BULL. AUSTRAL. MATH. SOC. 58C20, 46G05
VOL. 46 (1992) [127–138]

CONVEX SPACES: CLASSIFICATION BY DIFFERENTIABLE CONVEX FUNCTIONS

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The differentiability, of a specified strength, of a convex function at a point, is shown to be characterised by the convergence of subdifferentials in the appropriate topology on the dual space. This is used to prove that if each gauge is *densely* differentiable then so is each convex function. The *generic* version of this is equivalent to a conjecture which, for Gateaux differentiability and Banach spaces, is the long standing open question of whether $X \times \mathbb{R}$ is Weak Asplund whenever X is. Some progress is made towards a resolution.

0. INTRODUCTION

In this paper we continue our study of the classification of locally convex spaces as differentiability spaces. Our intention is both to synthesise and extend older ideas.

For a class β of bounded subsets of a locally convex space X, a real valued function f is said to be β differentiable at a point of X whenever the usual limit exists, is linear and continuous, and converges uniformly over β subsets of X.

We show that, if f is convex, β differentiability at a point is characterised by the convergence of subdifferentials in the topology on X^* generated by β. This implies, for example, that a continuous convex function f defined on an open subset of a Banach space is Fréchet (respectively Gateaux) differentiable at a point x if and only if there is a selection for the subdifferential map which is norm to norm (respectively norm to weak*) continuous at x. (See, for example, [**1**, Lemma 5] and [**9**, Proposition 2.8]).

The classification of locally spaces according to the dense or generic differentiability of convex functions continues work of Asplund [**1**], Larman and Phelps [**7**] and Namioka and Phelps [**8**].

Received 8th August 1991
Some of this work is contained in Bernice Sharp's doctoral thesis, supervisor Prof. J.R. Giles, University of Newcastle, NSW. The authors would also like to thank Prof. I. Namioka and Dr. S. Fitzpatrick for helpful comments.

1. Preliminaries

The term *function* is used for a *real valued* map. For a topological linear space X and an open convex subset D of X, a function f on D is said to be *convex* whenever for all $x, y \in D$ and for all $\lambda \in [0,1]$,

$$f(\lambda x + (1-\lambda)y) \leqslant \lambda f(x) + (1-\lambda) f(y).$$

We will assume that the domain of a convex function is nonempty, open and convex.

A spectrum of derivatives of f at x is defined as follows. Let X^* denote the continuous dual of a topological linear space X, let U be an open subset of X and let β be a *bornology* on X, that is, a class of bounded subsets containing all singletons.

A function f on U is β *differentiable* at $x \in U$ whenever there exists $u \in X^*$ such that, for all $M \in \beta$, for all $\varepsilon > 0$, there exists $\delta > 0$, such that for all $y \in M$, for all t such that $|t| \in (0, \delta)$,

$$\left| \frac{f(x+ty) - f(x)}{t} - u(y) \right| < \varepsilon.$$

The function u is denoted by $f'(x)$. If β is the class of all *bounded* (*singleton*) subsets of X then f is *Fréchet* (*Gateaux*) differentiable at x; if X is a normed space these definitions coincide with the usual ones. For the purposes of differentiation, there is no loss of generality in assuming that the sets in β are balanced, and we shall do so.

If f is a continuous *convex* function, to prove that f is β differentiable at x, it suffices to show *either* that, for all $M \in \beta$, for all $\varepsilon > 0$, there exists $\delta > 0$, such that for all $y \in M$, for all $t \in (0, \delta)$,

$$0 \leqslant f(x+ty) + f(x-ty) - 2f(x) < t\varepsilon,$$

or that there exists $u \in X^*$ such that for all $M \in \beta$, for all $\varepsilon > 0$, there exists $\delta > 0$, such that for all $y \in M$, for all $t \in (0, \delta)$,

$$0 \leqslant f(x+ty) - f(x) - u(ty) < t\varepsilon.$$

If β is a bornology on X, we denote by $\mathcal{T}_\beta$ the topology on X^* defined by uniform convergence over sets of β. For example if β is the family of finite, compact, convex balanced weakly compact, or bounded sets, then $\mathcal{T}_\beta$ will be respectively the weak*, compact-open, Mackey or strong topology on X^*. Bornologies β_1 and β_2 on X are said to be *equivalent*, written $\beta_1 \equiv \beta_2$, if $\mathcal{T}_{\beta_1} = \mathcal{T}_{\beta_2}$.

