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CONNECTEDNESS PROPERTIES OF LATTICES

BY

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ABSTRACT. Let L be a lattice and q a convergence structure (or a topology) finer than the interval topology of L. In case of compact maximal chains and continuous lattice translations, the connected components of the space (L, q) are characterized using lattice conditions only. Moreover, lattice conditions of L are related to connectedness conditions of the order convergence space (L, o). Throughout this note, maximal chain conditions and maximal chain techniques play an important role.

0. **Introduction**. In posets and lattices, convergence structures provide a more effective tool than topologies, which was demonstrated in for instance Ball [1], Erné [4] and Kent [7]. Therefore, this note is written in full generality to concern convergence structures rather than topologies only. By a *lattice convergence space* we shall mean a lattice endowed with a convergence structure which is finer than the interval topology of the lattice.

In Section 2, purely lattice theoretical lemmata on conditional completeness are given. It is proved i.a. that a lattice is conditionally complete, if it has complete maximal chains. Section 3 utilizes these results in a theorem characterizing the connected components of certain lattice convergence spaces. Finally, Section 4 deals with certain intrinsic topologies and convergence structures on conditionally complete lattices. It is proved that in structures finer than interval topology but coarser than order convergence, every maximal chain carries its own interval topology. This result is used in creating a connectedness theory for the structures in question.

1. Preliminaries

1.1 **Convergence structures** Let S be a set and let $F(S)$ denote the collection of all proper filters on S. A *convergence* (*structure*) on S is a map $q : S \to 2^{F(S)}$, which for every $x \in S$ satisfies

(1) $[x] \in q(x)$, where $[x]$ denotes the fixed ultrafilter generated by $\{x\}$.

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(2) If $\mathcal{F} \in q(x)$ and $\mathcal{G} \supseteq \mathcal{F}$, then $\mathcal{G} \in q(x)$.

(3) If $\mathcal{F} \in q(x)$, then $\mathcal{F} \cap [x] \in q(x)$.

The pair (S, q) is called a *convergence space*. Examples are topological spaces, pseudotopological spaces (G. Choquet) and limit spaces (H.J. Kowalsky, H.R. Fischer). The natural *topological modification* of the convergence q is the finest topology on S coarser than q; it is denoted by top(q). The notion of *open* (*closed*) set in a space (S, q) always refers to the topology top(q). There is a rich literature on convergences and applications; reference is made to Gähler [6].

Below, the necessary definitions of connectivity properties in convergence spaces are recapitulated from Gähler [6] and Vainio [9], [10]. A convergence space (S, q) is *connected*, if the topological modification $(S, \text{top}(q))$ is, i.e. if all continuous maps from (S, q) to the two-point discrete space are constant maps. A set in a given convergence space is a *connected set*, if the corresponding subspace is a connected convergence space. In general, a convergence space possesses fewer connected sets than its topological modification does. The maximal connected sets in (S, q) are called the (*connected*) *components* of the space; they coincide with the components of $(S, \text{top}\,(q))$. Finally, (S, q) is *locally connected*, if for every $x \in S$ every filter $\mathcal{F} \in q(x)$ has a subfilter $\mathcal{G} \in q(x)$ possessing a filter base of connected sets. It can be shown that if a convergence space is locally connected, then so is its topological modification.

1.2 **Convergences on posets and lattices** Partially ordered sets (posets) will be indicated by the letter P, lattices by the letter L. For a subset $A \subseteq P$, the set of all upper (lower) bounds is denoted by A^* (A^+). If $A \subseteq P$ has a least upper bound (l.u.b.), then this is denoted by $\bigvee A$. Dually, a greatest lower bound (g.l.b.) is indicated by $\bigwedge A$. In a lattice L, the notation $x \vee y$ $(x \wedge y)$ is used instead of $\bigvee\{x, y\}$ $(\bigwedge\{x, y\})$. For S a subset of L and $x \in L$, $x \vee S$ denotes the set $\{x \vee s : s \in S\}$. The set $x \wedge S$ is defined dually.

