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SUMMABILITY OF MATRIX TRANSFORMS OF SUBSEQUENCES

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ABSTRACT. D. F. Dawson has considered several questions of the following nature. Suppose T is a regular matrix summability method. If A is a regular matrix and x is a sequence having a finite limit point, then there exists a subsequence y of x such that each finite limit point of x is a T-limit point of Ay. In the present paper, we show the regularity condition for A may be replaced by the requirement that A be a limit preserving bv to c map. This leads to summability characterizations for several classes of sequences.

Following Dawson [4], we say the matrix A is semiregular if A is regular over the set of all convergent sequences of 0's and 1's. Thus $A=(a_{pq})$ is semiregular if and only if it satisfies the first two of the following three conditions for regularity:

$$(1)\quad \lim_p a_{pq}=0 \quad \text{for all } q,$$

$$(2)\quad \lim_p \sum_q a_{pq}=1,$$

and

$$(3)\quad \sup_p \sum_q |a_{pq}|<\infty.$$

In [8], this author proved that the matrix A is semiregular if and only if for each sequence x with finite limit point σ there exists a subsequence y of x such that the A-limit of y is σ. As an immediate consequence, we have the following results.

THEOREM 1. *Suppose T is a regular summability matrix. If A is a semiregular matrix and x is a sequence with finite limit point σ, then there exists a subsequence y of x such that Ay is T-summable to σ.*

COROLLARY 2. *The sequence x diverges to ∞ if there exist a regular matrix T and a semiregular matrix A such that $T(Ay)$ diverges to ∞ for every subsequence y of x.*

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In [9], we showed that if the matrix A is semiregular and x is a sequence having a finite limit point, then there exists a subsequence y of x such that each limit point of x is an A-limit point of y. Dawson [4] has proved that if T and A are regular and x is a sequence with a finite limit point, then there exists a subsequence y of x such that each finite limit point of x is a T-limit point of Ay. An analog to Theorem 1 might be expected which would provide that if T is a regular matrix, A is a semiregular matrix, and x is a sequence with a finite limit point, then there exists a subsequence y of x such that each finite limit point of x is a T-limit point of Ay. Such an analog fails to be true.

The following example provides a regular matrix T and a semiregular matrix A such that for each subsequence y of $x=(0, 1, 0, 1, 0, 1, \ldots)$, $T(Ay)$ either fails to exist or Ay is T-summable to 0 or 1. Let $t_{1q}=1/2^q$ for each q, $t_{pp}=1$ for $p>1$, and $t_{pq}=0$ otherwise. Let $a_{pp}=2^p$ and $a_{p,p+1}=1-2^p$ for all p, and $a_{pq}=0$ otherwise. If y is a convergent subsequence of x, then y is eventually constant, hence so is $T(Ay)$. But whenever $y_p=1$ and $y_{p+1}=0$, then $(Ay)_p=2^p$ and $t_{1p}(Ay)_p=1$. Therefore if y is not eventually constant, $T(Ay)$ fails to exist.

It is possible to use Theorem 1 of [9] and the associative property to obtain the following special case.

THEOREM 3. *Suppose T is a row finite regular summability matrix. If A is a row finite semiregular matrix and x is a sequence with a finite limit point, then there exists a subsequence y of x such that each limit point of x (finite or infinite) is a T-limit point of Ay.*

The above example illustrates the necessity of strengthening the requirement of semiregularity on A if row-finiteness is not assumed in Theorem 3. This may be accomplished by requiring A to be a limit preserving bv to c map.

