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THE BP-COACTION FOR PROJECTIVE SPACES

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1. Introduction. The Brown-Peterson spectrum BP has been used recently to establish some new information about the stable homotopy groups of spheres [**9**; **11**]. The best results have been achieved by using the associated homology theory $BP_*(\ )$, the Hopf algebra $BP_*(BP)$, and the Adams-Novikov spectral sequence

$$\mathrm{Ext}_{BP_*BP}(BP_*, BP_*(X)) \Rightarrow \pi_*{}^s(X)_{(p)}.$$

A knowledge of the stable homotopy groups of stunted real projective spaces $P_n = RP^\infty/RP^{n-1}$ is useful in studying the problem of immersing manifolds in Euclidean space [**7**]. One might hope that computing these groups via the BP-Adams-Novikov spectral sequence would provide insight which the classical Adams spectral sequence has missed (e.g. some elements have lower filtration).

As a first step in this program, we compute $BP_*(P_n)$ and the coaction $BP_*(P_n) \to BP_*(BP) \otimes_{BP_*} BP_*(P_n)$. In order to state our main result, we recall [**3**; **8**] that $BP_* = BP_*(pt) = \pi_*(BP) = \mathbf{Z}_{(2)}[v_1, v_2, \ldots]$ and $BP_*(BP) \simeq BP_*[t_1, t_2, \ldots]$, where v_i and t_i both have degree $2(2^i - 1)$. We use only the Brown-Peterson spectrum associated to the prime 2 [**3**; **6**].

1.1 THEOREM. i) *For $i \geqq n$ there are elements $\gamma_i \in BP_{2i+1}(P_{2n+1})$ such that there is an isomorphism of $\mathbf{Z}_{(2)}[v_2, v_3, \ldots]$-modules*

$$BP_*(P_{2n+1}) \approx \mathbf{Z}_{(2)}[v_2, v_3, \ldots](\gamma_n, \gamma_{n+1}, \ldots)/(2^{i+1-n}\gamma_i).$$

ii) *The coaction*

$$\Psi : BP_*(P_{2n+1}) \to BP_*(BP) \otimes_{BP_*} BP_*(P_{2n+1})$$

is given by

$$\Psi(\gamma_i) = \sum_{j=1}^{i} (S^j)_{(i-j)} \otimes \gamma_j,$$

where

a) *if T is a graded expression (such as S^j), $T_{(k)}$ denotes the component of T of degree $2k$,*

b) *S^j is the jth power of $S = 1 + S_1 + S_2 + \ldots$, where*

$$S_k = \frac{1}{k+1}\left(\left(\sum_{\nu=0}^{\infty} N_\nu\right)^{-k-1}\right)_{(k)},$$

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c) *$N_\nu \in BP_{2\nu}(BP)$ is defined inductively by*

$$\sum_{a,b} m_a t_b^{2^a} x^{2^{a+b}} = \sum_{f \geqq 0} m_f \Big(\sum_{\nu \geqq 0} N_\nu x^{\nu+1} \Big)^{2^f}$$

where x is an indeterminate, and

d) *$m_n \in \pi_{2(2^n-1)}(BP) \otimes Q$ is related to v_n by*

$$v_n = 2m_n - \sum_{i=1}^{n-1} v_{n-i}^{2^i} m_i$$

Thus the first few nonzero groups of $BP_*(P_{2n+1})$ are $BP_{2n+1}(P_{2n+1}) = \mathbf{Z}_2$, $BP_{2n+3}(P_{2n+1}) = \mathbf{Z}_4$, $BP_{2n+5}(P_{2n+1}) = \mathbf{Z}_8$, and $BP_{2n+7}(P_{2n+1}) = \mathbf{Z}_{16} \oplus \mathbf{Z}_2$, with $v_2\gamma_n$ generating the latter $\mathbf{Z}_2$-summand. The formula for the coaction is extremely complicated. The first few terms are

$$\begin{aligned}\Psi(\gamma_i) = 1 \otimes \gamma_i - (i-1)t_1 \otimes \gamma_{i-1} + \Big((2(i-2) + \binom{i-2}{2})t_1^2 \\ + (i-2)v_1 t_1 \Big) \otimes \gamma_{i-2} - \Big(\binom{i-3}{3} t_1^3 \\ + (i-3)((2i-3)t_1^3 + (i+1)v_1 t_1^2 + v_1^2 t_1 + t_2) \Big) \otimes \gamma_{i-3} + \ldots\end{aligned}$$