Interesting convergences on a poset P are for instance the *interval topology* $t(P)$ and the *order convergence* $o(P)$. The former is the coarsest topology on P for which all rays $[a, -]$, $[-, a]$ are closed sets and the latter is defined through (cf. Kent [7])

$$\mathcal{F} \in o(P)\,(x) \Leftrightarrow x = \bigwedge \mathcal{F}^* = \bigvee \mathcal{F}^+,$$

$$\mathcal{F}^* = (\cup F^* : F \in \mathcal{F}),\ \mathcal{F}^+ = (\cup F^+ : F \in \mathcal{F}).$$

For P a poset and $x \in P$, every filter in $o(P)\,(x)$ is *bounded*, i.e. contains a bounded set. Hence, if L is a *conditionally complete lattice*, i.e. a lattice in which every non-empty subset with an upper bound has a l.u.b., and dually, then order convergence on L can be described through

$$\mathcal{F} \in o(L)\,(x) \Leftrightarrow x = \bigwedge\{\bigvee F : F \in \mathcal{F}\} = \bigvee\{\bigwedge F : F \in \mathcal{F}\}.$$

It is known that in an arbitrary lattice L, the classical *order topology* of G. Birkhoff can be characterized as the finest topology coarser than $o(L)$, i.e. as top $(o(L))$. It is always true that

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$$t(L) \leq \mathrm{top}(o(L)) \leq o(L),$$

and hence all structures above yield lattice convergence spaces. Concerning order convergence, reference is made to Erné and Weck [5], and Kent [7].

2. **The compact maximal chain condition**. The compact maximal chain (c.m.c.) condition was explained in Ward [11], which deals with i.a. compactness and connectedness properties of chains contained in quasi ordered spaces. However, no one seems to have noticed that for lattice convergences (topologies) finer than interval topology, the c.m.c. condition implies conditional completeness of the lattice. This is a direct consequence of the following two lemmata, the first of which is due to Rennie [8].

LEMMA 1. *A lattice is conditionally complete, if every non-empty chain with an upper bound has a lowest upper bound.*

LEMMA 2. *A lattice with complete maximal chains (i.e. complete as lattices in their own right) is conditionally complete.*

PROOF. Let L be a lattice with complete maximal chains and let S be any subchain of L. Below is proved that the l.u.b. of S taken along any maximal chain containing S yields the same element of L.

Therefore, let I and J be maximal chains in L containing S and write $x_1 = \bigvee_I S$, $x_2 = \bigvee_J S$, the indices telling in which lattice the l.u.b. is formed. The L-infimum $x_1 \wedge x_2$ is related to every element in $I(J)$. To see this, take an arbitrary element $m \in I(J)$. If m is an upper bound of S, then $m \geq x_1 \geq x_1 \wedge x_2$. In other case, there is an element $s \in S$ for which $s > m$, and $x_1 \wedge x_2 \geq s > m$ follows. Since $I(J)$ is a maximal chain, $x_1 \wedge x_2 \in I(J)$, and thus $x_1 = x_2$.

Now, Lemma 1 is used to close the proof. Let S be a chain in L and I a maximal chain in L containing S. Assume that $s \in L$ is a lower upper bound of S than the element $\bigvee_I S$. There exists a maximal chain J containing S and s, and $\bigvee_J S < \bigvee_I S$, which contradicts the result at the beginning of the proof. The lemma follows. □

Finally, the following related result is proved.

LEMMA 3. *Let L be a conditionally complete lattice, I a maximal chain in L and S a bounded subset of I. Then*

$$\bigvee_L S = \bigvee_I S \ (\bigwedge_L S = \bigwedge_I S),$$

the indices telling in which lattice the l.u.b. (g.l.b.) is formed.

PROOF. The l.u.b. $\bigvee_L S$ exists, and is denoted by a. Let x be an arbitrary element of I. If x is an upper bound of S, then $x \geq a$ in L, and if x is not an upper bound of S, then there is an element $s \in S$ for which $x < s \leq a$. Since I is a maximal chain in L, it follows $a \in I$. Clearly, a is the lowest I upper bound of S, and thus $\bigvee_I S$ exists and equals a. □

3. **Connectedness properties of lattices with compact maximal chains**. As known, the interval topology of a chain is connected, if and only if the chain is order dense and conditionally complete. We shall use this result together with a compact maximal chain reasoning.

A lattice is *order dense* if, whenever $x < y$ in the lattice, there exists an element z in the lattice such that $x < z < y$. In the previous section, *conditionally complete* lattices were defined. Naturally, a lattice is order dense if and only if every maximal chain is order dense. Further, every maximal chain in a conditionally complete lattice is conditionally complete in its own right (cf. Lemma 3 above).

