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2-Clean Rings

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Abstract. A ring R is said to be n-clean if every element can be written as a sum of an idempotent and n units. The class of these rings contains clean rings and n-good rings in which each element is a sum of n units. In this paper, we show that for any ring R, the endomorphism ring of a free R-module of rank at least 2 is 2-clean and that the ring $B(R)$ of all $\omega \times \omega$ row and column-finite matrices over any ring R is 2-clean. Finally, the group ring RC_n is considered where R is a local ring.

1 Introduction

The question of when the automorphism group of a module additively generates its endomorphism ring has been of interest for many years. A ring is called n-good [15] if every element is a sum of n units. In 1953 and 1954, respectively, Wolfson [17] and Zelinsky [20] showed, independently, that every element of the ring of all linear transformations of a vector space over a division ring of characteristic not 2 is 2-good. In 1985 Goldsmith [4] proved that the endomorphism ring of a complete module over a complete discrete valuation ring is 2-good. In [16] Wans considered free R-modules where R is a *PID*, and showed that if the rank of M is finite and greater than 1, then $End_R(M)$ is 2-good. Goldsmith *et al.* [5] considered unit sum numbers of rings and modules. This was further developed by Meehan in [10]. Moreover, the above question is considered by many authors on abelian groups (see [2, 8, 9]) and on general rings with an identity (see [3, 7, 14]).

In 1977 Nicholson [12] introduced the concept of a clean ring (1-clean) which contains unit-regular rings and semiperfect rings, and showed that every clean ring must be an exchange ring. Camillo and Yu [1] further proved that a clean ring with 2 invertible is 2-good. Recently, Xiao and Tong [19] called a ring R n-clean if every element of R is the sum of an idempotent and n units. The class of these rings contains clean rings and n-good rings. In 1974 Henriksen [7] found that for any ring R and $n > 1$, the matrix ring $M_n(R)$ is 3-good. Moreover, Vámos [15] proved that for any ring R, the endomorphism ring of a free R-module of rank at least 2 is 3-good. Motivated by the result of Henriksen and Vámos, we conjectured that for any ring R, the endomorphism ring of a free R-module of rank at least 2 is 2-clean.

In this paper, we answer the question in the positive. In fact, we prove that for any ring R, the endomorphism ring of a free R-module of rank at least 2 is 2-clean. It is also proved that the ring $B(R)$ of all $\omega \times \omega$ row and column-finite matrices over any ring R is 2-clean. Finally, the group ring RC_n is considered where R is a local ring.

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Throughout this paper, rings are associative with identity and modules are unitary. $J(R)$ and $U(R)$ denote the Jacobson radical and the group of units of R, respectively.

2 Basic Properties of n-Clean Rings

An element of a ring is called n-clean if it can be written as the sum of an idempotent and n units. A ring is called n-clean if each of its elements is n-clean. In this section, some properties of n-clean rings are given.

Proposition 1 *Let R be a ring and let $a \in R$. Then the following statements hold:*

(1) *If a is n-clean, then it is also l-clean for all $n \leq l$.*
(2) *Every n-good ring is n-clean; if R is n-clean with $2 \in U(R)$, then it is $(n+1)$-good.*

Proof (1) We only need to prove that a is $n+1$-clean. Let $a \in R$ be n-clean: $a = e + u_1 + u_2 + \cdots + u_n$ where $e^2 = e \in R$ and $u_1, u_2, \ldots, u_n \in U(R)$. Note that $e = (1-e) + (2e-1)$, thus we have $a = (1-e) + (2e-1) + u_1 + \cdots + u_n$ where $2e - 1 \in U(R)$.

(2) It is clear that every n-good ring is n-clean.
The second statement is well known. ■

Let $S(R)$ be the nonempty set of all proper ideals of R generated by central idempotents. An ideal $P \in S(R)$ is called a Pierce ideal of R if P is a maximal (with respect to inclusion) element of the set $S(R)$. If P is a Pierce ideal of R, then the factor ring R/P is called a Pierce stalk of R. The next result shows that the n-clean property needs to be checked only for indecomposable rings or Pierce stalks.

Proposition 2 *Let R be a ring. Then the following are equivalent:*

(1) *R is n-clean.*
(2) *Every factor ring of R is n-clean.*
(3) *Every indecomposable factor ring of R is n-clean.*
(4) *Every Pierce stalk of R is n-clean.*

Proof $(1) \Rightarrow (2)$, $(2) \Rightarrow (3)$ and $(2) \Rightarrow (4)$ are directly verified.

