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THE NON-COMMUTATIVE SCHWARTZ SPACE IS WEAKLY AMENABLE

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Abstract. We show in a straightforward way that the non-commutative Schwartz space is weakly amenable. At the end, we leave an open problem.

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1. Introduction. The question of whether or not an algebra is amenable belongs to one of the very interesting and important questions in the field. We address this question in the case of one particular lmc Fréchet *-algebra, the so-called *non-commutative Schwartz space* $\mathcal{S}$. This is the algebra of linear and continuous operators from the space of tempered distributions into the Schwartz space of rapidly decreasing functions (see the next Section for details). In the present paper, we show that the non-commutative Schwartz space is weakly amenable (see Section 3 for the definition).

The algebra $\mathcal{S}$ has been reinvestigated recently by Ciaś [**2**], Domański (unpublished manuscript) and the author [**13**]. In the last paper, the author showed that $\mathcal{S}$ is boundedly approximately contractible. By [**12**, Theorem 9.7] and [**13**, Proposition 2] it is also known that $\mathcal{S}$ is not amenable.

The non-commutative Schwartz spaces is isomorphic to several, natural function (and sequence) spaces, for instance, $\mathcal{S}$ is isomorphic (as a locally convex space) to the space $C^\infty(M)$ of smooth functions on a compact smooth manifold M or to the Schwartz space $\mathcal{S}(\mathbb{R})$ of test functions for tempered distributions (again as an lcs). This last isomorphism justifies the name for $\mathcal{S}$ (the other name *algebra of smooth operators* – used e.g. by Ciaś – comes from the first mentioned isomorphism). These isomorphisms link our object with the structure theory of Fréchet spaces, especially with questions concerning nuclearity or splitting of short exact sequences – see [**10**, Part IV]. This algebra appears also in the context of K-theory – see [**3**, **11**] or in the context of cyclic cohomology for crossed products – see [**8**, **14**].

Another motivation comes from the theory of operator spaces and its locally convex analogues – see [**6**, **7**]. The non-commutative Schwartz space plays also a role in quantum mechanics, where it is called the *space of physical states* and its dual is the so-called *space of observables* – see [**5**] for details. This algebra shares also some nice features with C^*-algebras, e.g. it admits a functional calculus – see [**2**, Theorem 5.2] and all positive functionals as well as all derivations are automatically continuous – see [**13**, Theorems 11, 13]. On the other hand, it has no bounded approximate identity – see [**13**, Proposition 2] and it is not a local C*-algebra – see [**9**, Theorems 8.2, 8.3].

The paper is divided into four parts. The next Section contains basic notation and terminology. Section 3 serves as a preparation for the last Section where we show that the non-commutative Schwartz space is weakly amenable.

We refer the reader to [**10**] for the structure theory of Fréchet spaces and to [**4**] for the 'algebraic-in-flavour' aspects of the paper.

2. Notation and terminology. Let

$$s = \left\{ \xi = (\xi_j)_{j\in\mathbb{N}} \subset \mathbb{C}^{\mathbb{N}}\colon\ |\xi|_k^2 := \sum_{j=1}^{+\infty} |\xi_j|^2 j^{2k} < +\infty \text{ for all } k \in \mathbb{N} \right\}$$

denote the space of rapidly decreasing sequences. This is a Fréchet space with the basis $(U_k)_{k\in\mathbb{N}}$ of zero neighbourhoods defined by $U_k := \{\xi \in s\colon\ |\xi|_k \leqslant 1\}$. The topological dual of s is

$$s' = \left\{ \eta = (\eta_j)_{j\in\mathbb{N}} \subset \mathbb{C}^{\mathbb{N}}\colon\ |\eta|_k'^2 := \sum_{j=1}^{+\infty} |\eta_j|^2 j^{-2k} < +\infty \text{ for some } k \in \mathbb{N} \right\}.$$

The non-commutative Schwartz space $\mathcal{S}$ is the space $L(s', s)$ of linear and continuous operators from the dual of s into s with the topology of uniform convergence on bounded sets. It is a Fréchet space with the fundamental sequence $(\|\cdot\|_n)_{n\in\mathbb{N}}$ of norms given by

$$\|x\|_n := \sup\{|x\xi|_n\colon\ \xi \in U_n^\circ\},$$

where $U_n^\circ = \{\xi \in s'\colon\ |\xi|_n' \leqslant 1\}$. Continuity of the identity map $\iota\colon s \hookrightarrow s'$ defines multiplication in $\mathcal{S}$ by

$$xy := x \circ \iota \circ y, \quad x, y \in \mathcal{S}.$$

This multiplication is separately (therefore jointly – [**15**, Theorem 1.5]) continuous. Duality between s and s' given by

$$\langle \xi, \eta \rangle := \sum_{j\in\mathbb{N}} \xi_j \overline{\eta_j}, \quad \xi \in s, \eta \in s' \tag{1}$$

introduces an involution map defined by

$$\langle x^*\xi, \eta \rangle := \langle \xi, x\eta \rangle, \quad x \in \mathcal{S},\ \xi, \eta \in s'.$$