In Section 3 we use this coaction (in P_{n-4}^n) to prove that if S^n has 4 independent vector fields, then $n \equiv 7(8)$. This is of course a very elementary result which was known long before Adams' solution of the vector field problem [**2**]. However it illustrates with a minimum of computation the application of coalgebraic methods (and particularly BP_*) to geometric questions. The author has proved by similar methods the known result that if S^n has 10 independent vector fields, then $n \equiv 31(32)$, but the calculations involved are extraordinarily tedious.

Theorem 1.1 is not quite complete in that it does not give the action of $v_1 \in BP_*$ on our generators of $BP_*(P_{2n+1})$. (In order to use the coaction formula we must know the structure of $BP_*(P_{2n+1})$ as a BP_*-module.) Some partial information, sufficient for our application to vector fields, is given in Theorem 2.5. We conjecture that

$$v_1\gamma_i = -2\gamma_{i+1} - \sum_{j=2}^{[\log_2(i-n+2)]} v_j \gamma_{i+2-2}\, \gamma_{i+2-2^j}.$$

Theorem 1.1 (i) is a straightforward Adams spectral sequence computation, while Theorem 1.1 (ii) follows from Adams' formula [**3**] for the MU-coaction in CP^∞ using the Spanier-Whitehead dual of the canonical map $RP^\infty \to CP^\infty$. Thus the methods are not new, and indeed the result may be known to a few specialists. The thrust of the paper is to stimulate applications of BP in new directions. The author wishes to thank Haynes Miller and Steve Wilson for introducing him to BP and answering a few questions. After this paper had been written, the author was told that $BP_*(P_1)$ was computed in Ming's thesis [**10**].

2. Proof of Theorem 1.1. In this section we prove the analogues of Theorem 1.1 for $P_{2n+1}^{2m} = RP^{2m}/RP^{2n}$ and P_{2n+1}^{2m+1}. Theorem 1.1 is obtained by letting $m = \infty$ in P_{2n+1}^{2m}.

Brown and Peterson [**6**] showed that

$$H^*(BP; \mathbf{Z}_2) \approx \mathscr{A}/(\mathrm{Sq}^1) = \mathscr{A}/I = \mathscr{A}//E.$$

Here (Sq^1) denotes the 2-sided ideal generated by Sq^1, I denotes the left ideal generated by Milnor basis elements $\mathscr{P}_1{}^0, \mathscr{P}_2{}^0, \ldots$, and E the primitively generated exterior subHopf algebra of $\mathscr{A}$ generated by $\mathscr{P}_1{}^0, \mathscr{P}_2{}^0, \ldots$ Thus by the change-of-rings theorem, for any space X, $\mathrm{Ext}_{\mathscr{A}}(X \wedge BP) \approx \mathrm{Ext}_E(X)$. Here and throughout the paper we abbreviate $\mathrm{Ext}_A(H^*(X; Z_2), Z_2)$ to $\mathrm{Ext}_A(X)$, where $A = E$ or $\mathscr{A}$. In particular

$$\mathrm{Ext}_{\mathscr{A}}(BP) \approx \mathrm{Ext}_E(Z_2, Z_2) \approx Z_2[x_0, x_1, x_2, \ldots],$$
$$\text{where } x_i \in \mathrm{Ext}_E{}^{1,2^{i+1}-1}(Z_2, Z_2).$$

2.1 Proposition. *In* $H^*(RP^\infty; Z_2) \approx Z_2[\alpha]$, $\mathscr{P}_i{}^0(\alpha^j) = j\,\alpha^{j+2^i-1}$.

Proof. Write $\mathscr{P}_i{}^0 = \sum_I a_I \mathrm{Sq}^I$, where Sq^I are admissible (Adem) monomials of degree $2^i - 1$ and $a_I \in Z_2$. Applying ξ_i shows $a_{(2^{i-1},2^{i-2},\ldots,2,1)} = 1$. But $\mathrm{Sq}^{2^{i-1}} \ldots \mathrm{Sq}^2\mathrm{Sq}^1$ is the only admissible monomial of degree $2^i - 1$ which can be nonzero on a 1-dimensional class. Therefore, $\mathscr{P}_i{}^0\alpha = \alpha^{2^i}$.

