

On the approach of a series to its Cesàro limit

By J. M. HYSLOP.

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1.1. The object of this paper is to investigate some properties of series which satisfy conditions of the form

$$(1.11) \quad S_n^{(p)} = O(n^{p-\rho}),$$

or

$$(1.12) \quad S_n^{(p)} = o(n^{p-\rho}),$$

where $0 < \rho \leq p$. $S_n^{(p)}$ denotes, as usual, the n -th Cesàro sum of order p for the series $\sum a_n$ and $A_n^{(p)}$ the binomial coefficient $\binom{p+n}{n}$. It is convenient to state here some properties of $S_n^{(p)}$ and $A_n^{(p)}$ to which we must constantly refer in the sequel.

LEMMA 1. *The functions $S_n^{(p)}$ and $A_n^{(p)}$ possess the following properties:*

$$(1.13) \quad \sum_{\nu=0}^n A_\nu^{(p)} A_{n-\nu}^{(q)} = A_n^{(p+q+1)},$$

$$(1.14) \quad \sum_{\nu=0}^n S_\nu^{(p)} A_{n-\nu}^{(q)} = S_n^{(p+q+1)},$$

$$(1.15) \quad A_n^{(p)} \sim \frac{n^p}{\Gamma(p+1)},$$

where p is not a negative integer. If p is a negative integer $A_n^{(p)} = 0$ when $n \geq -p$.

These results are all well known.

The relations (1.11) and (1.12) may be regarded, from our point of view, as simplified forms of the more extended relations

$$(1.16) \quad S_n^{(p)} = sA_n^{(p)} + O(n^{p-\rho}),$$

$$(1.17) \quad S_n^{(p)} = sA_n^{(p)} + \lambda A_n^{(p-\rho)} + o(n^{p-\rho}),$$

where $0 < \rho \leq p$. More precisely we have the following lemma.

LEMMA 2. *If the series Σa_n satisfies (1.16) the series Σu_n , where*

$$(1.18) \quad u_n = a_n - s A_n^{(-1)},$$

satisfies (1.11). If Σa_n satisfies (1.17) the series Σv_n , where

$$(1.19) \quad v_n = a_n - s A_n^{(-1)} - \lambda A_n^{(-p-1)},$$

satisfies (1.12).

Let $T_n^{(p)}$ be the n -th Césàro sum of order p for the series Σv_n . Then, by (1.13) and (1.14),

$$\begin{aligned} T_n^{(p)} &= \sum_{\nu=0}^n A_{n-\nu}^{(p)} a_\nu - s \sum_{\nu=0}^n A_{n-\nu}^{(p)} A_\nu^{(-1)} - \lambda \sum_{\nu=0}^n A_{n-\nu}^{(p)} A_\nu^{(-p-1)} \\ &= S_n^{(p)} - s A_n^{(p)} - \lambda A_n^{(p-\rho)} \\ &= o(n^{p-\rho}). \end{aligned}$$

The proof of the first result is obvious.

1.2. It will be observed that series which satisfy (1.16) or (1.17) are “more than” summable (C, p) . Generally speaking, the theorems of this paper may be regarded as analogues, for series which are “more than” summable (C, p) of certain known¹ theorems for series which are “less than” summable (C, p) . It is convenient for purposes of reference to state these known theorems here.

THEOREM 1. *If $S_n^{(p)} = O(n^{p+\beta})$, $\beta > 0$, then the series $\Sigma n^{-\gamma} a_n$ is summable (C, p) for any $\gamma > \beta$.*

THEOREM 2. *If $S_n^{(p)} = o(n^{p+\beta})$, $\beta > 0$, then the series $\Sigma n^{-\beta} a_n$ is either summable (C, p) or not summable by any Césàro means.*

THEOREM 3. *If the series $\Sigma n^{-\beta} a_n$ is summable (C, p) and $\beta > 0$ then $S_n^{(p)} = o(n^{p+\beta})$.*

In each case the theorem is proved on the supposition that p is a positive integer. Whenever necessary in the interests of simplicity I shall impose the same restriction on p . All the results are probably true when p is merely restricted to be positive, but their proofs, in the general case, would, I imagine, follow broadly the lines of Andersen’s proof² of the Bohr-Hardy Theorem, and be quite as long and difficult.

¹ Hardy and Littlewood, 7, 432-435. Dr L. S. Bosanquet kindly pointed out this connection to me, and his suggestion has enabled me to make this paper more comprehensive.

² Bohr, 2, 61. Hardy, 5. Andersen, 1, 47.

It is clear that the most interesting cases of (1.16) or (1.17) occur when $0 < \rho \leq 1$. Indeed, if $\rho > 1$ these conditions are very artificial and it would be difficult to find series to satisfy them, although it is easy enough, as we shall see, to find series satisfying the more extended conditions

$$(1.21) \quad S_n^{(p)} = sA_n^{(p)} + \sum_{\sigma=1}^r \lambda_\sigma A_n^{p-\sigma} + O(n^{p-\rho}),$$

$$(1.22) \quad S_n^{(p)} = sA_n^{(p)} + \sum_{\sigma=1}^r \lambda_\sigma A_n^{(p-\sigma)} + \lambda A_n^{(p-\rho)} + o(n^{p-\rho}),$$

where $r = [\rho] \leq \rho$, and $0 < \rho \leq p$.

The theorems of this paper may be modified to apply to series satisfying (1.21) or (1.22), but the presence of the terms involving $\lambda_1, \lambda_2, \dots, \lambda_r$ involves a considerable amount of tedious algebraical work in their proofs. I have therefore considered it advisable to confine myself to the case $0 < \rho \leq 1$.

It will be shown that Theorem 1 remains true when $\beta < 0$, but is capable of generalisation, that Theorem 2 remains unaltered with $\beta < 0$, and that certain modifications have to be made in the case of Theorem 3.

The paper concludes with some remarks on Tauberian Theorems for series satisfying these special conditions.

2.1. I begin by proving two simple theorems for series which satisfy (1.16) or (1.17).

THEOREM 4. *If (1.17) holds the series Σa_n is summable $(C, p - \rho)$ to the sum s , while, if (1.16) holds Σa_n is bounded $(C, p - \rho)$.*

Suppose that (1.17) holds. Then, by (1.13) and (1.14),

$$\begin{aligned} S_{n..}^{(p-\rho)} &= \sum_{\nu=0}^n A_{n-\nu}^{(-\rho-1)} S_\nu^{(p)} \\ &= s \sum_{\nu=0}^n A_\nu^{(p)} A_{n-\nu}^{(-\rho-1)} + \lambda \sum_{\nu=0}^n A_\nu^{(p-\rho)} A_{n-\nu}^{(-\rho-1)} + o\left\{ \sum_{\nu=0}^n \nu^{p-\rho} (n-\nu+1)^{-\rho-1} \right\} \\ &= sA_n^{(p-\rho)} + \lambda A_n^{(p-2\rho)} + o(n^{p-\rho}) \\ &= sA_n^{(p-\rho)} + o(n^{p-\rho}). \end{aligned}$$

The proof is similar when (1.16) holds.