The following property of convex functions is proved under more general conditions by Borwein.

1.1. [**2**, Corollary 2.4] *Let f be a convex function with domain D in a locally convex space X. Suppose that for $x \in D$, U is a convex balanced neighbourhood of 0 contained in $D - x$, $r > 0$,*

$$|f[x+U] - f(x)| < r.$$

For all $u, v \in x + U/2$, for all $\alpha \in [0,1]$,

$$\textit{if} \quad u - v \in \frac{1}{3}\alpha U \quad \textit{then} \quad |f(u) - f(v)| < \alpha r.$$

2. Characterisations of Differentiability

In this section we show that for a continuous convex function, β differentiability is characterised by the $\mathcal{T}_\beta$ convergence of subdifferentials.

Suppose that X is a topological linear space, that f is a continuous, convex function with domain D in X, and that x is in D. The *subdifferential set of f at x*, denoted $\partial f(x)$, is the subset of X^* defined by

$$\partial f(x) = \{x^* \in X^* : \text{ for all } y \in D, \ \langle x^*, y - x\rangle \leqslant f(y) - f(x)\}.$$

For a continuous convex function f with domain in a locally convex space, $\partial f(x)$ is nonempty, and f is Gateaux differentiable at x if and only if $\partial f(x)$ is a singleton. These assertions are easily proved by Banach space methods (see for example [**9**, Propositions 1.11 and 1.8]).

2.1. *Let X be a locally convex space, β a bornology on X, f a continuous convex function with domain D in X, $x \in D$ and $x^* \in X^*$. The following are equivalent:*

(a) *f is β differentiable at x with derivative x^*;*

(b) *if $w_\alpha \to x$ and $w_\alpha^* \in \partial f(w_\alpha)$ then $w_\alpha^* \xrightarrow{\mathcal{T}_\beta} x^*$;*

(c) *if $w_\alpha \to x$ then for each α there exists $w_\alpha^* \in \partial f(w_\alpha)$ such that $w_\alpha^* \xrightarrow{\mathcal{T}_\beta} x^*$.*

Lemma. *Suppose that the conditions of the theorem statement hold, $y \in D$, λ is a positive real number and $(x + \lambda y)^* \in \partial f(x + \lambda y)$. If as $\lambda \downarrow 0$, $(x+\lambda y)^* \to x^*$ pointwise, then $x^* \in \partial f(x)$.*

Proof: Assume that, for all $z \in X$, as $\lambda \downarrow 0$, $\langle (x+\lambda y)^*, z\rangle \to \langle x^*, z\rangle$. If $z \in D$ then for λ sufficiently small,

$$\langle (x+\lambda y)^*, z - (x + \lambda y)\rangle \leqslant f(z) - f(x+\lambda y).$$

Suppose $\lambda \downarrow 0$; then, by the continuity of f and the linearity of $(x+\lambda y)^*$, $\langle x^*, z - x\rangle \leqslant f(z) - f(x)$, so $x^* \in \partial f(x)$. ∎

PROOF OF 2.1: It is immediate that (b) implies (c), since for each α, $\partial f(w_\alpha)$ is nonempty.

We will show that (c) implies (a). Fix $\varepsilon > 0$. Let $M \in \beta$; since M is bounded and balanced there exists $\gamma > 0$ such that for all $\lambda \in (0,\gamma)$, $(x + \lambda M) \subset D$. If $y \in M$ and $\lambda \downarrow 0$ then $(x + \lambda y) \to x$, so from (c), for each $\lambda \in (0,\gamma)$ there exists $(x+\lambda y)^* \in \partial f(x + \lambda y)$ and as $\lambda \downarrow 0$, $(x+\lambda y)^* \xrightarrow{\mathcal{T}_\beta} x^*$. From the lemma, $x^* \in \partial f(x)$; there exists $\delta > 0$ (without loss of generality, we assume that $\delta < \gamma$) such that for all $y \in M$, for all $\lambda \in (0,\delta)$,