In this section, let (L, q) be a *lattice convergence space* (i.e. $q \geq t(L)$) with *compact maximal chains* and *continuous lattice translations* (i.e. the maps $x \rightarrow a \vee x$ and $x \rightarrow a \wedge x$ are continuous maps from (L, q) to (L, q) for every $a \in L$). The assumption $q \geq t(L)$ ensures every maximal chain in L to be a closed set. We also note that $q \geq t(L)$, the c.m.c. condition and Lemma 2 together imply that L is a conditionally complete lattice.

THEOREM 4. *Let (L, q) be as above. Then the connected components of (L, q) can be characterized as the maximal conditionally complete and order dense convex sublattices of L.*

PROOF. Let K be a connected component of (L, q). The continuity condition alone implies that K is a convex sublattice of L. Namely, for any $a \in K$ it is $a \vee K \subseteq K$, $a \wedge K \subseteq K$, and hence K is a sublattice. To prove the convexity, take $a \leq b$ in K, and take $x \in L$ satisfying $a \leq x \leq b$. The set $x \wedge K$ is connected, both a and x are in $x \wedge K$, and hence $x \in K$. Moreover, for any $a < b$ in K the set $[a, b]$ is connected (only use that the map $x \rightarrow (a \vee x) \wedge b$ is continuous), and since every singleton set is closed in the subspace K, there is an element $x \in K$ for which $a < x < b$. Thus K is order dense. Every maximal chain of K can be embedded in a compact maximal chain of L, and is thus compact. Hence the subspace K is a lattice convergence space with compact maximal chains, which implies that K is conditionally complete (Lemma 2).

It remains to prove that every conditionally complete and order dense convex sublattice S of L is a connected subset of (L, q). Therefore, take $a, x \in S$ and prove that x is in a connected subset S_x of S containing a. Define S_x to be the L-interval $[a \wedge x, a \vee x]$, which is a subset of S, since S is a convex sublattice of L. All maximal chains in S_x are compact, and hence carrying their own interval topologies. Since all maximal chains in S_x are order dense and conditionally complete (Lemma 3), they are connected. Hence S_x is connected, and the proof is enclosed. □

COROLLARY 5. *Let the lattice convergence space (L, q) be as in the theorem. The space (L, q) is connected, if and only if the lattice L is order dense.*

In view of Theorem 6, the following conclusion holds.

COROLLARY 6. *Let L be a conditionally complete lattice, and let $s(L)$ be a convergence on L satisfying $t(L) \leq s(L) \leq o(L)$, and for which the lattice translations are continuous maps. Then the connected components of $(L, s(L))$ can be characterized as the maximal order dense convex sublattices of L.*

4. **Order convergence of conditionally complete lattices**. This section investigates conditionally complete lattices endowed with order convergence (cf. Section 1.2), and specially, the maximal chains of such spaces. The results obtained are used to yield sufficient conditions for a lattice to be connected or locally connected in its order convergence.

Let L be a lattice and J a chain in L. Again, let $o(L)$ denote order convergence and $t(L)$ interval topology. By $t(L)_J$ $(o(L)_J)$ is indicated the inherited structure from the interval topology (order convergence) of L to the subchain J. Among the standard results we note the following:

$$t(L) \leq o(L),\ o(J) = t(J) \leq t(L)_J.$$

LEMMA 7. *Let L be a conditionally complete lattice and J a maximal chain in L. Then $o(L)_J = t(J)$.*

PROOF. In order to prove $o(L)_J \leq t(J)$, take $x \in J$ and denote the $t(J)$-neighbourhood filter of x by $\mathcal{F}$. Since J is a conditionally complete lattice,

$$\bigvee_J \{\bigwedge_J F : F \in \mathcal{F}\} = \bigwedge_J \{\bigvee_J F : F \in \mathcal{F}\} = x.$$

According to Lemma 3, the index J in this formula can be replaced by the index L, and thus it follows that $\mathcal{F}$ is a filter base for a filter in $o(L)(x)$. Thus $o(L)_J \leq t(J)$ holds. □

COROLLARY 8. *Let L be a conditionally complete lattice and let J be a maximal chain in L. Take $x \in J$ and let W be a neighbourhood of x in the order topology of L. Then W contains some J-interval $[y, z]$ containing x.*

Lemma 7 can be strengthened to

THEOREM 9. *Let L be a conditionally complete lattice, J a maximal chain in L and I a convex subset of J. Suppose $s(L)$ is a convergence on L with $t(L) \leq s(L) \leq o(L)$. Then $s(L)_J = t(J)$ and $s(L)_I = t(I)$.*