When (1.16) is satisfied the series Σa_n is bounded $(C, p - \rho)$ and also summable (C, p) . It follows, by a theorem¹ due to Andersen,

¹ Andersen, 1, 56.

that Σa_n is summable $(C, p - \rho + \delta)$ for every positive δ . It also follows that the index of summability of series Σa_n , which satisfy (1.16) or (1.17), is not greater than $p - \rho$.

2.2. THEOREM 5. *If (1.16) or (1.17) is satisfied for a given value of p it is also satisfied when p is replaced by $p + \delta$ for any positive δ .*

Suppose that (1.17) is satisfied. Then, by Lemma 1,

$$\begin{aligned} S_n^{(p+\delta)} &= \sum_{\nu=0}^n A_{n-\nu}^{(\delta-1)} S_\nu^{(p)} \\ &= s \sum_{\nu=0}^n A_\nu^{(p)} A_{n-\nu}^{(\delta-1)} + \lambda \sum_{\nu=0}^n A_\nu^{(p-\rho)} A_{n-\nu}^{(\delta-1)} + o \left\{ \sum_{\nu=0}^n A_\nu^{(p-\rho)} A_{n-\nu}^{(\delta-1)} \right\} \\ &= s A_n^{(p+\delta)} + \lambda A_n^{(p+\delta-\rho)} + o(n^{p+\delta-\rho}). \end{aligned}$$

A similar proof holds in the case of (1.16).

2.3. As an illustration of these theorems and of some theorems which will be proved later on, it is interesting to consider the simple series $1 - 1 + 1 \dots$. For this series, easy calculations show that

(2.31) $S_n = \frac{1}{2} \{1 + (-1)^n\},$

(2.32) $S_n^{(1)} = \frac{1}{2} A_n^{(1)} + \frac{1}{4} \{1 + (-1)^n\},$

(2.33) $S_n^{(2)} = \frac{1}{2} A_n^{(2)} + \frac{1}{4} A_n^{(1)} + \frac{1}{8} \{1 + (-1)^n\},$

.....

(2.34) $S_n^{(p)} = \frac{1}{2} A_n^{(p)} + \frac{1}{2^2} A_n^{(p-1)} \dots + \frac{1}{2^p} A_n^{(1)} + \frac{1}{2^{p+1}} \{1 + (-1)^n\}.$

Moreover, if $0 < \delta < 1$, we see from (2.33) that

$$\begin{aligned} S_n^{(1+\delta)} &= \frac{1}{2} \sum_{\nu=0}^n A_{n-\nu}^{(\delta-2)} A_\nu^{(2)} + \frac{1}{4} \sum_{\nu=0}^n A_{n-\nu}^{(\delta-2)} A_\nu^{(1)} + \frac{1}{8} \sum_{\nu=0}^n A_{n-\nu}^{(\delta-2)} \{1 + (-1)^\nu\} \\ &= \frac{1}{2} A_n^{(1+\delta)} + \frac{1}{4} A_n^{(\delta)} + E, \end{aligned}$$

where

$$|E| < A \sum_{\nu=0}^n (\nu + 1)^{-\delta-2} < A.$$

Thus

(2.35) $S_n^{(1+\delta)} = \frac{1}{2} A_n^{(1+\delta)} + \frac{1}{4} A_n^{(\delta)} + O(1).$

Similarly it may easily be shown that

(2.36) $S_n^{(\delta)} = \frac{1}{2} A_n^{(\delta)} + O(1).$

3.1. In this section we consider the effect of the conditions of §1 on the summability of the series $\Sigma n^\rho a_n$ and similar series. Throughout, we shall denote by $S_{n,\rho}^{(p)}$ the n -th Cesàro sum of order p for the series $\Sigma n^\rho a_n$, and by $T_{n,\rho}^{(p)}$ the n -th Cesàro sum of order p for either

$\Sigma n^\rho u_n$ or $\Sigma n^\rho v_n$. First of all we state a well known and easily proved lemma which connects the Cesàro sums of the series $\Sigma n a_n$ and Σa_n .

3.2. LEMMA 3. *If p is positive we have*

$$(3.21) \quad S_{n,1}^{(p)} = (p + n + 1) S_n^{(p)} - (p + 1) S_n^{(p+1)}.$$

We now make some straightforward deductions from Lemma 3.

THEOREM 6. *If $1 = \rho \leq p$ and Σa_n satisfies (1.17) then the series $\Sigma n a_n$ is summable (C, p) to the sum $-\lambda$.*

From Lemma 3 and Theorem 5 we have

$$\begin{aligned} S_{n,1}^{(p)} &= (p + n + 1) \{sA_n^{(p)} + \lambda A_n^{(p-1)} + o(n^{p-1})\} \\ &\quad - (p + 1) \{sA_n^{(p+1)} + \lambda A_n^{(p)} + o(n^p)\} \\ &= \lambda \{(p + n + 1) A_n^{(p-1)} - (p + 1) A_n^{(p)}\} + o(n^p) \\ &= \lambda A_n^{(p)} \left\{ (p + n + 1) \frac{A_n^{(p-1)}}{A_n^{(p)}} - (p + 1) \right\} + o(n^p) \\ &= -\lambda A_n^{(p)} \left\{ (p + 1) - \frac{p(p + n + 1)}{p + n} \right\} + o(n^p) \\ &= -\lambda A_n^{(p)} + o(n^p). \end{aligned}$$

A similar, though simpler, proof suffices to prove the following theorem.

THEOREM 7. *If $1 = \rho \leq p$ and Σa_n satisfies (1.16), then the series $\Sigma n a_n$ is bounded (C, p) .*

If Σa_n is the particular series $1 - 1 + 1 - \dots$, we see from (2.32) and Theorem 7 that $\Sigma n a_n$ is bounded $(C, 1)$, while, from (2.33) and Theorem 6, $\Sigma n a_n$ is summable $(C, 2)$ to the sum $-\frac{1}{4}$. Using Andersen's Theorem¹ we at once obtain the familiar result that the series $1 - 2 + 3 - 4 \dots$ is summable $(C, 1 + \delta)$, to the sum $\frac{1}{4}$, for every positive δ .

3.3. We now prove two theorems similar to Theorems 6 and 7 for the case $0 < \rho < 1$. The proof of Theorem 8 may easily be constructed from that of Theorem 9, so that it is only necessary to prove the latter. In the proof of this and some succeeding theorems

¹ Andersen, 1, 56.

we employ a technique which was introduced¹ and developed by Andersen.