With these operations $\mathcal{S}$ becomes a locally multiplicatively convex (*lmc* for short) Fréchet *-algebra. It has several representations as the lmc Fréchet *-algebra – see e.g. [Domański (unpublished manuscript) Theorem 1.1]. Of particular, interest for us will be the *-algebra isomorphism $\mathcal{S} \simeq \mathcal{K}^p$ $(1 \leqslant p \leqslant +\infty)$, where

$$\mathcal{K}^p := \left\{ x = (x_{i,j})_{i,j\in\mathbb{N}}\colon\ \|x\|_{n,p} := \left(\sum_{i,j=1}^{\infty} |x_{ij}|^p (ij)^{pn} \right)^{\frac{1}{p}} < +\infty \text{ for all } n \in \mathbb{N} \right\}$$

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is the lmc Fréchet *-algebra endowed with the topology defined by the sequence $(\|\cdot\|_{n,p})_{n\in\mathbb{N}_0}$ of norms, multiplication of matrices and the 'star' operation being the matrix conjugate transpose. For $\mathcal{K}^\infty$, we define the respective supremum norms. The isomorphism $\mathcal{S}\simeq\mathcal{K}^\infty$ follows by [Domański (unpublished manuscript) Theorem 1.1] and $\mathcal{K}^\infty\simeq\mathcal{K}^p$ follows by the Grothendieck–Pietsch nuclearity criterion [**10**, Theorem 28.15]. Indeed, for any $p\in[1,+\infty)$ and any infinite matrix $x:=(x_{ij})_{i,j\in\mathbb{N}}$, we have

$$\|x\|_{n,\infty}\leqslant\|x\|_{n,p}=\left(\sum_{i,j=1}^{+\infty}|x_{ij}|^p(ij)^{p(n+2)}(ij)^{-2p}\right)^{\frac{1}{p}}\leqslant\zeta(2p)^{\frac{2}{p}}\|x\|_{n,\infty},$$

where ζ is the Riemann 'zeta' function.

The non-commutative Schwartz space is therefore an algebra of matrices. Since $\mathcal{S}$ is not unital ($\mathcal{S}\subset\mathcal{K}(\ell_2)$), by $\mathcal{S}_1$ we denote its unitization (identity map on ℓ_2 being the unit). By [Domański (unpublished manuscript) Theorem 2.3], an element $x\in\mathcal{S}_1$ is invertible in $\mathcal{S}_1$ if and only if it is invertible in $B(\ell_2)$. Since $\mathcal{S}B(\ell_2)\mathcal{S}\subset\mathcal{S}$, by [**1**, Proposition A.2.8] (cf. [Domański (unpublished manuscript) Corollary 2.5]), we have

$$\sigma_{\mathcal{S}_1}(x)=\sigma_{B(\ell_2)}(x),\quad x\in\mathcal{S}_1.$$

This implies that an element $x\in\mathcal{S}$ is positive in $\mathcal{S}$ if and only if it is positive as an operator on ℓ_2. Consequently, the order structure, multiplication and the 'star' operation in $\mathcal{S}$ and $\mathcal{S}_1$ are inherited from the C*-algebra $B(\ell_2)$. Moreover, all the three notions of positivity in $\mathcal{S}$ coincide, i.e. an operator $x\in\mathcal{S}$ is positive if $\sigma_{\mathcal{S}_1}(x)\subset[0,+\infty)$ iff $x=y^*y$ for some operator $y\in\mathcal{S}$ iff $\langle x\xi,\xi\rangle\geqslant 0$ for every $\xi\in s'$ – see [**13**, Proposition 3] for details. By [Domański (unpublished manuscript) Corollary 2.4], $\mathcal{S}_1$ is a Q-algebra and by [**9**, Theorems 8.2, 8.3], the topology of both $\mathcal{S}$ and $\mathcal{S}_1$ cannot be given by a sequence of C*-norms.