$$|\langle (x+\lambda y)^*, y\rangle - \langle x^*, y\rangle| < \varepsilon$$

and $$\langle x^*, \lambda y\rangle \leqslant f(x+\lambda y) - f(x).$$

Also $(x+\lambda y)^* \in \partial f(x+\lambda y)$:

$$\langle (x+\lambda y)^*, -\lambda y\rangle \leqslant f(x) - f(x+\lambda y).$$

Altogether, for all $y \in M$, for all $\lambda \in (0,\delta)$,

$$\langle x^*, \lambda y\rangle \leqslant f(x+\lambda y) - f(x) \leqslant \langle (x+\lambda y)^*, \lambda y\rangle < \langle x^*, \lambda y\rangle + \lambda\varepsilon;$$

hence $$0 \leqslant f(x+\lambda y) - f(x) - \langle x^*, \lambda y\rangle < \lambda\varepsilon$$

and f is β differentiable at x with derivative x^*.

It remains to show that (a) implies (b). Using 1.1, since f is continuous at x, there is a convex balanced neighbourhood U of 0, such that for all $\gamma \in (0,1]$, for all $u, v \in (x + U/2) \subset D$,

$$(*) \qquad \text{if} \quad u - v \in \frac{1}{3}\gamma U \quad \text{then} \quad |f(u) - f(v)| < \gamma.$$

Let $w_\alpha \to x$ and for all α, let $w_\alpha{}^* \in \partial f(w_\alpha)$. Let $M \in \beta$ be balanced and choose $\varepsilon \in (0,1)$. Assuming (a), there exists $\delta \in (0,1)$ such that for all $\lambda \in (0,\delta)$, for all $y \in M$,

$$(**) \qquad |\langle x^*, \lambda y\rangle - f(x+\lambda y) + f(x)| < \frac{1}{3}\lambda\varepsilon.$$

Let $k \in (0,\delta)$ be such that $kM \subset U/6$. For $y \in M$ and α sufficiently large that $w_\alpha \in x + (k\varepsilon U)/9$, using $(*)$ with $\gamma = (k\varepsilon)/3$, and $(**)$,

$$\begin{aligned} k(\langle w_\alpha^*, y\rangle - \langle x^*, y\rangle) &= \langle w_\alpha^*, ky\rangle - \langle x^*, ky\rangle \\ &\leqslant f(w_\alpha + ky) - f(w_\alpha) - \langle x^*, ky\rangle \\ &= -\big(\langle x^*, ky\rangle - f(x+ky) + f(x)\big) \\ &\qquad + \big(f(w_\alpha + ky) - f(x+ky)\big) \\ &\qquad + \big(f(x) - f(w_\alpha)\big) \\ &< \frac{1}{3}k\varepsilon + \frac{1}{3}k\varepsilon + \frac{1}{3}k\varepsilon, \end{aligned}$$

hence $\langle w_\alpha^*, y\rangle - \langle x^*, y\rangle < \varepsilon.$

Since M is balanced, $|\langle w_\alpha^*, y\rangle - \langle x^*, y\rangle| < \varepsilon.$

∎

It follows from 2.1 that equivalent bornologies give the same differentiation theory, that is for $\beta_1 \equiv \beta_2$ and f a continuous convex function, f is β_1 differentiable at x if and only if f is β_2 differentiable at x.

3. Dense Differentiability.

In this section we use 2.1 to develop ideas from [**9**, Section 6] for locally convex spaces, dense and generic sets and any strength of differentiability.

The Minkowski gauges of convex neighbourhoods of the origin in a locally convex space X (we shall call these simply *gauges*) are precisely the continuous non-negative functions on X such that for all $x, y \in X$, for all $t \geqslant 0$, $g(x+y) \leqslant g(x) + g(y)$ and $g(tx) = tg(x)$. This last property is *positive homogeneity*. A *seminorm* is a gauge which is *absolutely homogeneous*, that is, for all $x \in X$, for all $t \in \mathbb{R}$, $g(tx) = |t|g(x)$.

