CONE CHARACTERIZATION OF REFLEXIVE BANACH LATTICES

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Abstract. We prove that a Banach lattice X is reflexive if and only if X_+ does not contain a closed normal cone with an unbounded closed dentable base.

Suppose that X is a Banach space and P a cone of X (i.e. $P \subseteq X$, $\lambda P + \mu P = P$ for each $\lambda, \mu \in \mathbb{R}_+$ and $P \cap (-P) = \{0\}$). The cone P is *normal* (or *self-allied*) if there exists $a \in \mathbb{R}_+$ such that for each $x, y \in P$, $x \leq y$ implies $\|x\| \leq a \|y\|$. A convex subset B of P is a base for P if for each $x \in P$, $x \neq 0$, there exists a unique number $f(x) \in \mathbb{R}_+$ such that $(f(x))^{-1}x \in B$. For each $D \subseteq X$, denote by $\overline{\text{co}}\, D$ the closed convex hull of D. A subset K of X is *dentable* if for each $\varepsilon \in \mathbb{R}_+$ there exists $x_\varepsilon \in K$ such that $x_\varepsilon \notin \overline{\text{co}}\{x \in K \mid \|x - x_\varepsilon\| \geq \varepsilon\}$.

We say that the *cone* P of X is *isomorphic* (or according to [**6**] and [**7**], *locally isomorphic*) to a cone Q of a Banach space Y if there exists an one-to-one, additive, positive homogeneous map T of P onto Q and T, T^{-1} are continuous in the induced topologies. Denote by c_0 the space of convergent to zero real sequences with the supremum norm and by l_1 the space of absolutely summing real sequences $\xi = (\xi(i))$ with the norm $\|\xi\| = \sum_{i=1}^{\infty} |\xi(i)|$. The cones

$$c_0^+ = \{x = (x(i)) \in c_0 \mid x(i) \in \mathbb{R}_+ \text{ for each } i\},$$

$$l_1^+ = \{x = (x(i)) \in l_1 \mid x(i) \in \mathbb{R}_+ \text{ for each } i\},$$

are the positive cones of c_0, l_1 respectively. If l_1^+ (respectively c_0^+) is isomorphic to a closed cone $D \subseteq P$, then we say that l_1^+ (respectively c_0^+) is embeddable in P. Cones isomorphic to l_1^+ are studied in [**6**]. For notation and terminology on convex sets we refer to [**2**].

THEOREM 1 ([**5**, Theorem 1]). *Let X be a reflexive Banach space. Then X does not contain a closed normal cone with an unbounded closed dentable base.*

Let X be a Banach lattice. By G. Lozanovskii's Theorem, see [**4**] or [**1**, p. 240], X is reflexive if and only if neither c_0 or l_1 is lattice embeddable in X.

THEOREM 2. *Let X be a Banach lattice. Then the following statements are equivalent:*

(i) *X is reflexive,*

(ii) *l_1^+ is not embeddable in X_+,*

(iii) *X_+ does not contain a closed normal cone P with an unbounded closed dentable base.*

Proof. By Theorem 1, (i)$\Rightarrow$(iii). Let the statement (iii) be true. Suppose that the statement (ii) does not hold. Then there exists a closed cone P of X isomorphic to l_1^+ and let $T: l_1^+ \to P$ be an isomorphism. By the continuity of T and T^{-1} at zero, there exist a, $b \in \mathbb{R}_+$ such that

$$a\,\|x\| \leq \|T(x)\| \leq b\,\|x\|, \qquad \text{for each } x \in l_1^+.$$

Let $f = (\xi(k))$ with $\xi(k) = k^{-1}$ for each $k \in \mathbb{N}$. The set $B = \{x \in l_1^+ \mid f(x) = 1\}$ is a closed base for l_1^+ and $T(B)$ a closed base for P. The cone P is normal because it is contained in X. The base B is unbounded because $ke_k \in B$ for each $k \in \mathbb{N}$; therefore $T(B)$ is also unbounded. The functional $g = (q(i))$ with $q(1) = 1$ and $q(k) = -1$ for each $k \neq 1$, strongly exposes the point $e_1 = (1, 0, 0, \ldots)$ in B. This holds because if $x \in B$ with $x \neq e_1$, then

