cdot y$ is conjugate to $y \cdot x^t$,

$$x^t \cdot y^2 \cdot x^t = y \cdot x^{2t} \cdot y. \tag{16}$$

Applying this to

$$x^{s+t} \cdot y^2 \cdot x^t = y \cdot x^{s+2t} \cdot y \tag{3}$$

gives

$$x^s \cdot y \cdot x^{2t} \cdot y = y \cdot x^{s+2t} \cdot y,$$

and cancelling $x^{2t} \cdot y$ on the right gives the main result:

Theorem 17 *If in the law*

$$x^{s+t} \cdot y^2 \cdot x^t = y \cdot x^{s+2t} \cdot y \tag{3}$$

the exponent s is coprime to t, then s-th powers are central.

Padmanabhan's challenge is the case $s = 1$, and the slight generalisation of Padmanabhan's problem mentioned in the introduction is the special case $s = 2$ of this. Finally it should be remarked that the centrality of s-th powers can be viewed as a semigroup law.

School of Mathematical Sciences
Australian National University
ACT 0200
Australia