A *generic* set in D is a set which *contains* a dense G_δ subset of D.

We classify a locally convex space X according to the β differentiability properties of the specified class of continuous convex functions with domain in X:

(1) βDS (β differentiability space): every continuous convex function is β differentiable on a dense subset of its domain;
(2) βMDS (β Minkowski differentiability space): every gauge is β differentiable on a dense subset;
(3) "[gen]" added to either of the above indicates that the differentiability occurs on a generic set.

If β is the class of bounded (singleton) subsets, then the spaces in (1) are known as FDS (GDS) and in (2) as FMDS (MDS). FDS[gen] and GDS[gen] are known as ASP and WASP (for Asplund and Weak Asplund).

The proof of 3.0 is an easy adaptation of [**5**, Section 3.3 Theorem 3].

3.0. *A β differentiability point of the sum of two convex functions is a β differentiability point of each of the summands.*

If g is a gauge on X then h defined by $h(x) = g(x) + g(-x)$ is a seminorm on X. Thus in the definitions of βMDS and βMDS[gen] we can equivalently replace "gauge" by "seminorm".

For Banach spaces the following results are known. FMDS is equivalent to ASP, because FMDS coincides with FDS [**10**, Theorem 1.28] and the set of Fréchet differentiability points of a continuous convex function is always a G_δ set [**9**, Proposition 1.25].

However, Čoban and Kenderov [**3**] have given examples to show that even when the set of Gateaux differentiability points of the norm is dense, it need not contain a G_δ set. MDS is equivalent to GDS [**9**, Corollary 6.6]; it is an open question whether there is a space which is GDS but not WASP.

We show that for a locally convex space X, not necessarily complete, βMDS and βDS are equivalent. If X is Q-complete and bound covering, for example a Banach space, then FDS and ASP coincide [**4**, Theorem 3.4], so our result subsumes all of those in the preceding paragraph.

The continuous dual $(X \times \mathbb{R})^*$ of $X \times \mathbb{R}$ is isomorphic with $X^* \times \mathbb{R}$; we shall use the pairing $\langle (x^*, r^*), (x, r)\rangle = \langle x^*, x\rangle + r^* r$.

In the proofs of 3.1 and 3.2 we shall use bornologies on X and on $X \times \mathbb{R}$ which correspond in a natural way. For a bornology β on X we take any bornology on $X \times \mathbb{R}$ corresponding to the product of $\mathcal{T}_\beta$ and the usual, and only, Hausdorff linear topology on $\mathbb{R}$, for example, $\{B \times \{a\} : B \in \beta, a \in \mathbb{R}\}$ and $\{B \times I : B \in \beta, I$ a bounded interval in $\mathbb{R}\}$ are such bornologies. For a bornology β on $X \times \mathbb{R}$ the projections onto X form a bornology on X: routine calculations show that this "projection" bornology gives rise to bornologies on $X \times \mathbb{R}$ which are equivalent to bornology β with which we started.

For a subset A of a topological linear space we define the *spray* of A by spray $A = \bigcup_{\lambda>0} \lambda A$ and by A is *radial* we mean that $A = \operatorname{spray} A$.

Lemma. *If X is a locally convex Baire space and A a generic radial subset of $X \times \mathbb{R}$ then A contains a dense set C which is a countable intersection of open radial sets.*

Proof: By hypothesis there is a dense set B in A such that $B = \bigcap_1^\infty O_n$, where each O_n is open, and $O_1 = X \times \mathbb{R}$. Let $\{r_1, r_2, r_3, \ldots\}$ be an enumeration of the positive rationals, let $I = (-1, 1) \subset \mathbb{R}$, $J = \overline{I}$. Let

$$X_{j,k} = X \times \left(\left(\{-r_j, r_j\} + \frac{1}{k} I \right) \setminus \{0\} \right) \qquad j, k \in \mathbb{N}$$

$$G_{i,j,k} = \operatorname{spray}(O_i \cap X_{j,k}) \qquad i, j, k \in \mathbb{N}.$$

Each $G_{i,j,k}$ is open, dense and radial in $X \times \mathbb{R}$.