3. Preparation. Recall that an algebra A is *weakly amenable* if every derivation δ from A into the dual module $A^\#$ (space of linear functionals) is *inner* that is, there exists an element $\phi\in A^\#$ such that

$$\delta(a)=\mathrm{ad}_\phi(a):=a\cdot\phi-\phi\cdot a\qquad(a\in A).$$

If A is a topological algebra, then we consider A' (space of linear and continuous functionals) instesd of $A^\#$ and continuous derivations. By [**13**, Theorem 13], every derivation of $\mathcal{S}$ is automatically continuous.

Assume now that A is a Fréchet lmc-algebra and assume (for technical reasons) that its underlying Fréchet space is distinguished (see [**10**, p. 300] for the definition). Suppose now that $A=\mathrm{proj}_n A_n$, a projective limit of Banach algebras. Then by [**10**, Remark 25.13], its topological dual A' is an (LB)-space. More explicitly, $A'=\mathrm{ind}_n A_n'$. By the Grothendieck Factorization Theorem [**10**, Theorem 24.33], for every linear and continuous map $T\colon A\to A'$, we can find an index $n\in\mathbb{N}$ such that $T\colon A_n\to A_n'$ is continuous. Therefore, if every A_n is weakly amenable, then so is A itself. In other words, to conclude that a Fréchet lmc-algebra is weakly amenable, it is enough to

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express this algebra as a projective limit of weakly amenable Banach algebras. We state this easy observation below.

PROPOSITION 1. *Let A be a Fréchet lmc-algebra whose underlying Fréchet space is distinguished. Let $A = \mathrm{proj}_n A_n$ be a projective limit of Banach algebras. If all the projective steps A_n, $n \in \mathbb{N}$ are weakly amenable, then so is A.*

Now, we come back to the non-commutative Schwartz space $\mathcal{S}$. Recall from the previous Section that $\mathcal{S}$ is isomorphic (as a Fréchet *-algebra) to the algebra $\mathcal{K}^1$ of rapidly decreasing matrices, i.e. we can represent $\mathcal{S}$ as the projective limit $\mathcal{S} = \mathrm{proj}_n \mathcal{K}^1_n$, where

$$\mathcal{K}^1_n := \left\{ x = (x_{i,j})_{i,j\in\mathbb{N}} \colon \ \|x\|_{n,1} := \sum_{i,j=1}^{\infty} |x_{ij}|(ij)^n < +\infty \right\} \quad (n \in \mathbb{N}).$$

The dual Banach spaces are

$$(\mathcal{K}^1_n)' = \left\{ \phi = (\phi_{i,j})_{i,j\in\mathbb{N}} \colon \ \|\phi\|'_{n,\infty} := \sup_{i,j\in\mathbb{N}} |\phi_{ij}|(ij)^{-n} < +\infty \right\} \quad (n \in \mathbb{N})$$

and the duality bracket $\langle \cdot, \cdot \rangle$ is given by

$$\langle x, \phi \rangle := \sum_{i,j=1}^{+\infty} x_{ij}\phi_{ij}, \quad x \in \mathcal{K}^1_n, \ \phi \in (\mathcal{K}^1_n)'.$$

Let $(e_{ij})_{i,j\in\mathbb{N}}$ be matrix units in $\mathcal{K}^1_n$ and $(\mathcal{K}^1_n)'$ for every $n \in \mathbb{N}$, respectively. If $\delta \colon \mathcal{K}^1_n \to (\mathcal{K}^1_n)'$ is a derivation, then by Proposition 2(i) below, one has

$$\langle e_{pq}, \delta(e_{ij}) \rangle = \begin{cases} 0 & \text{if } \ p \neq j, q \neq i \\ \langle e_{iq}, \delta(e_{ii}) \rangle & \text{if } \ p = j, q \neq i \\ \langle e_{pj}, \delta(e_{jj}) \rangle & \text{if } \ p \neq j, q = i \\ \langle e_{ji}, \delta(e_{ij}) \rangle & \text{if } \ p = j, q = i \ . \end{cases} \tag{2}$$

In other words, the only non-zero entries of the matrix $\delta(e_{ij})$ lie in the j-row and i-th column. Below we collect two technical equalities, proofs of which are a matter of straightforward computation. For convenience, we enclose them below.

PROPOSITION 2. *Let $n \in \mathbb{N}$, $x \in \mathcal{K}^1_n$, $\phi \in (\mathcal{K}^1_n)'$ and let $\delta \colon \mathcal{K}^1_n \to (\mathcal{K}^1_n)'$ be a derivation. Then,*

(*i*) $\forall\, i, j, k \in \mathbb{N} \colon \ \langle e_{jk}, \delta(e_{kj}) \rangle = \langle e_{ik}, \delta(e_{ki}) \rangle + \langle e_{ji}, \delta(e_{ij}) \rangle$,
(*ii*) $\forall\, i, j \in \mathbb{N} \colon \ \langle e_{ji}, \delta(e_{ij} + e_{ii}) \rangle = 0$.