Assume the Proposition proved for $j - 1$. Since $\mathscr{P}_i{}^0$ is primitive

$$\mathscr{P}_i{}^0(\alpha^j) = \mathscr{P}_i{}^0(\alpha^{j-1}) \cup \alpha + \alpha^{j-1} \cup \mathscr{P}_i{}^0(\alpha)$$
$$= (j-1)\alpha^{j+2^i-1} + \alpha^{j+2^i-1} = j\,\alpha^{j+2^i-1}.$$

2.2 Lemma. *Suppose B is a subalgebra of A such that A is a free B-module on generators of degree* 0 *and d. Suppose M is a bounded-below A-module such that* $\mathrm{Ext}_B{}^{s,t}(M, \mathbf{Z}_2) = 0$ *whenever $t - s$ is odd. Then there is an isomorphism of* $\mathbf{Z}_2[x]$*-modules*

$$\mathrm{Ext}_A{}^{*,*}(M, \mathbf{Z}_2) \approx \mathbf{Z}_2[x] \otimes \mathrm{Ext}_B{}^{*,*}(M, \mathbf{Z}_2), \quad \textit{where } x \in \mathrm{Ext}_A{}^{1,d}(\mathbf{Z}_2, \mathbf{Z}_2).$$

Proof. We use the exact sequence of [**4**, 3.2]:

$$\rightarrow \mathrm{Ext}_A{}^{s-1,t-d}(M, \mathbf{Z}_2) \xrightarrow{x} \mathrm{Ext}_A{}^{s,t}(M, \mathbf{Z}_2) \rightarrow \mathrm{Ext}_B{}^{s,t}(M, \mathbf{Z}_2)$$
$$\rightarrow \mathrm{Ext}_A{}^{s,t-d}(M, \mathbf{Z}_2).$$

We first note that $\mathrm{Ext}_A{}^{s,t}(M, \mathbf{Z}_2) = 0$ when $t - s$ is odd, for the first such nonzero element would have to induce an element in $\mathrm{Ext}_B{}^{s,t}(M, \mathbf{Z}_2)$, where none exists. Thus the exact sequence above is in fact short exact. This implies that $\mathrm{Ext}_A{}^{s,t}(M, \mathbf{Z}_2)$ contains a subset S which maps isomorphically onto

$\text{Ext}_B^{s,t}(M, \mathbf{Z}_2)$, and

$$S \oplus xS \oplus x^2S \oplus \ldots \subset \text{Ext}_A^{s,t}(M, \mathbf{Z}_2).$$

To show the inclusion is actually equality, consider the smallest degree element not in the sum and use the exact sequence to find one of smaller degree.

2.3 COROLLARY. *$\text{Ext}_E^{*,*}(P_{2n+1}^{2m})$ is a free $\mathbf{Z}_2[x_1, x_2, \ldots]$-module on generators $g_n, g_{n+1}, \ldots, g_{m-1}$, where $g_i \in \text{Ext}_E^{0,2i+1}(P_{2n+1}^{2m})$.*

Proof. E may be constructed by adding one generator $\mathscr{P}_i^0$ at a time, and Lemma 2.2 may be applied. The induction is begun by noting that if $\mathscr{A}_0$ is the subalgebra of $\mathscr{A}$ generated by $\mathscr{P}_1^0$, then

$$\text{Ext}_{\mathscr{A}_0}^{s,t}(P_{2n+1}^{2m}) = \begin{cases} \mathbf{Z}_2 & s = 0, t = 2i+1, n \leqq i < m \\ 0 & \text{otherwise.} \end{cases}$$