Let $C = \bigcap G_{i,j,k}$; since $X \times \mathbb{R}$ is Baire, C is dense. Since spray $B \subset A$ the proof is complete if we show that $C \subset \operatorname{spray} B$: let $(x, s) \in C$ (so $s \neq 0$). It suffices to construct inductively a nest $K_1 \supset K_2 \supset K_3 \supset \ldots$ of non empty closed sets in $\mathbb{R}$, with $\operatorname{diam} K_i \to 0$, such that

$$K_i(x, s) = \{(tx, ts) : t \in K_i\} \subset O_i.$$

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For then, since $\mathbb{R}$ is complete, there exists $\lambda \in \bigcap K_i$; $\lambda(x,s) \in \bigcap O_i = B$ so $(x,s) \in$ spray B.

Let $\lambda_1 = 1$, $\varepsilon_1 \in (0,1/2)$ and suppose that $\lambda_2, \lambda_3, \dots \lambda_n$ and $\varepsilon_2, \varepsilon_3, \dots \varepsilon_n$ have been defined so that $\varepsilon_i < \varepsilon_{i-1}/2$, $\lambda_i \in \lambda_{i-1} + \varepsilon_i/2J$, $K_i(x,s) \subset O_i$, (where K_i denotes $\lambda_i + \varepsilon_i J$). Choose $k_{n+1}, j_{n+1} \in \mathbb{N}$ so that $4/k_{n+1} < \varepsilon_n|s|$ and $r_{j_{n+1}} \in \lambda_n|s| + 1/k_{n+1}I$. Now $(x,s) \in G_{n+1,j_{n+1},k_{n+1}}$ so there exists $\lambda_{n+1} > 0$ such that

$$\lambda_{n+1}(x,s) \in O_{n+1} \cap X_{j_{n+1},k_{n+1}}.$$

Hence $\lambda_{n+1} \in \lambda_n + \varepsilon_n/2J$. Since O_{n+1} is open and $t \mapsto (tx, ts)$ is continuous, there exists $\varepsilon_{n+1} \in (0, \varepsilon_n/2)$ such that $K_{n+1}(x,s) \subset O_{n+1}$ (where K_{n+1} denotes $\lambda_{n+1} + \varepsilon_{n+1}J$).

Then $\operatorname{diam} K_n < 2^{-n}$ and for $n > 1$,

$$K_{n+1} = \lambda_{n+1} + \varepsilon_{n+1}J \subset \lambda_n + \varepsilon_n/2J + \varepsilon_{n+1}J \subset \lambda_n + \varepsilon_n J = K_n,$$

which completes the proof. ∎

3.1. *Suppose X is a locally convex space. If $X \times \mathbb{R}$ is βMDS then X is βDS; if X is also Baire and $X \times \mathbb{R}$ is βMDS[gen] then X is βDS[gen].*

PROOF: Suppose that f is a continuous convex function with domain D in X. We will assume, without loss of generality, that $0 \in D$ and that $f(0) = -1$. The graph G of f is $\{(x, f(x)) : x \in D\}$; the epigraph is

$$\operatorname{epi} f = \{(x,t) : x \in D, f(x) \leqslant t\};$$

let $\mu : X \times \mathbb{R} \to \mathbb{R}$ be defined by

$$\mu(x,r) = \inf\{\lambda > 0 : (x,r) \in \lambda \operatorname{epi} f\},$$

and let D_μ denote the β differentiability points of μ. Then μ is a gauge, so by hypothesis D_μ is dense (generic) in $X \times \mathbb{R}$ and by positive homogeneity D_μ is radial. Hence $D_\mu \cap G$ is dense (generic) in G: the *dense* assertion is straightforward, *generic* follows from the preceding lemma.