LEMMA 4. *If s is any positive integer and r is any real number², then*

$$(3.31) \quad \sum_{p=0}^s \Delta_{\nu+p}^{s+1} \epsilon_{\nu} A_{n-\nu-p}^{(r-1)} = \sum_{q=0}^s \binom{s+1}{q} \Delta^q \epsilon_{\nu} A_{n-\nu-q}^{(r+q-s-1)},$$

where

$$\Delta^{s+1} \epsilon_{\nu} = \epsilon_{\nu} - \binom{s+1}{1} \epsilon_{\nu+1} + \binom{s+1}{2} \epsilon_{\nu+2} + \dots + (-1)^{s+1} \epsilon_{\nu+s+1},$$

$$\Delta_{\nu+p}^{s+1} \epsilon_{\nu} = \epsilon_{\nu} - \binom{s+1}{1} \epsilon_{\nu+1} + \dots + (-1)^p \binom{s+1}{p} \epsilon_{\nu+p}.$$

LEMMA 5. *If r is any real number, and p is a positive integer, we have*

$$(3.32) \quad \sum_{\nu=\mu}^n A_{n-\nu}^{(r)} A_{\nu-\mu}^{(-p-2)} \epsilon_{\nu} = \sum_{q=0}^{p+1} \binom{p+1}{q} A_{n-\mu-q}^{(r+q-p-1)} \Delta^q \epsilon_{\mu}.$$

We have

$$\begin{aligned} \sum_{\nu=\mu}^n A_{n-\nu}^{(r)} A_{\nu-\mu}^{(-p-2)} \epsilon_{\nu} &= \sum_{\sigma=0}^{n-\mu} A_{n-\mu-\sigma}^{(r)} A_{\sigma}^{(-p-2)} \epsilon_{\mu+\sigma} \\ &= \sum_{\sigma=0}^{n-\mu} A_{\sigma}^{(-p-2)} \epsilon_{\mu+\sigma} \sum_{\tau=0}^{n-\mu-\sigma} A_{\tau}^{(r-1)} \\ &= \sum_{\tau=0}^{n-\mu} A_{\tau}^{(r-1)} \Delta_{n-\tau}^{p+1} \epsilon_{\mu} \\ &= \sum_{\tau=0}^{n-\mu} A_{n-\mu-\tau}^{(r-1)} \Delta_{\mu+\tau}^{p+1} \epsilon_{\mu} \\ &= \sum_{\tau=0}^p A_{n-\mu-\tau}^{(r-1)} \Delta_{\mu+\tau}^{p+1} \epsilon_{\mu} + \Delta^{p+1} \epsilon_{\mu} \sum_{\tau=p+1}^{n-\mu} A_{n-\mu-\tau}^{(r-1)}, \end{aligned}$$

and, by Lemma 4, this is equal to

$$\begin{aligned} &\sum_{q=0}^p \binom{p+1}{q} A_{n-\mu-q}^{(r+q-p-1)} \Delta \epsilon_{\mu} + A_{n-\mu-p-1}^{(r)} \Delta^{p+1} \epsilon_{\mu} \\ &= \sum_{q=0}^{p+1} \binom{p+1}{q} A_{n-\mu-q}^{(r+q-p-1)} \Delta^q \epsilon_{\mu}. \end{aligned}$$

The essence of this result is contained in Andersen's dissertation, but it is not quite stated in the form (3.32) which is most convenient for our purpose. In these circumstances I considered it advisable to insert the above proof.

¹ Andersen, 1, 47.

² Andersen, 1, 42.

THEOREM 8. *If p is a positive integer and if Σa_n satisfies (1.17), then, for $0 < \delta < \rho \leq 1 \leq p$, the series $\Sigma n^{\rho-\delta} a_n$ is summable $(C, p - \delta)$ to the sum of the convergent series*

$$(3.33) \quad \lambda \sum_{\nu=0}^{\infty} A_{\nu}^{(-\rho-1)} \nu^{\rho-\delta} + \sum_{\nu=0}^{\infty} \eta_{\nu} \Delta^{p+1} \nu^{\rho-\delta},$$

where η_n denotes the error term in (1.17).

THEOREM 9. *If p is a positive integer and if Σa_n satisfies (1.16) then, for $0 < \delta < \rho \leq 1 \leq p$, the series $\Sigma n^{\rho-\delta} a_n$ is bounded $(C, p - \delta)$, and summable $(C, p - \delta + \epsilon)$, for any positive ϵ , to the sum of the convergent series*

$$(3.34) \quad \sum_{\nu=0}^{\infty} \beta_{\nu} \Delta^{p+1} \nu^{\rho-\delta},$$

where β_n denotes the error term in (1.16).

It should be observed that these theorems are also true when $\delta = \rho$ and have already been proved as parts of Theorem 4. In this case, of course, the sum of the resulting series, namely Σa_n , is s , so that (3.33) and (3.34) do not apply.

For the series Σu_n , defined as in (1.18), we have

$$T_n^{(p)} = \beta_n = O(n^{p-\rho}),$$

and

$$\begin{aligned} T_{n, \rho-\delta}^{(p-\delta+\epsilon)} &= \sum_{\nu=0}^n A_{n-\nu}^{(p-\delta+\epsilon)} \nu^{\rho-\delta} a_{\nu} \\ &= \sum_{\nu=0}^n A_{n-\nu}^{(p-\delta+\epsilon)} \nu^{\rho-\delta} \sum_{\mu=0}^{\nu} A_{\nu-\mu}^{(-p-2)} T_{\mu}^{(p)} \\ &= \sum_{\mu=0}^n T_{\mu}^{(p)} \sum_{\nu=\mu}^n \nu^{\rho-\delta} A_{n-\nu}^{(p-\delta+\epsilon)} A_{\nu-\mu}^{(-p-2)} \\ &= E_1 + E_2, \end{aligned}$$

where

$$E_1 = \sum_{\mu=n-p}^n T_{\mu}^{(p)} \sum_{\nu=\mu}^n A_{n-\nu}^{(p-\delta+\epsilon)} \nu^{\rho-\delta} A_{\nu-\mu}^{(-p-2)} = O(n^{p-\delta}),$$

and

$$E_2 = \sum_{\mu=0}^{n-p-1} \beta_{\mu} \sum_{\nu=\mu}^n A_{n-\nu}^{(p-\delta+\epsilon)} \nu^{\rho-\delta} A_{\nu-\mu}^{(-p-2)}.$$

By Lemma 5, we may write

$$E_2 = \sum_{q=0}^{p+1} \binom{p+1}{q} F_q,$$

where

$$F_q = \sum_{\mu=0}^{n-p-1} \beta_\mu A_{n-\mu-q}^{(q+\epsilon-\delta-1)} \Delta^q \mu^{\rho-\delta}.$$

Since

$$\Delta^q \mu^{\rho-\delta} = O(\mu^{\rho-\delta-q}),$$

we have

$$\begin{aligned} F_0 &= O\left\{ \sum_{\mu=0}^{n-p-1} \mu^{\rho-\delta} (n-\mu)^{\epsilon-\delta-1} \right\} \\ &= O\left\{ n^{\rho-\delta} \sum_{\mu=0}^{\infty} (\mu+1)^{\epsilon-\delta-1} \right\} \\ &= O(n^{\rho-\delta}), \end{aligned}$$

if we suppose, as we may without loss of generality, that $0 < \epsilon < \delta$.

Again, if $1 \leq q \leq p$,

$$\begin{aligned} F_q &= O\left\{ \sum_{\mu=0}^{n-q} A_\mu^{(p-\delta-q)} A_{n-q-\mu}^{(q+\epsilon-\delta-1)} \right\} + O(n^{\rho-\delta}) \\ &= O\{A_{n-q}^{(p+\epsilon-2\delta)}\} + O(n^{\rho-\delta}) \\ &= O(n^{\rho-\delta}), \end{aligned}$$

since $0 < \epsilon < \delta$.