$$g(x) = x(1) - \sum_{k=2}^{\infty} x(k) < x(1) < f(x) = g(e_1).$$

Also, if $x_n \in B$ with $g(x_n) = x_n(1) - \sum_{k=2}^{\infty} x_n(k) \to 1$, then $x_n(1) \to 1$ and $\sum_{k=2}^{\infty} x_n(k) \to 0$; therefore $\|e_1 - x_n\| \to 0$. Let $z_1 = T(e_1)$ and $h(y) = g(T^{-1}(y))$, for each $y \in P$. Then $h(y) < h(z_1)$ for each $y \in T(B)$ with $y \neq z_1$. For each sequence $y_n = T(x_n)$ of $T(B)$ with $h(y_n) \to h(z_1)$ we have that $g(x_n) \to g(e_1)$; therefore $x_n \to e_1$ and so $y_n \to z_1$. Thus for each $\varepsilon \in \mathbb{R}_+$ there exists $\rho = \rho(\varepsilon) \in \mathbb{R}_+$ such that $h(y) < h(z_1) - \rho$, for each $y \in T(B)$ with $\|y - z_1\| \geq \varepsilon$. Since h is additive, positive homogeneous and continuous we have

$$h(y) \leq h(z_1) - \rho, \qquad \text{for each } y \in \overline{\text{co}}\{z \in T(B) \mid \|z - z_1\| \geq \varepsilon\},$$

therefore $T(B)$ is dentable. This is a contradiction; therefore (iii)$\Rightarrow$(ii).

Suppose now that the statement (ii) holds. Since X_+ does not contain l_1^+ we have that l_1 is not lattice embeddable in X. Let $b_n = \sum_{i=1}^{n} e_i$, where (e_n) is the usual (Schauder) basis of c_0. Then (b_n) is a basis of c_0 because for each $x = (x(i)) \in c_0$ we have

$$\sum_{i=1}^{n} (x(i) - x(i+1))b_i = \sum_{i=1}^{n} x(i)e_i - x(n+1)b_n \quad \text{and} \quad \lim_{n\to\infty} x(n+1)b_n = 0.$$

The basis (b_n) is of type l_+ (i.e. (b_n) is bounded and there exists $k \in \mathbb{R}_+$, $k \neq 0$ such that $\left\|\sum_{i=1}^{n} a_i b_i\right\| \geq k \sum_{i=1}^{n} a_i$, for each finite sequence $a_1, a_2, \ldots, a_n$, of positive real numbers); therefore by [**7**, Theorem II.10.2, p. 323], the positive cone

$$C = \left\{\sum_{i=1}^{\infty} \lambda_i b_i \in c_0 \mid \lambda_i \in \mathbb{R}_+ \text{ for each } i\right\} \subseteq c_0^+$$

of the basis (b_n), is isomorphic to l_1^+. (C is the set of decreasing real sequences convergent to zero.) This shows that c_0 is not lattice embeddable in X; therefore X is reflexive.

Remark 1. In the proof of the previous theorem we have also show that l_1^+ is isomorphic to the cone $C \subseteq c_0^+$, of decreasing real sequences convergent to zero; therefore l_1^+ is embeddable in c_0^+.

It is known [**1**, Theorem 14.12, p. 226] that a Banach lattice X is a KB-space (i.e. X has the property: every increasing, norm bounded, sequence of X_+ is norm convergent) if and only if c_0 is not lattice embeddable in X. Also c_0^+ is not embeddable in the positive cone X_+ of a KB-space. This holds because if we suppose that a closed cone $P \subseteq X_+$ is isomorphic to c_0^+, and T: $c_0^+ \to P$ is an isomorphism then we have: the sequence $s_n = T(b_n)$, where $b_n = \sum_{i=1}^{n} e_i \in c_0^+$, is norm bounded because $\|T(b_n)\| \leq A\, \|b_n\| = A$, for each n. (s_n) is also increasing; therefore (s_n) is norm convergent to a point s of P. If $T(e) = s$, then $b_n \to e$, which is a contradiction; therefore c_0^+ is not embeddable in X_+. Now, using Theorem 2 and the above remarks we obtain the following characterization of Banach lattices X in terms of the embeddability of the cones l_1^+ and c_0^+ in X_+.

THEOREM 3. *A Banach lattice X is a non-reflexive KB-space if and only if l_1^+ is embeddable in X_+ and c_0^+ is not embeddable in X_+.*

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