2.4 THEOREM. *As a module over $\text{Ext}_E^{*,*}(\mathbf{Z}_2, \mathbf{Z}_2)$, $\text{Ext}_E^{*,*}(P_{2n+1}^{2m})$ is generated by the elements g_i of 2.3 with the only relations being consequences of*

$$R_i : 0 = \sum_{\nu=0}^{[\log_2(i-n+1)]} x_\nu g_{i-2^\nu+1}, \quad n \leqq i < m.$$

Proof. That R_i is a relation follows readily from the cobar resolution [**1**]. To see this, let $H_* = H_*(P_{2n+1}^{2m})$ and let E_* denote the dual of E. E_* is a primitively generated exterior algebra on classes ξ_i of degree $2^i - 1$. Let $\bar{E}_* = E_*/E_0$. $\text{Ext}_E^{1,*}(P_{2n+1}^{2m})$ is ker d_2/im d_1 in

$$H_* \xrightarrow{d_1} \bar{E}_* \otimes H_* \xrightarrow{d_2} \bar{E}_* \otimes \bar{E}_* \otimes H_*,$$

where

$$d_1(\hat{\alpha}_{2i-1}) = 0, \quad d_1(\hat{\alpha}_{2i}) = \sum_{\nu=1}^{[\log_2(2i-2n)]} \xi_\nu \otimes \hat{\alpha}_{2i-2^\nu+1},$$

$$\text{and } d_2(\xi_i \otimes \hat{\alpha}_j) = \xi_i \otimes d_1\hat{\alpha}_j.$$

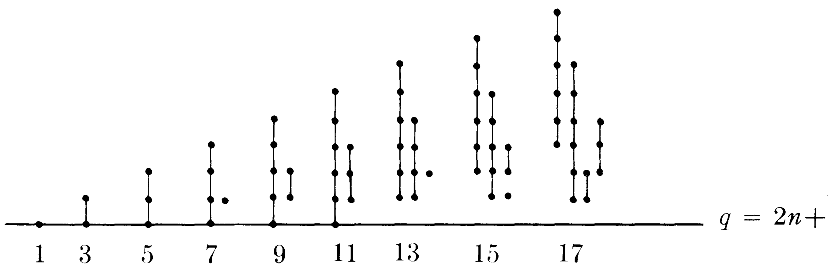
Then $x_{\nu-1}g_{i-2^{\nu-1}+1}$ corresponds to $\xi_\nu \otimes \hat{\alpha}_{2i-2^\nu+3}$, so that the relation R_i is due to $d_1(\hat{\alpha}_{2i+2})$.

That these are the only relations follows by induction from the exact sequence

$$\longrightarrow \text{Ext}_{E_i}^{s-1,t-2^i+1}(P_{2n+1}^{2m}) \xrightarrow{x_{i-1}} \text{Ext}_{E_i}^{s,t}(P_{2n+1}^{2m}) \longrightarrow \text{Ext}_{E_{i-1}}^{s,t}(P_{2n+1}^{2m}) \longrightarrow$$

where E_i is the exterior algebra generated by $\mathscr{P}_1^0, \ldots, \mathscr{P}_i^0$.

Since $\text{Ext}_{\mathscr{A}}^{s,t}(P_{2n+1}^{2m} \wedge BP) = 0$ for $t - s$ odd, there can be no nonzero differentials in the Adams spectral sequence converging to $\pi_*^s(P_{2n+1}^{2m} \wedge BP) \simeq BP_*(P_{2n+1}^{2m})$. For example, the Adams spectral sequence chart, (see e.g. [**7**]) for $\pi_q(P_{2n+1}{}^{2n+12} \wedge BP)$ begins



Here vertical lines indicate multiplication by x_0 in Ext ( ) which corresponds (up to elements of higher filtration) to multiplication by 2 in $\pi_*(\ )$.