Finally

$$\frac{F_{p+1}}{A_{n-p-1}^{(p-\delta+\epsilon)}} = \frac{1}{A_{n-p-1}^{(p-\delta+\epsilon)}} \sum_{\mu=0}^{n-p-1} \beta_\mu \Delta^{p+1} \mu^{\rho-\delta} A_{n-p-1-\mu}^{(p-\delta+\epsilon)},$$

which, by the consistency theorem for Cesàro summability, tends to the sum of the convergent series

$$\sum_{\nu=0}^{\infty} \beta_\nu \Delta^{p+1} \nu^{\rho-\delta}.$$

The second part of the theorem now follows almost at once.

It has in fact been shown that

$$S_{n,\rho-\delta}^{(p-\delta+\epsilon)} = A_n^{(p-\delta+\epsilon)} \sum_{\nu=0}^{\infty} \beta_\nu \Delta^{p+1} \nu^{\rho-\delta} + O(n^{\rho-\delta}),$$

and the first part of the theorem follows from this and Theorem 4.

It is interesting at this point to compare Theorem 9 with the case of Theorem 1 when $\beta < 0$. Translating Theorem 1 with $-1 < \beta < 0$ into the notation of the present paper it would become:—

If $\sum a_n$ satisfies (1.11) then, for $0 < \delta \leq \rho < 1 \leq p$, where p is a positive integer, the series $\sum n^{\rho-\delta} a_n$ is summable (C, p) .

It will be observed that Theorem 9 is more general than this result.

4.1. It is easy to see that the hypotheses of Theorems 8 and 9 do not necessarily imply that $\sum n^\rho a_n$ is summable or bounded (C, p) . If, for example,

$$a_n = A_n^{(-\frac{1}{2})},$$

we have

$$S_n^{(1)} = A_n^{(\frac{1}{2})},$$

and the series $\sum n^{\frac{1}{2}} a_n$, being properly divergent, is neither summable nor bounded $(C, 1)$. The example indeed shows that, in general, it is not possible to assert that if $\sum a_n$ satisfies (1.16), or (1.17), the series $\sum n^\rho a_n$ will be bounded, or summable, by some Cesàro means of sufficiently high order. In this respect the cases $0 < \rho < 1$ and $\rho = 1$ of (1.16) and (1.17) present a marked contrast.

4.2. In view of these remarks the following theorems are of interest.

THEOREM 10. *If p is a positive integer, $0 < \rho < 1 \leq p$, and $\sum a_n$ satisfies (1.16), then the series $\sum n^\rho a_n$ is either bounded (C, p) or not bounded by any Cesàro means.*

THEOREM 11. *If p is a positive integer, $0 < \rho < 1 \leq p$, and $\sum a_n$ satisfies (1.17), with $\lambda = 0$, then the series $\sum n^\rho a_n$ is either summable (C, p) or not summable by any of Cesàro's means.*

It is well known and, in any case, follows readily from Lemma 3 and the consistency theorem for Cesàro summability, that necessary and sufficient conditions for a series $\sum a_n$ to be summable (C, p) are that $\sum a_n$ should be summable by some Cesàro method and that the sequence na_n should be summable $(C, p + 1)$ to zero. A similar result holds for series to be bounded (C, p) .

It should be observed that, if we take the series $\sum v_n$ instead of $\sum a_n$, Theorem 11 corresponds exactly to the case of Theorem 2 when $-1 < \beta < 0$. Hardy and Littlewood's proof extends to this case without alteration but, for the sake of completeness, I give a brief proof on the lines of the proof of Theorem 9. Theorem 10 may, of course, be proved by the same method.

It is sufficient to show that

$$S_{n, \rho+1}^{(p)} = o(n^{p+1}).$$

Taking as our hypothesis

$$S_n^{(p)} = sA_n^{(p)} + \eta_n, \quad \eta_n = o(n^{p-\rho}),$$

we have

$$\begin{aligned}
 S_{n,\rho+1}^{(p)} &= \sum_{\nu=0}^n A_{n-\nu}^{(p)} \nu^{\rho+1} \sum_{\mu=0}^{\nu} A_{\nu-\mu}^{(-p-2)} S_{\mu}^{(p)} \\
 &= \sum_{\mu=0}^n \eta_{\mu} \sum_{\nu=\mu}^n A_{n-\nu}^{(p)} A_{\nu-\mu}^{(-p-2)} \nu^{\rho+1} \\
 &= o(n^{\rho+1}) + \sum_{\mu=0}^{n-p-1} \eta_{\mu} \sum_{\nu=\mu}^n A_{n-\nu}^{(p)} A_{\nu-\mu}^{(-p-2)} \nu^{\rho+1}.
 \end{aligned}$$

By Lemma 5, the second term may be written in the form

$$\sum_{q=0}^{p+1} \binom{p+1}{q} F_q,$$

where

$$F_q = \sum_{\mu=0}^{n-p-1} \eta_{\mu} A_{n-\mu-q}^{(q-1)} \Delta^q \mu^{\rho+1}, \quad q = 0, 1, 2, \dots, p+1.$$

Clearly $F_0 = 0$ and, for $1 \leq q \leq p+1$,

$$F_q = o\left\{ \sum_{\mu=0}^{n-p-1} A_{n-\mu-q}^{(q-1)} \mu^{p+1-q} \right\} = o(n^{\rho+1}).$$

When Σa_n satisfies (1.17) and $\lambda \neq 0$ we have the following more precise theorem.

THEOREM 12. *If p is a positive integer, $0 < \rho < 1 \leq p$, and Σa_n satisfies (1.17), with $\lambda \neq 0$, then the series $\Sigma n^{\rho} a_n$ is not summable by Cesàro means of any order.*

If k is any number $\geq p$ and η_n , as usual, denotes the error term in (1.17), we have, by arguments similar to those used in the proofs of Theorems 9 and 11,

$$\begin{aligned}
 S_{n,\rho}^{(k)} &= \lambda \sum_{\nu=0}^n A_{n-\nu}^{(k)} A_{\nu}^{(-\rho-1)} \nu^{\rho} + \sum_{\mu=0}^{n-p-1} \eta_{\mu} A_{n-p-1-\mu}^{(k)} \Delta^{p+1} \mu^{\rho} + o(n^k) \\
 &= \lambda \sum_{\nu=0}^n A_{n-\nu}^{(k)} A_{\nu}^{(-\rho-1)} \nu^{\rho} + \sum_{\nu=p+1}^n \epsilon_{\nu-p-1} (\nu-p-1)^{p-\rho} A_{n-\nu}^{(k)} \Delta^{p+1} (\nu-p-1)^{\rho} + o(n^k),
 \end{aligned}$$

where $\epsilon_n \rightarrow 0$ as $n \rightarrow \infty$. Now

$$\begin{aligned}
 A_{\nu}^{(-\rho-1)} &= A \nu^{-\rho-1} + O(\nu^{-\rho-2}), \\
 (\nu-p-1)^{p-\rho} \Delta^{p+1} (\nu-p-1)^{\rho} &= B \nu^{-1} + O(\nu^{-2}),
 \end{aligned}$$

where A and B are non-zero constants. Given $\epsilon \left(< \frac{A\lambda}{B} \right)$, we can

determine $N (> p + 1)$, such that $|\epsilon_{\nu-p-1}| < \epsilon$ for $\nu \geq N$. Thus, since $A_n^{(-\rho-1)} < 0$ for $n \geq 1$, we have

$$\begin{aligned} |S_{n,\rho}^{(k)}| &\geq |A\lambda \sum_{\nu=N}^n A_{n-\nu}^{(k)} \nu^{-1}| - |\sum_{\nu=N}^n B\epsilon_{\nu-p-1} A_{n-\nu}^{(k)} \nu^{-1}| - O(n^k) \\ &\geq \{|A\lambda| - |B|\epsilon\} \sum_{\nu=N}^n A_{n-\nu}^{(k)} \nu^{-1} - O(n^k). \end{aligned}$$

Since the series $\Sigma \nu^{-1}$ is properly divergent, the result follows.