The projection of G onto D is a homeomorphism. The proof is complete if we show that if $(x, f(x)) \in D_\mu$ then f is β differentiable at x. Suppose $(x, f(x)) \in D_\mu$ with derivative $(x^*, f(x)^*)$ and let $w_\alpha \to x$; then $(w_\alpha, f(w_\alpha)) \to (x, f(x))$ and from 2.1(c) for each α there exists $(w_\alpha^*, f(w_\alpha)^*) \in \partial\mu(w_\alpha, f(w_\alpha))$ such that $(w_\alpha^*, f(w_\alpha)^*) \xrightarrow{\mathcal{T}_\beta} (x^*, f(x)^*)$.

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For all $(z, s_\alpha) \in X \times \mathbb{R}$,

$$\langle (w_\alpha^*, f(w_\alpha)^*), (z, s_\alpha) \rangle \leqslant \mu(w_\alpha + z, f(w_\alpha) + s_\alpha) - \mu(w_\alpha, f(w_\alpha)).$$

For each α, for each $z \in D - w_\alpha$, let $s_\alpha = f(w_\alpha + z) - f(w_\alpha)$; then, since for all $y \in D$, $\mu(y, f(y)) = 1$,

$$(*) \qquad \langle w_\alpha^*, z \rangle \leqslant -f(w_\alpha)^* \left(f(w_\alpha + z) - f(w_\alpha) \right).$$

We show that for large α, $-f(w_\alpha)^* > 0$. Since $(x^*, f(x)^*) \in \partial\mu(x, f(x))$,

$$\langle (x^*, f(x)^*), (0,1) \rangle \leqslant \mu(x, f(x) + 1) - \mu(x, f(x)),$$

but $\mu(x, f(x)) > \mu(x, f(x) + 1)$, so $f(x)^* < 0$; for sufficiently large α, $-f(w_\alpha)^* > 0$. From $(*)$,

$$-\frac{1}{f(w_\alpha)^*} w_\alpha^* \in \partial f(w_\alpha) \quad \text{and} \quad -\frac{1}{f(w_\alpha)^*} w_\alpha^* \to -\frac{1}{f(x)^*} x^*$$

with uniform convergence over β sets, so f is β differentiable at x. ∎

3.2. *If a locally convex space X is βDS then $X \times \mathbb{R}$ is βDS.*

PROOF: Suppose f is a continuous convex function on a subset D of $X \times \mathbb{R}$, and $(x_0, t_0) \in D$. Let U be an open neighbourhood of (x_0, t_0) in D. There exists $M > 0$ such that $|f(x_0, t_0)| < M$; there exist an open balanced convex neighbourhood N of x_0 in X and $a > 0$ such that for $I = (t_0 - a, t_0 + a)$,

$$N \times I \subset f^{-1}[(-M, M)] \cap U.$$

Choose a differentiable function g on I which is non-positive and such that $g(t_0) = 0$ and $g(r) \to -\infty$ as $r \to t_0 \pm a$. Define $h : N \to \mathbb{R}$ by

$$h(x) = \sup\{f(x, t) + g(t) : t \in I\};$$

then h is continuous and convex; so if X is βDS there exists $x_1 \in N$ which is a β differentiability point of h. There exists $b \in (0, a)$ such that for $J = [t_0 - b, t_0 + b]$, $f(x_1, t) + g(t)$ attains its supremum on I at $t_1 \in J$.

Fix $\varepsilon > 0$ and let $B \in \beta$. Since h is β differentiable at x_1, for sufficiently small δ, for all $(y, s) \in B$, for all $\lambda \in (0, \delta)$,

$$\begin{aligned} 0 &\leqslant f(x_1 + \lambda y, t_1 + \lambda s) + f(x_1 - \lambda y, t_1 - \lambda s) - 2f(x_1, t_1) \\ &\leqslant [h(x_1 + \lambda y) + h(x_1 - \lambda y) - 2h(x_1)] \\ &\qquad - [g(t_1 + \lambda s) + g(t_1 - \lambda s) - 2g(t_1)] \\ &\leqslant \lambda\varepsilon; \end{aligned}$$

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hence f is β differentiable at $(x_1, t_1) \in U$. ∎

3.3. *A locally convex space X is β DS if and only if it is β MDS.*

PROOF: To see the nontrivial result, let Y be a closed hyperplane in X; then X is isomorphic to $Y \times \mathbb{R}$ [**6**, p.156, (2)]. If X is β MDS then from 3.1 Y is β DS; by 3.2, X is β DS. ∎

We would have liked to prove a *generic* version of 3.2 which, if true, would turn the conjectures of Section 4 into theorems.