2.5 THEOREM. *Suppose* $\{\gamma_i \in BP_{2i+1}(P_{2n+1}^{2m}) : n \leqq i < m\}$ *is any collection of filtration zero generators. Then as a graded abelian group* $BP_*(P_{2n+1}^{2m})$ *has generators* $v_2{}^{i_2} \ldots v_r{}^{i_r}\gamma_i$ *and* $v_1{}^{i_1}v_2{}^{i_2} \ldots v_r{}^{i_r}\gamma_{m-1}$ *of degree* $2i + 1 + \sum_{\nu=1}^{r} 2i_\nu(2^\nu - 1)$ *and filtration* $\sum_{\nu=1}^{r} i_\nu$ *truncated by* $2^{i+1-n}v_1{}^{i_1} \ldots v_r{}^{i_r}\gamma_i = 0$, *for all* $n \leqq i < m$, $i_\nu \geqq 0$, $r \geqq 0$. *Moreover, for* $n \leqq i < m$, $\sum_{\nu=0}^{[\log_2(i-n+1)]} v_\nu\gamma_{i-2^\nu+1}$ *has filtration* $\geqq 2$.

Remark. We shall soon give a specific set of generators γ_i. The last part of the theorem gives a partial description of the action of v_1 on the γ_i. Theorem 1.1 (i) follows from this theorem by letting m become infinite.

Proof. The generators $v_i \in \pi_{2(2^i-1)}(BP)$ must have filtration 1 and must be represented in $\mathrm{Ext}_E(\mathbf{Z}_2, \mathbf{Z}_2)$ by x_i. Similarly γ_i must be represented in $\mathrm{Ext}(P_{2n+1}^{2m} \wedge BP)$ by g_i. The relation $x_0{}^{i+1-n}g_i = 0$ in $\mathrm{Ext}_E(P_{2n+1}^{2m})$ is established by induction on i using the relation R_i. Since there are no elements of filtration greater than that of $x_0{}^{i+1-n}g_i$, this implies $2^{i+1-n}\gamma_i = 0$. The final statement of the theorem follows from the Ext relation R_i.

2.6 PROPOSITION. $BP_*(P_{2n+1}^{2m+1}) \approx BP_*(P_{2n+1}^{2m}) \oplus BP_*(S^{2m+1})$
$BP_*(P_{2n}^{2m}) \approx BP_*(P_{2n+1}^{2m}) \oplus BP_*(S^{2n})$.

Proof. This follows easily from the exact BP-homology sequences of the relevant cofibrations.

In fact, the splitting of homotopy groups comes from a splitting of spaces.

2.7 PROPOSITION. $P_{2n+1}^{2m+1} \wedge BP \simeq P_{2n+1}^{2m} \wedge BP \vee S^{2m+1} \wedge BP$
$P_{2n}^{2m} \wedge BP \simeq P_{2n+1}^{2m} \wedge BP \vee S^{2_1} \wedge BP$.

Proof. To prove the first we let

$$S^{2m+1} \xrightarrow{f} P_{2n+1}^{2m+1} \wedge BP$$

be a map such that the homotopy class of

$$S^{2m+1} \xrightarrow{f} P_{2n+1}^{2m+1} \wedge BP \xrightarrow{k} S^{2m+1} \wedge BP$$

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is a generator. Then

$$S^{2m+1} \wedge BP \xrightarrow{f \wedge BP} P_{2n+1}^{2m+1} \wedge BP \wedge BP \xrightarrow{P \wedge \mu} P_{2n+1}^{2m+1} \wedge BP \xrightarrow{k} S^{2m+1} \wedge B$$

induces an isomorphism of $\mathbf{Z}_2$-cohomology groups and hence of homotopy groups. Thus so does

$$P_{2n+1}^{2m} \wedge BP \vee S^{2m+1} \wedge BP \xrightarrow{(i \wedge BP) \vee ((P \wedge \mu)(f \wedge BP))} P_{2n+1}^{2m+1} \wedge BP$$

Thus it is a homotopy equivalence by J. H. C. Whitehead's theorem.

For the second, we note that by G. W. Whitehead's duality theorem [**13**]

$$[P_{2n+1}^{2m}, S^{2n+1} \wedge BP] \approx \pi_{2^L-2n-2}(P_{2^L-2m-1}^{2^L-2n-2} \wedge BP) = 0.$$

Thus the cofibration sequence

$$P_{2n}^{2m} \wedge BP \xrightarrow{i} P_{2n+1}^{2m} \wedge BP \rightarrow S^{2n+1} \wedge BP$$

implies that there is a map

$$P_{2n+1}^{2m} \xrightarrow{f} P_{2n}^{2m} \wedge BP$$

such that $if = 1 \wedge \iota$. As before,

$$P_{2n+1}^{2m} \wedge BP \vee S^{2n} \wedge BP \xrightarrow{(1 \wedge \mu)(f \wedge BP) \vee (i \wedge BP)} P_{2n}^{2m} \wedge BP$$

is a homotopy equivalence. This completes the proof.