4.3 We have seen from Theorem 9 that, if Σa_n satisfies (1.16) and $0 < \delta \leq \rho < 1 \leq p$, the series $\Sigma n^{\rho-\delta} a_n$ is summable $(C, p - \delta + \epsilon)$ for every positive ϵ . Also, from Theorem 7, we know that, if Σa_n satisfies (1.16) when $p \geq \rho = 1$, the series Σna_n is bounded (C, p) . It might be plausible then to suppose that the hypothesis of Theorem 7 would imply the summability $(C, p + \epsilon)$ of Σna_n for every positive ϵ . Such a surmise however would be erroneous for it is possible to construct a series Σa_n , which satisfies (1.16) in the particular case $p = \rho = 1$, and for which the series Σna_n is not summable $(C, 1 + \epsilon)$ for any positive ϵ .

Consider, for example, the series Σa_n for which

$$S_n^{(1)} = \lambda_n,$$

where λ_n is the bounded function¹

$$\begin{aligned} \lambda_n &= 1, & 2^{2\nu} \leq n < 2^{2\nu+1} \\ &= 0, & 2^{2\nu+1} \leq n < 2^{2\nu+2}. \end{aligned}$$

As in the proof of Theorem 9, we have

$$\begin{aligned} S_{n,1}^{(2)} &= \sum_{\mu=0}^n \lambda_\mu \sum_{\nu=\mu}^n \nu A_{n-\nu}^{(2)} A_{\nu-\mu}^{(-3)} \\ &= \sum_{\mu=0}^n \lambda_\mu (3\mu - 2n). \end{aligned}$$

It is not difficult to show that

$$n^{-1} \sum_{\mu=0}^n \lambda_\mu \rightarrow \frac{1}{\frac{2}{3}}, \quad n^{-2} \sum_{\mu=0}^n \mu \lambda_\mu \rightarrow \frac{1}{\frac{10}{5}},$$

according as n tends to infinity through the even or the odd powers of 2. It follows that

$$\lim_{n \rightarrow \infty} \frac{S_{n,1}^{(2)}}{A_n^{(2)}} = 2 \lim_{n \rightarrow \infty} \frac{S_{n,1}^{(2)}}{n^2} = \left. \begin{matrix} -11/15 \\ -4/15 \end{matrix} \right\},$$

¹The series $\Sigma(\lambda_n - \lambda_{n-1})$ possesses many interesting properties. See Hardy, 4, 6.

according as n tends to infinity through the even or the odd powers of 2. Thus the series $\sum na_n$ is not summable $(C, 2)$ and *a fortiori* is not summable $(C, 1 + \epsilon)$ for $0 < \epsilon < 1$. Moreover $\sum na_n$ is not summable $(C, 1 + \epsilon)$ for $\epsilon > 1$, for, if it were, it would necessarily be summable $(C, 2)$ since, by Theorem 7, it is bounded $(C, 1)$.

4.4. We may appropriately conclude this section by remarking that, if we apply Theorem 9 to the series $1 - 1 + 1 \dots$, taking as hypothesis the relation (2.32), we obtain the following familiar result: For $0 < \delta \leq 1$, the series

$$1^{1-\delta} - 2^{1-\delta} + 3^{1-\delta} \dots$$

is bounded $(C, 1 - \delta)$ and summable $(C, 1 - \delta + \epsilon)$ for every positive ϵ .

5.1. We next consider the case of Theorem 3 when $-1 \leq \beta < 0$. The problem discussed is the converse of the type of result which is contained in Theorems 6, 7, 8 and 9; that is, we assume as hypothesis the summability or boundedness (C, p) of the series $\sum n^p a_n$ and the conclusion involves a property of $\sum a_n$. One well known result in this connection is that the summability (C, p) of $\sum n^p a_n$ implies the summability $(C, p - \delta)$ of $\sum n^{p-\delta} a_n$, where $0 < \delta \leq p \leq p$, and the results of this section may be regarded as generalisations of theorems of this kind for the particular case $0 < \delta = \rho \leq 1$.

5.2. First we establish a lemma which serves to simplify the proofs of the theorems which follow.

LEMMA 6. *If $\gamma > 0$ and*

$$(5.21) \quad \phi_m(\gamma, a) = \sum_{\mu=1}^m \{A_m^{(\gamma)} - A_{m-\mu}^{(\gamma)}\} \epsilon_\mu \mu^{-a-1}, \quad \epsilon_\mu \rightarrow 0,$$

then

$$= o(m^{\gamma-a}) \quad , \quad 0 < a < 1,$$

$$(5.22) \quad \phi_m(\gamma, a) = o(m^{\gamma-1} \log m), \quad a = 1,$$

$$= O(m^{\gamma-1}) \quad , \quad a > 1.$$

We have

$$\begin{aligned} \phi_m(\gamma, a) &= \sum_{\mu=1}^m \epsilon_\mu \mu^{-a-1} \sum_{\tau=m+1-\mu}^m A_\tau^{(\gamma-1)} \\ &= \sum_{\tau=1}^m A_\tau^{(\gamma-1)} \sum_{\mu=m+1-\tau}^m \epsilon_\mu \mu^{-a-1}. \end{aligned}$$

Given ϵ , there exists N such that, for $\mu \geq N$, $|\epsilon_\mu| < \epsilon$. Thus

$$|\phi_m(\gamma, \alpha)| \leq \left| \sum_{\tau=1}^{m+1-N} A_\tau^{(\gamma-1)} \sum_{\mu=m+1-\tau}^m \epsilon_\mu \mu^{-\alpha-1} \right| + \sum_{\tau=m+2-N}^m A_\tau^{(\gamma-1)} \sum_{\mu=m+1-\tau}^m \epsilon_\mu \mu^{-\alpha-1}$$

$$< \epsilon \sum_{\tau=1}^{m+1-N} A_\tau^{(\gamma-1)} \sum_{\mu=m+1-\tau}^\infty \mu^{-\alpha-1} + O(m^{\gamma-1}).$$