PROOF. Lemma 7 gives $t(L)_J \leq o(L)_J = t(J)$, and thus $t(L)_J = t(J)$. Using Lemma 7 once more, the assumption governing $s(L)$ yields $s(L)_J = t(J)$. Furthermore, $s(L)_I = t(J)_I = t(I)$, where the last equality employs the convexity assumption on I. (cf. Erné [2], Lemma 2). □

Another application of Lemma 7 is the following

THEOREM 10. *Let X be a set, Y a lattice convergence space, and denote the lattice of all functions from X to Y endowed with order convergence structure by* $F_o(X, Y)$. *If Y has compact maximal chains, then so has* $F_o(X, Y)$.

PROOF. Let M be a maximal chain in $F(X, Y)$. Since Y has compact maximal chains, $F(X, Y)$ is conditionally complete, and hence according to Lemma 7 the subspace M of $F_o(X, Y)$ carries its own interval topology. Since the c.m.c. condition of Y implies that M is complete, the theorem follows. □

REMARK 11. Theorem 10 gives conditions for a totally ordered pointwise convergent net of functions to be order convergent. Naturally, the theorem does not hold if $F(X, Y)$ is endowed with continuous convergence. (Dini's theorem in Calculus does not work without compactness conditions on the space X.)

Finally, Lemma 7 and Theorem 9 are applied to connectivity theory of order convergence spaces. It is recapitulated that the interval topology of a chain is connected, if and only if the chain is conditionally complete and order dense.

THEOREM 12. *If the lattice L is order dense and conditionally complete, then the order convergence space* $(L, o(L))$ *is connected. The converse statement is not true.*

PROOF. Take $a, b \in L$, let J_a be a maximal chain through a and $a \vee b$, and J_b a maximal chain through b and $a \vee b$. The interval topologies of J_a and J_b are connected, and hence a and b belong to the same connected component of the space $(L, o(L))$. The second part of the theorem is proved through a simple counterexample. Both lattices below are built up by intervals of the real axis. The direction into which the order relation grows is indicated by arrows. Although both lattices are connected in their order convergences, the first one is not order dense and the second one is not conditionally complete. □

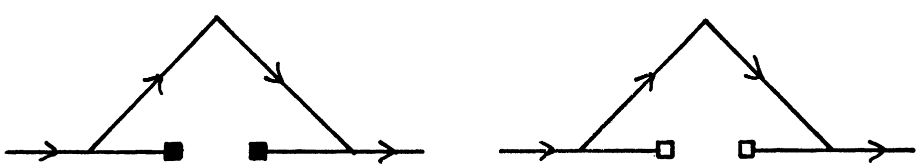

FIGURE.

COROLLARY 13. *Let X be a set and L an order dense and conditionally complete lattice. Then the space* $F_o(X,L)$ *is connected.*

REMARK 14. The theorem provides sufficient conditions for the order topology, the interval topology and the Lawson topology to be connected. Trivially, the Scott topology of any lattice is connected.

A lattice convergence space (L, q) is called *order connected*, if for every order related pair x, y the set $[x, y]$ is connected. Order connectedness of q always implies order density of L. Theorem 12 is strengthened through

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THEOREM 15. *If the lattice L is order dense and conditionally complete, then the space $(L, o(L))$ is order connected and hence locally connected.*

PROOF. The first part is proved using Theorem 9. According to Theorem 3.8 of Erné and Weck [5], it holds for arbitrary lattices L that for every $x \in L$ and every $\mathcal{F} \in o(L)(x)$ there is a subfilter $\mathcal{G} \in o(L)(x)$ possessing a base of intervals in L of the form $[a, b]$. Hence the second part follows. □

In some cases, there is a very close connection between the convergence $o(L)$ and the topology $t(L)$.

REMARK 16. Let L be a lattice. If $t(L)$ is Hausdorff, $t(L) = \text{top}(o(L))$ (cf. Erné and Weck [5]). Thus it follows from Theorem 15 that Hausdorff interval topologies of order dense and conditionally complete lattices always are locally connected. Moreover, in this case the connected components of the interval topology coincide with those of the order convergence. However, the assumption of Hausdorff interval topology is restrictive. For such interval topologies in lattices, T_2 coincides with T_3, and if the lattice is complete even with T_4 (cf. Erné [3]).

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