Adams [**3**, Lemma 2.14] has defined generators $\beta_i \in BP_{2i}(CP^\infty)$. We use these to define $\gamma_i \in BP_{2i+1}(RP)$. There are canonical maps

$$RP_m{}^n \xrightarrow{h_m{}^n} CP_{[(m+1)/2]}^{[n/2]}$$

which are compatible with respect to inclusions and collapsings. The Spanier-Whitehead $(2^L - 1)$-dual [**12**; **5**] is a map

$$\Sigma CP_{2^{L-1}-1-[n/2]}^{2^{L-1}-1-[(m+1)/2]} \xrightarrow{D(h_m{}^n)} RP_{2^L-1-n}^{2^L-1-m}$$

which induces an epimorphism in $\mathbf{Z}_2$-cohomology. Reindexing, we have maps

$$g_{2n-1}^{2m+\epsilon} : \Sigma CP_{n-1}^m \rightarrow RP_{2n-1}^{2m+\epsilon}, \quad \epsilon = 0 \text{ or } 1,$$

compatible with respect to inclusions and collapsings, and inducing epimorphisms in $\mathbf{Z}_2$-cohomology. Consideration of the induced homomorphism in $\mathrm{Ext}_E(\ )$ shows that

$$BP_*(\Sigma CP_{n-1}^m) \xrightarrow{g_{2n-1*}^{2m+\epsilon}} BP_*(P_{2n-1}^{2m+\epsilon})$$

is surjective.

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2.8 *Definition.* $\gamma_i = g_{2n-1\,*}^{2m+\epsilon}(s\beta_i) \in BP_{2i+1}(P_{2n-1}^{2m+\epsilon})$.

Theorems 2.5 and 2.6 describe the structure of $BP_*(P_{2n-1}^{2m+\epsilon})$ as a BP_*-module with respect to these generators. The coaction formula of Theorem 1.1(ii), valid either in finite- or infinite-dimensional real projective space, follows now from the analogous formula for the β_i.

Proof of Theorem 1.1(ii). The following diagram is commutative

$$\begin{array}{ccc} MU_*(CP^\infty) & \xrightarrow{\Psi_1} & MU_*(MU) \otimes_{MU_*} MU_*(CP^\infty) \\ \downarrow \pi' & & \downarrow \pi \otimes \pi' \\ BP_*(CP^\infty) & \xrightarrow{\Psi} & BP_*(BP) \otimes_{BP_*} BP_*(CP^\infty) \end{array}$$

and $\pi'(\beta_i{}^{MU}) = \beta_i$. Thus

$$\gamma(\beta_i) = (\pi \otimes \pi')\Psi_1\beta_i{}^{MU} = \sum_{j=1}^{i} \pi\left(\sum_{k\geqq 0} b_k\right)^j_{(i-j)} \otimes \beta_j$$

by [**3**; 11.4]. Thus S_k of Theorem 1.1(ii) is Adams' πb_k. Adams does not give an expression for the πb_k; however, he does give an expression for πM_k, where

$$b_k = \frac{1}{k+1}\left(\sum_{i=0}^{\infty} M_i\right)^{-k-1}_{(k)} \quad [\mathbf{3}, 7.5].$$

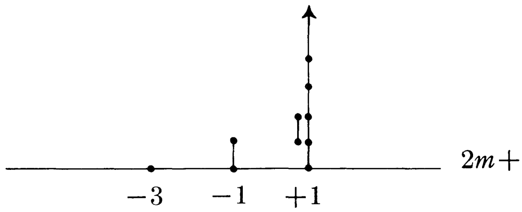
Letting $N_k = \pi M_k$, our 1.1(ii)(c) is Adams' 16.3. The relation 1.1(ii)(d) between v_i and m_i was proved in [**8**].