THEOREM 4. *Suppose T is a regular summability matrix. If A is a limit preserving bv to c map and x is a sequence with a finite limit point, then there exists a subsequence y of x such that each finite limit point of x is a T-limit point of Ay.*

The sequence x is an element of bv if $\sum_n |x_n - x_{n+1}| < \infty$. The matrix A is said to be a limit preserving bv to c map if $\lim_p (Ax)_p = \lim_n x_n$ whenever x is in bv. Such matrices may be characterized by the three conditions [7]

$$\text{(A)}\quad \lim_p a_{pq} = 0 \quad \text{for all } q,$$

$$\text{(B)}\quad \lim_p \sum_q a_{pq} = 1,$$

and

$$\text{(C)}\quad \sup_{p,n} \left| \sum_{q=1}^{n} a_{pq} \right| < \infty.$$

If T is regular and A is a limit preserving bv to c map, then TA is necessarily semiregular. Theorem 4 thus follows as an immediate consequence of Theorem 1 in [9] if associativity is assumed. The treatment presented here is without any benefit of associativity.

The proof of the following form of Theorem 1 in [8] will be omitted.

LEMMA 5. *The matrix A is semiregular if and only if for each sequence x with finite limit point σ and for each $\varepsilon>0$ there exists a subsequence y of x such that the A-limit of y is σ and*

$$\sup_{p,n,m}\left|\sum_{q=n}^{m} a_{pq}y_q-\sigma\sum_{q=n}^{m} a_{pq}\right|<\varepsilon.$$

Proof of Theorem 4. Using the separability of the complex plane we find a sequence u such that each finite limit point of x is either a term of u or a limit point of u. Let $V=\{u_1; u_1, u_2; u_1, u_2, u_3; \ldots\}$ and $K=\sup_{p,k}|\sum_{q=k}^{\infty} a_{pq}|$. It follows that $\sup_{p,k,m}|\sum_{q=k}^{m} a_{pq}|\le 2K$. By Lemma 5 there exists a subsequence $y_1=\{y(1,n)\}_{n=1}^{\infty}$ of x such that $\lim_p (Ay_1)_p=V_1$ and

$$\sup_{p,n,m}\left|\sum_{q=n}^{m} a_{pq}y(1,q)-V_1\sum_{q=n}^{m} a_{pq}\right|<1,$$

hence

$$\sup_{p,m}\left|\sum_{q=1}^{m} a_{pq}y(1,q)\right|\le(2K|V_1|+1). \tag{1}$$

Let $p(1)\ge 1$ such that $|[T(Ay_1)]_{p(1)}-V_1|<\frac{1}{4}$ and $q(1)\ge 1$ such that

$$\left|\sum_{q=1}^{q(1)} t_{p(1),q}(Ay_1)_q-V_1\right|<\tfrac{1}{4} \tag{2}$$

and

$$\left[1+\sum_{i=1}^{2}(2K|V_i|+1)\right]\sum_{q=q(1)+1}^{\infty}|t_{pq}|<\tfrac{1}{64} \tag{3}$$

for all $p\le p(1)$. Let $k(1)\ge 1$ such that

$$\left|\sum_{q=1}^{q(1)} t_{p(1),q}\sum_{i=1}^{k(1)} a_{qi}y(1,i)-\sum_{q=1}^{q(1)} t_{p(1),q}(Ay_1)_q\right|<\tfrac{1}{8}, \tag{4}$$

$$|V_2|\sup_{q\le q(1),m}\left|\sum_{i=k(1)+1}^{m} a_{qi}\right|<\tfrac{1}{2}, \tag{5}$$

and

$$|V_2|\sum_{q=1}^{q(1)}\sup_{p\le p(1),m}|t_{pq}|\left|\sum_{i=k(1)+1}^{m} a_{qi}\right|<\tfrac{1}{32}. \tag{6}$$