Proof. (i): $\langle e_{jk}, \delta(e_{kj}) \rangle = \langle e_{jk}, e_{ki} \cdot \delta(e_{ij}) + \delta(e_{ki}) \cdot e_{ij} \rangle = \langle e_{ji}, \delta(e_{ij}) \rangle + \langle e_{ik}, \delta(e_{ki}) \rangle$.
(ii): If $i = j$, then by [**4**, Proposition 1.8.2(i)],

$$\langle e_{jj}, \delta(e_{jj}) \rangle = \langle e_{jj}, e_{jj} \cdot \delta(e_{jj}) \cdot e_{jj} \rangle = 0.$$

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If $i \neq j$, then $e_{ii} + e_{jj}$ is an idempotent therefore by [**4**, Proposition 1.8.2(i)] and (i) above, we have

$$\begin{aligned} 0 &= (e_{ii} + e_{jj}) \cdot \delta(e_{ii} + e_{jj}) \cdot (e_{ii} + e_{jj}) \\ &= e_{ii} \cdot \delta(e_{ii} + e_{jj}) \cdot e_{jj} + e_{jj} \cdot \delta(e_{ii} + e_{jj}) \cdot e_{ii} \\ &= \langle e_{ji}, \delta(e_{jj} + e_{ii})\rangle e_{ji} + \langle e_{ij}, \delta(e_{jj} + e_{ii})\rangle e_{ij}. \end{aligned} \tag{3}$$

□

4. Main result. THEOREM 3. *For every $n \in \mathbb{N}$, the Banach *-algebra $\mathcal{K}_n^1$ is weakly amenable.*

REMARK. Since $\mathcal{K}_n^1$ is an algebra of matrices, it is natural to expect that for a given derivation $\delta\colon \mathcal{K}_n^1 \to (\mathcal{K}_n^1)'$, there is an infinite matrix ϕ which satisfies $\delta = \mathrm{ad}_\phi$. The point is to guarantee that $\phi \in (\mathcal{K}_n^1)'$.

Proof. We fix $n \in \mathbb{N}$ and a continuous derivation $\delta\colon \mathcal{K}_n^1 \to (\mathcal{K}_n^1)'$. Since matrix units form a Schauder basis in $\mathcal{K}_n^1$, it is enough to find a functional $\phi \in (\mathcal{K}_n^1)'$ satisfying $\delta(e_{ij}) = \mathrm{ad}_\phi(e_{ij})$ for all $i, j \in \mathbb{N}$. By (2), we have

$$\delta(e_{ij}) = \begin{pmatrix} 0 & \dots & 0 & \langle e_{1j}, \delta(e_{jj})\rangle & 0 & \dots \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots \\ 0 & \dots & 0 & \langle e_{j-1,j}, \delta(e_{jj})\rangle & 0 & \dots \\ \langle e_{i1}, \delta(e_{ii})\rangle & \dots & \langle e_{i,i-1}, \delta(e_{ii})\rangle & \langle e_{ji}, \delta(e_{ij})\rangle & \langle e_{i,i+1}, \delta(e_{ii})\rangle & \dots \\ 0 & \dots & 0 & \langle e_{j+1,j}, \delta(e_{jj})\rangle & 0 & \dots \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

On the other hand, taking any infinite scalar matrix $\phi = (\phi_{ij})_{i,j\in\mathbb{N}}$ and applying (formally) ad_ϕ to matrix units, we get

$$\mathrm{ad}_\phi(e_{ij}) = \begin{pmatrix} 0 & \dots & 0 & \langle e_{1j}, \phi\rangle & 0 & \dots \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots \\ 0 & \dots & 0 & \langle e_{j-1,j}, \phi\rangle & 0 & \dots \\ -\langle e_{i1}, \phi\rangle & \dots & -\langle e_{i,i-1}, \phi\rangle & \langle e_{jj} - e_{ii}, \phi\rangle & -\langle e_{i,i+1}, \phi\rangle & \dots \\ 0 & \dots & 0 & \langle e_{j+1,j}, \phi\rangle & 0 & \dots \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