Hence, if $0 < \alpha < 1$,

$$\phi_m(\gamma, \alpha) = o \left\{ \sum_{\tau=1}^{m+1} A_\tau^{(\gamma-1)} A_{m+1-\tau}^{(-\alpha)} \right\} + O(m^{\gamma-1})$$

$$= o(m^{\gamma-\alpha}),$$

while, if $\alpha > 1$,

$$\phi_m(\gamma, \alpha) = o \left\{ \sum_{\tau=1}^m A_\tau^{(\gamma-1)} (m+1-\tau)^{-\alpha} \right\} + O(m^{\gamma-1})$$

$$= o \left\{ \left[\sum_{\tau=1}^{\frac{1}{2}m} + \sum_{\tau=\frac{1}{2}m}^m \right] A_\tau^{(\gamma-1)} (m+1-\tau)^{-\alpha} \right\} + O(m^{\gamma-1})$$

$$= o \left\{ m^{-\alpha} \sum_{\tau=1}^{\frac{1}{2}m} A_\tau^{(\gamma-1)} \right\} + o \left\{ m^{\gamma-1} \sum_{\tau=1}^\infty \tau^{-\alpha} \right\} + O(m^{\gamma-1})$$

$$= O(m^{\gamma-1}).$$

Finally, if $\alpha = 1$,

$$\phi_m(\gamma, \alpha) = o \left\{ m^{-1} \sum_{\tau=1}^{\frac{1}{2}m} A_\tau^{(\gamma-1)} \right\} + o \left\{ m^{\gamma-1} \sum_{\tau=1}^{\frac{1}{2}m} \tau^{-1} \right\} + O(m^{\gamma-1})$$

$$= o(m^{\gamma-1}) + o(m^{\gamma-1} \log m) + O(m^{\gamma-1})$$

$$= o(m^{\gamma-1} \log m).$$

5.3. THEOREM 13. *If p is a positive integer, $0 < \rho < 1 \leq p$, and the series $\sum n^\rho a_n$ is bounded (C, p), that is,*

$$S_{n,\rho}^{(p)} = r_n n^\rho, \quad r_n = O(1),$$

then $\sum a_n$ satisfies (1.16), where

$$s = a_0 + \sum_{\nu=1}^\infty r_\nu \nu^p \Delta^{p+1} \nu^{-\rho}.$$

THEOREM 14. *If p is a positive integer, $0 < \rho < 1 \leq p$, and the series $\sum n^\rho a_n$ is summable (C, p), that is,*

$$S_{n,\rho}^{(p)} = \sigma A_n^{(p)} + \epsilon_n n^\rho, \quad \epsilon_n \rightarrow 0,$$

then $\sum a_n$ satisfies (1.17), where $\lambda = 0$ and

$$s = a_0 - \sigma \sum_{\nu=1}^{p+1} \nu^{-\rho} A_\nu^{(-p-2)} + \sum_{\nu=1}^\infty \epsilon_\nu \nu^p \Delta^{p+1} \nu^{-\rho}.$$

We shall prove only Theorem 14 since the proof of Theorem 13 is similar.

We have

$$\begin{aligned}
 S_n^{(p)} &= a_0 A_n^{(p)} + \sum_{\nu=1}^n A_{n-\nu}^{(p)} \nu^{-\rho} \nu^\rho a_\nu \\
 &= a_0 A_n^{(p)} + \sum_{\nu=1}^n A_{n-\nu}^{(p)} \nu^{-\rho} \sum_{\mu=1}^{\nu} A_{\nu-\mu}^{(-p-2)} S_{\mu, \rho}^{(p)} \\
 &= a_0 A_n^{(p)} + \sigma \sum_{\nu=1}^n A_{n-\nu}^{(p)} \nu^{-\rho} \left\{ \sum_{\mu=0}^{\nu} A_{\nu-\mu}^{(-p-2)} A_\mu^{(p)} - A_\nu^{(-p-2)} \right\} \\
 &\quad + \sum_{\mu=1}^n \epsilon_\mu \mu^p \sum_{\nu=\mu}^n A_{n-\nu}^{(p)} \nu^{-\rho} A_{\nu-\mu}^{(-p-2)} \\
 &= a_0 A_n^{(p)} + E_1 + E_2,
 \end{aligned}$$

say. Clearly

$$\begin{aligned}
 E_1 &= -\sigma \sum_{\nu=1}^{p+1} A_{n-\nu}^{(p)} \nu^{-\rho} A_\nu^{(-p-2)} \\
 &= -\sigma A_n^{(p)} \sum_{\nu=1}^{p+1} \nu^{-\rho} A_\nu^{(-p-2)} + O(n^{p-1}) \\
 &= -\sigma A_n^{(p)} \sum_{\nu=1}^{p+1} \nu^{-\rho} A_\nu^{(-p-2)} + o(n^{p-\rho}).
 \end{aligned}$$

Also, by Lemma 5, we have

$$\begin{aligned}
 E_2 &= o(n^{p-\rho}) + \sum_{\mu=1}^{n-p-1} \epsilon_\mu \mu^p \sum_{\nu=\mu}^n A_{n-\nu}^{(p)} A_{\nu-\mu}^{(-p-2)} \nu^{-\rho} \\
 &= o(n^{p-\rho}) + \sum_{q=0}^{p+1} \binom{p+1}{q} F_q,
 \end{aligned}$$

where

$$F_q = \sum_{\mu=1}^{n-p-1} \epsilon_\mu \mu^p A_{n-\mu-q}^{(q-1)} \Delta^q \mu^{-\rho}.$$

Now $F_0 = 0$, and, for $q = 1, 2, \dots, p$,

$$F_q = o \left\{ \sum_{\mu=0}^{n-q} A_\mu^{(p-\rho-q)} A_{n-q-\mu}^{(q-1)} \right\} = o(n^{p-\rho}).$$

Also we may write

$$\begin{aligned}
 F_{p+1} &= A_{n-p-1}^{(p)} \sum_{\mu=1}^{\infty} \epsilon_\mu \mu^p \Delta^{p+1} \mu^{-\rho} - A_{n-p-1}^{(p)} \sum_{\mu=n-p}^{\infty} \epsilon_\mu \mu^p \Delta^{p+1} \mu^{-\rho} \\
 &\quad - \sum_{\mu=1}^{n-p-1} \{A_{n-p-1}^{(p)} - A_{n-p-1-\mu}^{(p)}\} \epsilon_\mu \mu^p \Delta^{p+1} \mu^{-\rho} \\
 &= E_{2,1} - E_{2,2} - E_{2,3},
 \end{aligned}$$

say. Now

$$E_{2,1} = A_n^{(p)} \sum_{\mu=1}^{\infty} \epsilon_{\mu} \mu^p \Delta^{p+1} \mu^{-\rho} + O(n^{p-1}),$$

$$E_{2,2} = o\{n^p \sum_{\mu=n}^{\infty} \mu^{-\rho-1}\} = o(n^{p-\rho}),$$

and

$$E_{2,3} = \sum_{\mu=1}^{n-p-1} \{A_{n-p-1}^{(p)} - A_{n-p-1-\mu}^{(p)}\} \{A \epsilon_{\mu} \mu^{-\rho-1} + \epsilon_{\mu}^* \mu^{-\rho-2}\},$$

where A is a definite non-zero constant and $\epsilon_n^* \rightarrow 0$. It follows, by Lemma 6, that

$$E_{2,3} = o(n^{p-\rho}) + O(n^{p-1}) = o(n^{p-\rho}).$$

The theorem is therefore proved.

5.4. We now discuss the analogues of Theorems 13 and 14 when $\rho = 1$, and we shall see that there is a slight difference between this case and the case $0 < \rho < 1$.