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Define the first $k(1)$ terms of the subsequence $y=\{y(n)\}_{n=1}^{\infty}$ by $y(i)=y(1,i)$ for $1\le i\le k(1)$. Let $r(1)\ge 1$ such that $y(k(1))$ is the $r(1)$ term of x. Since A is semiregular, the submatrix A_1 of A formed by deleting the first $k(1)$ columns of A is also semiregular. By Lemma 5 there exists a subsequence $z_2=\{z(2,n)\}_{n=k(1)+1}^{\infty}$ of $\{x_n\}_{n=r(1)+1}^{\infty}$ such that $\lim_p(A_1z_2)_p=V_2$ and

$$\sup_{p,k(1)<n,m}\left|\sum_{q=n}^{m} a_{pq}z(2,q)-V_2\sum_{q=n}^{m} a_{pq}\right|<1, \tag{7}$$

hence

$$\sup_{p,m}\left|\sum_{q=k(1)+1}^{m} a_{pq}z(2,q)\right|<(2K|V_2|+1). \tag{8}$$

By (5) and (6), z_2 may also be selected such that

$$\sup_{q\le q(1),m}\left|\sum_{i=k(1)+1}^{m} a_{qi}z(2,i)\right|<\tfrac{1}{2} \tag{9}$$

and

$$\sum_{q=1}^{q(1)}\sup_{p\le p(1),m}|t_{pq}|\left|\sum_{i=k(1)+1}^{m} a_{qi}z(2,i)\right|<\tfrac{1}{32}. \tag{10}$$

Let $y_2=\{y(2,n)\}_{n=1}^{\infty}$ be the subsequence of x determined by $y(2,n)=y(n)$ for $1\le n\le k(1)$ and $y(2,n)=z(2,n)$ otherwise. Since each column of A is null, $\lim_p(Ay_2)_p=V_2$.

Let $p(2)>p(1)$ such that $|[T(Ay_2)]_{p(2)}-V_2|<\frac{1}{8}$ and $q(2)>q(1)$ such that

$$\left|\sum_{q=1}^{q(2)} t_{p(2),q}(Ay_2)_q-V_2\right|<\tfrac{1}{8} \tag{11}$$

and

$$\left[\sum_{i=1}^{3}(2K|V_i|+1)\right]\sum_{q=q(2)+1}^{\infty}|t_{pq}|<\tfrac{1}{256} \tag{12}$$

for all $p\le p(2)$. Let $k(2)>k(1)$ such that

$$\left|\sum_{q=1}^{q(2)} t_{p(2),q}\sum_{i=1}^{k(2)} a_{qi}y(2,i)-\sum_{q=1}^{q(2)} t_{p(2),q}(Ay_2)_q\right|<\tfrac{1}{16}, \tag{13}$$

$$|V_3|\sup_{q\le q(2),m}\left|\sum_{i=k(2)+1}^{m} a_{qi}\right|<\tfrac{1}{4}, \tag{14}$$

and

$$|V_3|\sum_{q=1}^{q(2)}\sup_{p\le p(2),m}|t_{pq}|\left|\sum_{i=k(2)+1}^{m} a_{qi}\right|<\tfrac{1}{128}. \tag{15}$$

Let $y(i)=y(2,i)$ for $k(1)<i\le k(2)$ and let $r(2)>r(1)$ such that $y(k(2))$ is the $r(2)$ term of x. Let A_2 denote the submatrix of A formed by deleting the first $k(2)$ columns of A. By Lemma 5 there exists a subsequence $z_3=\{z(3,n)\}_{n=k(2)+1}^{\infty}$ of $\{x_n\}_{n=r(2)+1}^{\infty}$ such that $\lim_p (A_2z_3)_p=V_3$ and

$$(16)\qquad \sup_{p,k(2)<n,m}\left|\sum_{q=n}^{m} a_{pq}z(3,q)-V_3\sum_{q=n}^{m} a_{pq}\right|<1,$$

hence

$$(17)\qquad \sup_{p,m}\left|\sum_{q=k(2)+1}^{m} a_{pq}z(3,q)\right|<(2K\,|V_3|+1).$$