3. Application to vector fields on spheres. It is well-known [**2**] that if S^n has k independent vector fields, there is a map

$$S^n \xrightarrow{f} P^n_{n-k}$$

such that following it by the collapsing map yields (up to homotopy) 1_{S^n}. Consideration of the induced map in $H_*(\ ;Z)$ or $BP_*(\ )$ shows n must be odd, say $n = 2m + 1$. Let X denote a generator of $BP_n(S^n)$. Then $\Psi(X) = 1 \otimes X$, for there are no elements in $BP_*(S^n)$ of smaller degree. Thus $\Psi(f_*X) = 1 \otimes f_*X$ and $f_*X = \gamma_m +$ terms involving lower γ_i. This enables us to obtain restrictions on n, although the computations become extremely tedious for $k \geqq 10$.

We illustrate by showing if S^n has 4 vector fields, then $n \equiv 7(8)$, by showing if there exists a degree 1 map $S^{2m+1} \to P^{2m+1}_{2m-3}$, then $m \equiv 3(4)$. Of course, this is easily established using the Steenrod operations Sq^2 and Sq^4, but this proof illustrates our method with a minimum of computation. $BP_*(P^{2m+1}_{2m-3})$ begins



(i.e. its first generators are γ_{m-2}, γ_{m-1}, $v_1\gamma_{m-1}$, and γ_m, of order 2, 4, 4, and ∞). We show that if

(3.1) $\Psi(\gamma_m + Nv_1\gamma_{m-1}) = 1 \otimes (\gamma_m + Nv_1\gamma_{m-1})$

then $m \equiv 3(4)$.

The left-hand-side of (3.1) is evaluated by 1.1(ii). In evaluating the right-hand-side, we note that there is a homomorphism $\eta_R : BP_* \to BP_*(BP)$ such that in $BP_*(BP) \otimes_{BP_*} BP_*(X)$, $t \otimes v \cdot \gamma = \eta_R(v) \cdot t \otimes \gamma$ (see [**3**, Proof of 16.1 (v)]). η_R is defined by

$$\eta_R(m_k) = \sum_{i=0}^{k} m_i t_{k-i}^{2^i} \quad [\mathbf{3}, 16.1(i)].$$

The behavior of η_R on the v_i is then determined using 1.1(ii)(d). In particular

$$\eta_R(v_1) = v_1 + 2t_1, \quad \eta_R(v_2) = v_2 + 2t_2 - 5v_1t_1^2 - 3v_1^2t_1 - 4t_1^3.$$

Ignoring cancelling terms, (3.1) becomes

$$-(m-1)t_1 \otimes \gamma_{m-1} + \left(\binom{m-2}{2}t_1^2 + mv_1t_1\right) \otimes \gamma_{m-2} + Nv_1(-(m-2)t_1 \otimes \gamma_{m-2}) = N2t_1 \otimes \gamma_{m-1}.$$

By Theorem 2.5 $2\gamma_{m-1} = -v_1\gamma_{m-2}$, for there are no terms of higher filtration. Thus the right-hand-side becomes

$$-Nt_1 \otimes v_1\gamma_{m-2} = -N(v_1 + 2t_1)t_1 \otimes \gamma_{m-2} = -Nv_1t_1 \otimes \gamma_{m-2},$$

and the equation becomes

$$-(m-1)t_1 \otimes \gamma_{m-1} + \left(\binom{m-2}{2}t_1^2 + (m - N(m-3))v_1t_1\right) \otimes \gamma_{m-2} = 0.$$

The only possible way to eliminate the first term is to have $m = 2l + 1$, so that the first term becomes

$$-lt_1 \otimes 2\gamma_{m-1} = lt_1 \otimes v_1\gamma_{m-2} = lv_1t_1 \otimes \gamma_{m-2},$$

and the equation becomes

$$\left(\binom{2l-1}{2}t_1^2 + (3l + 1 - N(2l-2))v_1t_1\right) \otimes \gamma_{m-2} = 0.$$

This implies that both coefficients must be even, i.e. l is odd, and hence $m \equiv 3(4)$.

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