THEOREM 15. *If p is a positive integer and the series $\sum na_n$ is bounded (C, p), that is,*

$$S_{n,1}^{(p)} = r_n n^p, \quad r_n = O(1),$$

then the series $\sum a_n$ satisfies the relation

$$(5.41) \quad S_n^{(p)} = sA_n^{(p)} + O(n^{p-1}),$$

where

$$s = a_0 + \sum_{\mu=1}^{\infty} r_{\mu} \mu^p \Delta^{p+1} \mu^{-1}.$$

THEOREM 16. *If the series $\sum na_n$ is summable (C, p), p a positive integer, to the sum σ , that is*

$$S_{n,1}^{(p)} = \sigma A_n^{(p)} + \epsilon_n n^p,$$

then the series $\sum a_n$ satisfies the relation

$$(5.42) \quad S_n^{(p)} = sA_n^{(p)} + \lambda A_n^{(p-1)} + o(n^{p-1}),$$

where

$$s = a_0 - \sigma \sum_{\mu=1}^{p+1} A_{\mu}^{(-p-2)} \mu^{-1} + \sum_{\mu=1}^{\infty} \epsilon_{\mu} \mu^p \Delta^{p+1} \mu^{-1},$$

$$\lambda = -\sigma.$$

It will be observed that the enunciations of Theorems 15 and 16 are practically the same as those of Theorems 13 and 14. I have thought it worth while to state the two former theorems in full for

the purpose of comparison. Moreover the details of the proofs of the two pairs of theorems are essentially different, the case $\rho = 1$ containing complications which are absent when $0 < \rho < 1$. It is sufficient to prove Theorem 16 since the proof of Theorem 15 is similar.

As in previous arguments we have

$$\begin{aligned} S_n^{(p)} &= a_0 A_n^{(p)} + \sum_{\nu=1}^n A_{n-\nu}^{(p)} \nu^{-1} \sum_{\mu=1}^{\nu} A_{\nu-\mu}^{(-p-2)} S_{\mu,1}^{(p)} \\ &= a_0 A_n^{(p)} + \sum_{\nu=1}^n A_{n-\nu}^{(p)} \nu^{-1} \sum_{\mu=0}^{\nu} A_{\nu-\mu}^{(-p-2)} \sigma A_{\mu}^{(p)} - \sigma \sum_{\nu=1}^n A_{n-\nu}^{(p)} \nu^{-1} A_{\nu}^{(-p-2)} \\ &\quad + \sum_{\nu=1}^n A_{n-\nu}^{(p)} \nu^{-1} \sum_{\mu=1}^{\nu} A_{\nu-\mu}^{(-p-2)} \epsilon_{\mu} \mu^p \\ &= a_0 A_n^{(p)} - \sigma \sum_{\nu=1}^{p+1} A_{n-\nu}^{(p)} \nu^{-1} A_{\nu}^{(-p-2)} + \sum_{\mu=1}^n \epsilon_{\mu} \mu^p \sum_{\nu=\mu}^n A_{n-\nu}^{(p)} A_{\nu-\mu}^{(-p-2)} \nu^{-1}. \end{aligned}$$

Now

$$A_{n-r}^{(p)} - A_n^{(p)} = - \sum_{\nu=n-r+1}^n A_{\nu}^{(p-1)} = -r A_n^{(p-1)} + o(n^{p-1}),$$

whence

$$\begin{aligned} S_n^{(p)} &= a_0 A_n^{(p)} - \sigma A_n^{(p)} \sum_{\nu=1}^{p+1} A_{\nu}^{(-p-2)} \nu^{-1} + \sigma A_n^{(p-1)} \sum_{\nu=1}^{p+1} A_{\nu}^{(-p-2)} \\ &\quad + \sum_{\mu=1}^n \epsilon_{\mu} \mu^p \sum_{\nu=\mu}^n A_{n-\nu}^{(p)} A_{\nu-\mu}^{(-p-2)} \nu^{-1} + o(n^{p-1}) \\ &= \{a_0 - \sigma \sum_{\nu=1}^{p+1} A_{\nu}^{(-p-2)} \nu^{-1}\} A_n^{(p)} - \sigma A_n^{(p-1)} + o(n^{p-1}) + E_n, \end{aligned}$$

where

$$\begin{aligned} E_n &= \left\{ \sum_{\mu=n-p}^n + \sum_{\mu=1}^{n-p-1} \right\} \epsilon_{\mu} \mu^p \sum_{\nu=\mu}^n A_{n-\nu}^{(p)} A_{\nu-\mu}^{(-p-2)} \nu^{-1} \\ &= o(n^{p-1}) + \sum_{q=0}^{p+1} \binom{p+1}{q} F_q, \end{aligned}$$

where

$$F_q = \sum_{\mu=1}^{n-p-1} \epsilon_{\mu} \mu^p A_{n-\mu-q}^{(q-1)} \Delta^q \mu^{-1}.$$

Now $F_0 = 0$ and, for $1 \leq q \leq p - 1$,

$$F_q = o \left\{ \sum_{\mu=0}^{n-q} A_{n-q-\mu}^{(q-1)} A_{\mu}^{(p-q-1)} \right\} = o(n^{p-1}),$$

so that

$$E_n = F_{p+1} + (p + 1) F_p + o(n^{p-1}).$$

We now write

$$F_{p+1} = G_1 - G_2 - G_3,$$

where

$$G_1 = A_{n-p-1}^{(p)} \sum_{\mu=1}^{\infty} \epsilon_{\mu} \mu^p \Delta^{p+1} \mu^{-1},$$

$$G_2 = A_{n-p-1}^{(p)} \sum_{\mu=n-p}^{\infty} \epsilon_{\mu} \mu^p \Delta^{p+1} \mu^{-1},$$

$$G_3 = \sum_{\mu=1}^{n-p-1} \epsilon_{\mu} \mu^p \Delta^{p+1} \mu^{-1} \{A_{n-p-1}^{(p)} - A_{n-p-1-\mu}^{(p)}\}.$$

Clearly

$$G_1 = A_n^{(p)} \sum_{\mu=1}^{\infty} \epsilon_{\mu} \mu^p \Delta^{p+1} \mu^{-1} - (p+1) A_n^{(p-1)} \sum_{\mu=1}^{\infty} \epsilon_{\mu} \mu^p \Delta^{p+1} \mu^{-1} + o(n^{p-1}),$$

and

$$G_2 = o\{n^p \sum_{\mu=n}^{\infty} \mu^{-2}\} = o(n^{p-1}).$$

Thus

$$S_n^{(p)} = s A_n^{(p)} - \{\sigma + (p+1) \sum_{\mu=1}^{\infty} \epsilon_{\mu} \mu^p \Delta^{p+1} \mu^{-1}\} A_n^{(p-1)} + H_n + o(n^{p-1}),$$

where

$$H_n = \sum_{\mu=1}^{n-p-1} \epsilon_{\mu} \mu^p [A_{n-p-\mu}^{(p-1)} (p+1) \Delta^p \mu^{-1} - \{A_{n-p-1}^{(p)} - A_{n-p-1-\mu}^{(p)}\} \Delta^{p+1} \mu^{-1}]$$

$$= (p+1)! \sum_{\mu=1}^{n-p-1} \frac{\epsilon_{\mu} \mu^p}{\mu(\mu+1) \dots (\mu+p+1)} \frac{[(\mu+p+1) A_{n-p-\mu}^{(p-1)} - \{A_{n-p-1}^{(p)} - A_{n-p-1-\mu}^{(p)}\}]}{[(\mu+p+1)(n-p-\mu+1)(n-p-\mu+2) \dots (n-1-\mu)]}$$