4. The Generic Case.

We conjecture:

4.0. *If a locally convex Baire space X is β DS[gen] then so is $X \times \mathbb{R}$.*

That the converse is true follows from 3.1. Equivalent formulations of the conjecture are given in the diagram at the end of this section. For the Gateaux bornology, this is a long standing open Banach space question; the Fréchet version is known to be true for a large class of locally convex spaces [**4**, Proposition 3.2].

In 4.3 we show that for a locally convex Baire space, $X \times \mathbb{R}$ is MDS[gen] if and only if X is WASP. This is a new result even for Banach spaces. We are then tantalisingly close to 4.0 for the Gateaux case: if X is WASP, then from 3.2 and 4.3, $X \times \mathbb{R}$ is both GDS and MDS[gen].

It turns out that the conjecture is equivalent to the coincidence of β MDS[gen] and β DS[gen]. With this in mind we define a new class of spaces which is formally between these, intuitively appears close to β MDS[gen], but which turns out to coincide with β DS[gen].

We define an *asymptotic seminorm* on a locally convex space X to be a continuous function f for which there is a seminorm g satisfying:

(1) if $x_\alpha \to x$ and $\lambda \to \infty$ then $f(\lambda x_\alpha)/\lambda \to g(x)$;
(2) for all $x \in X$, $f(x) = f(-x)$ and $f(x) \geqslant g(x)$.

An asymptotic seminorm is convex and every seminorm is an asymptotic seminorm.

A locally convex space is defined to be β ADS[gen] when each asymptotic seminorm is generically β differentiable. Clearly

$$\beta\,\text{DS[gen]} \implies \beta\,\text{ADS[gen]} \implies \beta\,\text{MDS[gen]}.$$

4.1. *A locally convex space X is β ADS[gen] if and only if for every seminorm p on $X \times \mathbb{R}$, $p(\cdot, 1)$ is generically β differentiable.*

PROOF: Suppose that X is β ADS[gen] and let p be a seminorm on $X \times \mathbb{R}$. Using 3.0, it suffices to prove that $q : X \to \mathbb{R}$, defined by $q(x) = (p(x,1) + p(-x,1))/2$, is

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generically β differentiable. It is easily seen that q is an asymptotic seminorm with associated seminorm $p(\cdot,0)$. So, by hypothesis, q is generically β differentiable.

Conversely suppose that for each seminorm p on $X\times\mathbb{R}$, $p(\cdot,1)$ is generically β differentiable. Let f be an asymptotic seminorm with associated seminorm g. Routine but nasty calculations show that defining

$$p(x,t)=\begin{cases} f\left(\frac{x}{t}\right) & t\neq 0\\ g(x) & t=0\end{cases}$$

makes p a seminorm on $X\times\mathbb{R}$ such that $f(x)=p(x,1)$. It follows that f is generically β differentiable. ∎

4.2. *A locally convex Baire space X is βADS[gen] if and only if $X\times\mathbb{R}$ is βMDS[gen].*

For the proof we need the following lemma which is easily verified.

Lemma. *Suppose Y is a locally convex space and f is a continuous convex function on $Y\times\mathbb{R}\times\mathbb{R}$. Define h_t and g_s on $Y\times\mathbb{R}$ by*

$$h_t(x,s)=f(x,t,s)\quad\text{and}\quad g_s(x,t)=f(x,t,s).$$

Then f is β differentiable at (x,t,s) if and only if h_t is β differentiable at (x,s) and g_s is β differentiable at (x,t).

Proof of 4.2: Suppose $X\times\mathbb{R}$ is βMDS[gen]. Then, by 3.1, X is βDS[gen] and so βADS[gen].