By (14) and (15), z_3 may also be selected such that

$$(18)\qquad \sup_{q\le q(2),m}\left|\sum_{i=k(2)+1}^{m} a_{qi}z(3,i)\right|<\tfrac{1}{4},$$

and

$$(19)\qquad \sum_{q=1}^{q(2)}\sup_{p\le p(2),m}|t_{pq}|\left|\sum_{i=k(2)+1}^{m} a_{qi}z(3,i)\right|<\tfrac{1}{128}.$$

This selection process may be continued such that

$$\begin{aligned}\left|\sum_{q=1}^{\infty} t_{p(1),q}(Ay)_q-V_1\right| &\le \left|\sum_{q=1}^{q(1)} t_{p(1),q}(Ay_1)_q-V_1\right|\\ &\quad+\left|\sum_{q=1}^{q(1)} t_{p(1),q}\left(\sum_{i=1}^{k(1)} a_{qi}y(1,i)\right)-\sum_{q=1}^{q(1)} t_{p(1),q}(Ay_1)_q\right|\\ &\quad+\sum_{j=1}^{\infty}\sum_{q=1}^{q(1)}|t_{p(1),q}|\left|\sum_{i=k(j)+1}^{k(j+1)} a_{qi}y(i)\right|+\left|\sum_{q=q(1)+1}^{\infty} t_{p(1),q}(Ay)_q\right|\\ &<\tfrac{1}{4}+\tfrac{1}{8}+\tfrac{1}{16}+\left|\sum_{q=q(1)+1}^{\infty} t_{p(1),q}(Ay)_q\right|\end{aligned}$$

by (2), (4), (10), and (19), where (10) and (19) constitute the cases $j=1$ and $j=2$ respectively. But for all $p\le p(1)$

$$\begin{aligned}\left|\sum_{q=q(1)+1}^{\infty} t_{pq}(Ay)_q\right| &\le \sum_{i=1}^{\infty}\sum_{q=q(i)+1}^{q(i+1)}|t_{pq}|\,|(Ay)_q|\\ &\le \sum_{i=1}^{\infty}\sum_{q=q(i)+1}^{q(i+1)}|t_{pq}|\left[\sum_{j=1}^{i+1}(2K\,|V_j|+1)+\left|\sum_{j=k(i+1)+1}^{\infty} a_{qj}y(j)\right|\right]\\ &<\tfrac{1}{32}\end{aligned}$$

where the second inequality follows from the pattern established by (1), (8), and (17), and the third inequality follows from the pattern established by (3) and (12) since $\left|\sum_{j=k(i+1)+1}^{\infty} a_{qj}y(j)\right|<1$ whenever $q<q(i+1)$ by the pattern

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established in (9) and (18). It follows that $|[T(Ay)]_{p(1)} - V_1| < \frac{1}{2}$ and $(T(Ay))_p$ converges for all $p \le p(1)$. Similar arguments show $|[T(Ay)]_{p(i)} - V_i| < 2^{-i}$ for each i and $[T(Ay)]_p$ exists for all p. Thus the proof is complete.

The form of Theorem 4 was chosen in order to simplify the details of the proof. Actually a slightly more general result may be obtained using the basic structure of the above proof and requiring A to be a semiregular matrix having the property that there exists an increasing sequence of positive integers $\{q(i)\}_{i=1}^{\infty}$ such that $\sup_i |\sum_{q=q(i)}^{\infty} a_{pq}|$ is finite.

THEOREM 6. *The sequence x is bounded if there exist a regular matrix T and a semiregular matrix A such that $T(Ay)$ is bounded for every subsequence y of x.*

Proof. If x is unbounded, then by an argument contained in the proof of Theorem 1 in [4] both T and A must be row-finite, hence $T(Ay) = (TA)y$. But this implies TA is a row-finite semiregular matrix, and the proof follows from Theorem 4 of [5].

COROLLARY 7. *The sequence x converges if there exist a regular matrix T and a limit preserving bv to c map A such that $T(Ay)$ converges for every subsequence y of x.*

Proof. The sequence x must be bounded by Theorem 6. Convergence then follows by Theorem 4.

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