$$= (p+1)! \sum_{\mu=1}^{n-p-1} \frac{\epsilon_{\mu} \mu^p}{\mu(\mu+1) \dots (\mu+p+1)} \frac{[(\mu+p+1)(n-p-\mu+1)(n-p-\mu+2) \dots (n-1-\mu)]}{(p-1)!}$$

$$- \left\{ \frac{(n-p)(n-p+1) \dots (n-1) - (n-p-\mu)(n-p-\mu+1) \dots (n-1-\mu)}{p!} \right\}$$

$$= (p+1)! \sum_{\mu=1}^{n-p-1} \frac{\epsilon_{\mu} \mu^p}{\mu(\mu+1) \dots (\mu+p+1)} \sum_{r+s \leq p} \beta_{r,s} n^r \mu^s,$$

where $\beta_{r,s}$ is a constant independent of n and μ ,

$$\beta_{p,0} = \beta_{p-1,1} = 0,$$

and

$$\beta_{p-1,0} = \frac{p+1}{(p-1)!}.$$

We now write

$$H_n = H_n^{(1)} + H_n^{(2)},$$

where

$$H_n^{(1)} = \frac{(p+1)}{(p-1)!} n^{p-1} \sum_{\mu=1}^{n-p-1} \epsilon_{\mu} \mu^p \Delta^{p+1} \mu^{-1},$$

$$H_n^{(2)} = (p+1)! \sum_{\mu=1}^{n-p-1} \frac{\epsilon_{\mu} \mu^p}{\mu(\mu+1) \dots (\mu+p+1)} \sum_{r,s} \beta_{r,s} n^r \mu^s,$$

the summation in $H_n^{(2)}$ extending over all values of r and s for which $0 \leq r \leq p - 2, 0 \leq s \leq p, r + s \leq p$.

As $n \rightarrow \infty$ we clearly have

$$H_n^{(1)} = A_n^{(p-1)}(p + 1) \sum_{\mu=1}^{\infty} \epsilon_{\mu} \mu^p \Delta^{p+1} \mu^{-1} + o(n^{p-1}),$$

and if we write

$$H_n^{(2)} = H_n^{(2,1)} + H_n^{(2,2)} + H_n^{(2,3)},$$

where $H_n^{(2,1)}$ contains all terms for which $s = 0, 0 \leq r \leq p - 2, H_n^{(2,2)}$ those for which $s = 1, 0 \leq r \leq p - 2$ and $H_n^{(2,3)}$ those for which $2 \leq s \leq p, 0 \leq r \leq p - 2, r + s \leq p$, then

$$H_n^{(2,1)} = o\left\{ \sum_r n^r \sum_{\mu=1}^n \mu^{-2} \right\} = \sum_r o(n^r) = o(n^{p-2}),$$

$$H_n^{(2,2)} = o\left\{ \sum_r n^r \sum_{\mu=1}^n \mu^{-1} \right\} = \sum_r o(n^r \log n) = o(n^{p-1}),$$

and

$$H_n^{(2,3)} = o\left\{ \sum_{r,s} n^r \sum_{\mu=1}^n \mu^{s-2} \right\} = \sum_{r,s} o(n^{r+s-1}) = o(n^{p-1}).$$

The theorem is therefore proved.

5.5. We conclude this section by considering briefly the truth or falsity of the following theorem.

THEOREM 17. *A necessary and sufficient condition for the series $\sum a_n$ to satisfy a relation of the form (1.17), where p is a positive integer and $0 < \rho \leq 1$, is that the series $\sum n^{\rho} a_n$ should be summable (C, p) .*

That this theorem is true for the case $\rho = 1$ follows at once from Theorems 6 and 16. When $0 < \rho < 1$ we see, from Theorem 14, that the sufficiency part of Theorem 17 is true and also that the λ of relation (1.17) is zero. When $\lambda \neq 0$, Theorem 12 shows that the necessity part of Theorem 17 is false. It is natural therefore to examine the following assertion.

If $0 < \rho < 1, p$ is a positive integer and the series $\sum a_n$ satisfies the relation

$$S_n^{(p)} = sA_n^{(p)} + \eta_n, \eta_n = o(n^{p-\rho}),$$

then the series $\sum n^{\rho} a_n$ is summable (C, p) .

An examination of the proof of Theorem 9 shows that the truth or falsity of this assertion depends, in the last resort, on whether or not the expression

$$\frac{1}{A_n^{(p)}} \sum_{\mu=0}^n A_{n-\mu}^{(p)} \eta_{\mu} \Delta^{p+1} \mu^{\rho}$$

tends to a definite limit; that is, it depends on whether or not the

series $\sum \eta_\mu \Delta^{p+1} \mu^\rho$ is summable (C, p) . It is clear that this series need not be summable (C, p) . Indeed, if

$$\eta_\mu = \frac{\mu^{p-\rho}}{\log(\mu+1)},$$

it is properly divergent. Hence the necessity part of Theorem 17 is not true for $0 < \rho < 1$ even when $\lambda = 0$. In other words, either of the alternatives in the conclusion of Theorem 11 may occur.

6.1. It is natural to expect that series which satisfy (1.16) and (1.17) should have a less restrictive Tauberian condition than $a_n = O(n^{-1})$. It was proved¹ by Hardy that series, for which

$$S_n^{(1)} = sA_n^{(1)} + o(n^{\frac{1}{2}}),$$

are summable by Borel's method. Hence a sufficient Tauberian condition for such series is

$$a_n = O(n^{-\frac{1}{2}}).$$

More generally², it has been proved³ recently that, if the series $\sum a_n$ satisfies (1.17) with $\lambda = 0$, it is summable by Valiron's method⁴ of order $1 - \rho/p$, where $0 < \rho < p$. An examination of the argument used in the proof of this theorem shows at once that the theorem remains true when $\sum a_n$ satisfies (1.17) with $\lambda \neq 0$. Now the Tauberian condition for summability (V, α) , where $0 < \alpha < 1$, is known⁵ to be

$$a_n = O(n^{-\alpha}).$$

Hence we have the theorem.

THEOREM 18. *If $\sum a_n$ satisfies (1.17), where p is a positive integer, and $0 < \rho < p$, $0 < \rho \leq 1$, and if*

$$a_n = O\{n^{(\rho-p)/p}\},$$

the series $\sum a_n$ is convergent.

This theorem may be proved directly by making use of an argument due to Hardy and Littlewood⁶, and the same type of

¹ Hardy, 3.

² Hyslop, 8.

³ Hyslop, 9.

⁴ Valiron, 10.

⁵ Valiron, 10.

⁶ Hardy and Littlewood, 7, 428.

argument may also be used to prove an analogous theorem for series satisfying (1.16).

THEOREM 19. *If Σa_n satisfies (1.16), where p is a positive integer and $0 < \rho < p$, $0 < \rho \leq 1$, and if*

$$a_n = o\{n^{(\rho-p)/p}\},$$

the series Σa_n is convergent.

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THE UNIVERSITY, GLASGOW.