Conversely, since X is Hausdorff we may write $X=Y\times\mathbb{R}$; let p be a seminorm on $X\times\mathbb{R}=Y\times\mathbb{R}\times\mathbb{R}$; define p_s and q_t on $Y\times\mathbb{R}$ by

$$p_s(y,t)=p(y,s,t)\quad\text{and}\quad q_t(y,s)=p(y,s,t).$$

Define sets G' and G'' by

$$G'=\{(z,u,w)\in Y\times\mathbb{R}\times\mathbb{R}: p_u \text{ is } \beta \text{ differentiable at } (z,w)\},$$
$$G''=\{(z,u,w)\in Y\times\mathbb{R}\times\mathbb{R}: q_w \text{ is } \beta \text{ differentiable at } (z,u)\}.$$

Let G_p (G_q) denote the set of β differentiability points of p_1 (q_1); it follows easily from the absolute homogeneity of p that for $s\neq 0$, p_s is β differentiable at (sy,st) if and only if p_1 is β differentiable at (y,t) (and analogously for q) so

$$G'=\{(sy,s,st)\in Y\times\mathbb{R}\times\mathbb{R}: s\neq 0,\ (y,t)\in G_p\}$$
$$G''=\{(ty,st,t)\in Y\times\mathbb{R}\times\mathbb{R}: t\neq 0,\ (y,s)\in G_q\}.$$

For any generic set S in X, $\{(tx,t) : t \neq 0, x \in S\}$ is generic in $X \times \mathbb{R}$; it follows from 4.1 that G_p and G_q are generic in X hence G' and G'' are generic in $X \times \mathbb{R}$. From the lemma, $G' \cap G''$ is the set of β differentiability points of p and, since X is a Baire space, it is generic. ∎

From 4.2 and 3.1 we have 4.3 and 4.4.

4.3. *A locally convex space Baire space X is β DS[gen] if and only if $X \times \mathbb{R}$ is β MDS[gen].*

In particular, for X a locally convex Baire space, X is WASP if and only if $X \times \mathbb{R}$ is MDS[gen]. Warren Moors has shown us an alternative proof of this.

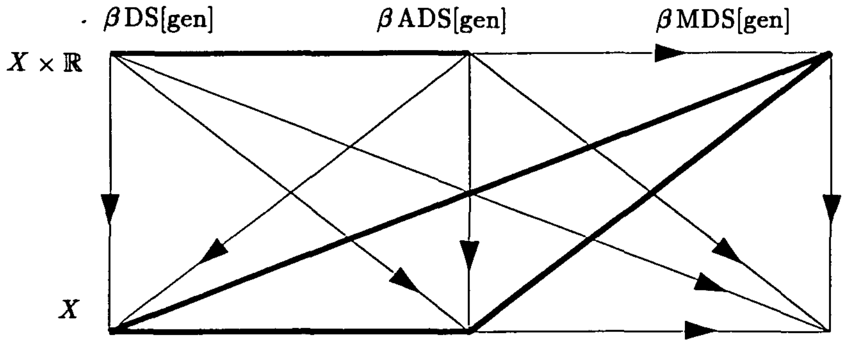
If, in 4.3, it were also true that $X \times \mathbb{R} \times \mathbb{R}$ is βMDS[gen] then from 3.1 the conjecture would be affirmed.

4.4. *A locally convex space Baire space is β DS[gen] if and only if it is β ADS[gen].*

It is not known whether there are spaces which are GDS but not WASP; in 4.5 we characterise such spaces.

4.5. *A locally convex space X is GDS and not WASP if and only if every seminorm is Gateaux differentiable on a dense set and there exists an asymptotic seminorm which is not generically Gateaux differentiable.*

The *generic* results of Sections 3 and 4 are summarised in the following diagram: for the *dense* case all classes are equivalent. If X is bound covering and Q complete (see [4]), for example if X is Banach, and β is the Fréchet bornology, then all classes (both dense and generic) coincide.



Heavy lines denote equivalences (for example, X is βDS[gen] if and only if $X \times \mathbb{R}$ is βMDS[gen]). For the light lines, down and/or right are true (for example, if $X \times \mathbb{R}$ is βADS[gen] then X is βMDS[gen]) and up and/or left are open questions (for example, our conjecture is up on the extreme left).

Further, all open questions are logically equivalent.

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