Defining

$$\phi := \begin{pmatrix} 0 & -\langle e_{12}, \delta(e_{11})\rangle & -\langle e_{13}, \delta(e_{11})\rangle & -\langle e_{14}, \delta(e_{11})\rangle & -\langle e_{15}, \delta(e_{11})\rangle & \dots \\ \langle e_{21}, \delta(e_{11})\rangle & \langle e_{21}, \delta(e_{12})\rangle & -\langle e_{23}, \delta(e_{22})\rangle & -\langle e_{24}, \delta(e_{22})\rangle & -\langle e_{25}, \delta(e_{22})\rangle & \dots \\ \langle e_{31}, \delta(e_{11})\rangle & \langle e_{32}, \delta(e_{22})\rangle & \langle e_{31}, \delta(e_{13})\rangle & -\langle e_{34}, \delta(e_{33})\rangle & -\langle e_{35}, \delta(e_{33})\rangle & \dots \\ \langle e_{41}, \delta(e_{11})\rangle & \langle e_{42}, \delta(e_{22})\rangle & \langle e_{43}, \delta(e_{33})\rangle & \langle e_{41}, \delta(e_{14})\rangle & -\langle e_{45}, \delta(e_{44})\rangle & \dots \\ \langle e_{51}, \delta(e_{11})\rangle & \langle e_{25}, \delta(e_{22})\rangle & \langle e_{53}, \delta(e_{33})\rangle & \langle e_{54}, \delta(e_{44})\rangle & \langle e_{51}, \delta(e_{15})\rangle & \dots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix},$$

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we get $\delta = \mathrm{ad}_\phi$. Indeed, fix $i, j \in \mathbb{N}$ and apply Proposition 2 to each of the five cases below:

(i) $p < j$:

$$\langle e_{pi}, \mathrm{ad}_\phi(e_{ij})\rangle = \phi_{pj} = -\langle e_{pj}, \delta(e_{pp})\rangle = \langle e_{pj}, \delta(e_{jj})\rangle = \langle e_{pi}, \delta(e_{ij})\rangle,$$

(ii) $p = j$:

$$\langle e_{ji}, \mathrm{ad}_\phi(e_{ij})\rangle = \phi_{jj} - \phi_{ii} = \langle e_{j1}, \delta(e_{1j})\rangle - \langle e_{i1}, \delta(e_{1i})\rangle = \langle e_{ji}, \delta(e_{ij})\rangle = \langle e_{ji}, \delta(e_{ij})\rangle,$$

(iii) $p > j$:

$$\langle e_{pi}, \mathrm{ad}_\phi(e_{ij})\rangle = \phi_{pj} = \langle e_{pj}, \delta(e_{jj})\rangle = \langle e_{pi}, \delta(e_{ij})\rangle,$$

(iv) $q < i$:

$$\langle e_{jq}, \mathrm{ad}_\phi(e_{ij})\rangle = -\phi_{iq} = -\langle e_{iq}, \delta(e_{qq})\rangle = \langle e_{iq}, \delta(e_{ii})\rangle = \langle e_{jq}, \delta(e_{ij})\rangle,$$

(v) $q > i$:

$$\langle e_{jq}, \mathrm{ad}_\phi(e_{ij})\rangle = -\phi_{iq} = \langle e_{iq}, \delta(e_{ii})\rangle = \langle e_{jq}, \delta(e_{ij})\rangle.$$

It remains to show that $\phi \in (\mathcal{K}_n^1)'$, i.e. $\sup_{i,j}\{|\phi_{ij}|(ij)^{-n}\}$ is finite. To this end, fix $i, j \in \mathbb{N}$. Then,

$$\begin{aligned}|\phi_{ij}| = |\langle e_{ij}, \phi\rangle| &= |\langle e_{i1}e_{1j} - e_{1j}e_{i1}, \phi\rangle| = |\langle e_{i1}, \mathrm{ad}_\phi(e_{1j})\rangle| \\ &= |\langle e_{i1}, \delta(e_{1j})\rangle| \leqslant \|\delta\| \|e_{1j}\|_{n,1} \|e_{i1}\|_{n,1} = \|\delta\|(ij)^n.\end{aligned}$$

Consequently,

$$\|\phi\|'_{n,\infty} = \sup\{|\phi_{ij}|(ij)^{-n} : \ i, j \in \mathbb{N}\} \leqslant \|\delta\| < +\infty.$$

□

Combining Proposition 1 and Theorem 3, we can finally derive the main result.

THEOREM 4. *The non-commutative Schwartz space is weakly amenable.*

REMARK. It is interesting to know whether the converse of Proposition 1 is true or not. Therefore, we finish the paper with the following problem.

QUESTION 5. Is it true that every weakly amenable Fréchet lmc-algebra may be expressed as a reduced projective limit of a sequence of weakly amenable Banach algebras? If the answer is 'yes', is it true that if A is a weakly amenable Fréchet algebra, then for every expression of A as a reduced projective limit of a sequence of Banach algebras the projective steps are weakly amenable?

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