$(3) \Rightarrow (1)$. Suppose that (3) holds and R is not n-clean, then there is an element $a \in R$ which is not n-clean. Now let $\mathcal{S}$ be the set of all proper ideals I of R such that $\overline{a}$ is not n-clean in R/I. Clearly, $0 \in \mathcal{S}$ and the set $\mathcal{S}$ is not empty. Define a partial ordering on $\mathcal{S}$ by $\subseteq$. If $\{I_\alpha : \alpha \in \Lambda\}$ is a chain in $\mathcal{S}$, let $I = \bigcup_{\alpha\in\Lambda} I_\alpha$. We will show that $\overline{a}$ is not n-clean in R/I. Suppose that $\overline{a}$ is n-clean in R/I. Then there exist $\overline{u}_1, \overline{u}_2, \ldots, \overline{u}_n \in U(R/I)$ (with inverses $\overline{v}_1, \overline{v}_2, \ldots, \overline{v}_n$, respectively) and $\overline{e}^2 = \overline{e} \in R/I$ such that $\overline{a} = \overline{e} + \overline{u}_1 + \overline{u}_2 + \cdots + \overline{u}_n$. Note that $e^2 - e \in \cup_{\alpha\in\Lambda} I_\alpha$ and $u_i v_i - 1,\ v_i u_i - 1 \in \bigcup_{\alpha\in\Lambda} I_\alpha$, so $e^2 - e \in I_{\alpha_0},\ u_i v_i - 1 \in I_{\alpha_i}$ and $v_i u_i - 1 \in I_{\alpha_i'}$ for $\alpha_0, \alpha_i, \alpha_i' \in \Lambda$. Because $\{I_\alpha : \alpha \in \Lambda\}$ is a chain in $\mathcal{S}$, there is a maximal I_s in the set $\{I_{\alpha_0}, I_{\alpha_1}, \ldots, I_{\alpha_n}, I_{\alpha_1'}, I_{\alpha_1'}, \ldots, I_{\alpha_n'}\}$ such that $I_{\alpha_0}, I_{\alpha_i}, I_{\alpha_i'} \subseteq I_s$. That is , $\overline{a}$ is n-clean in R/I_s, a contradiction. This implies that $I \in \mathcal{S}$ is an upper bound of the chain. Thus $\mathcal{S}$ is an inductive set and, by Zorn's Lemma, $\mathcal{S}$ has a maximal element I_0. By (3) R/I_0 is decomposable as a ring. Write $R/I_0 \cong R/I_1 \oplus R/I_2$ where both the ideals I_1 and I_2 strictly contain I_0, and so by the choice of I_0, $\overline{a}$ is n-clean in R/I_1 and R/I_2. But then $\overline{a}$ is n-clean in R/I_0, a contradiction.

(4) $\Rightarrow$ (1). Let $\mathcal{S}$ be the set of all proper ideals I of R such that I is generated by central idempotents and the ring R/I is not n-clean. Assume that R is not n-clean. Then $0 \in \mathcal{S}$ and the set $\mathcal{S}$ is not empty. It is directly verified as above that the union of every ascending chain of ideals from $\mathcal{S}$ belongs to $\mathcal{S}$. By Zorn's Lemma, the set $\mathcal{S}$ contains a maximal element P. By condition (4), it is sufficient to prove that P is a Pierce ideal. Assume the contrary. By the definition of the Pierce ideal, there is a central idempotent e of R such that $P + eR$ and $P + (1 - e)R$ are proper ideals of R which properly contain the ideal P. Since ideals $P+eR$ and $P+(1-e)R$ do not belong to $\mathcal{S}$ and are generated by central idempotents, $R/(P + eR)$ and $R/(P + (1 - e)R)$ are n-clean. Note that $R/P \cong (R/(P+eR)) \times (R/(P+(1-e)R))$, and it now follows that R is n-clean. ■

3 Matrix Rings and Endomorphism Rings of Free Modules

In this section, we will consider the 2-cleanness of the endomorphism ring of a free R-module of rank at least 2. First we give the following simple and interesting decomposition.

Lemma 3 *Over any ring, the 2×2 and 3×3 matrices are 2-clean.*

Proof Let R be a ring and let

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \in M_2(R). \quad \text{Put } E = \begin{pmatrix} a_{11} - 1 & 2 - a_{11} \\ a_{11} - 1 & 2 - a_{11} \end{pmatrix}.$$

It is checked easily that then $E^2 = E$. Thus we have

$$A - E = \begin{pmatrix} 1 & a_{12} + a_{11} - 2 \\ a_{21} - a_{11} + 1 & a_{22} + a_{11} - 2 \end{pmatrix}.$$

Now there exist invertible matrices P and Q such that

$$P(A - E)Q = \begin{pmatrix} 1 & 0 \\ 0 & c \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & -1 \\ -1 & c \end{pmatrix},$$

for an appropriate c and thus is a sum of two units. Hence A is 2-clean.

Now let

$$B = \begin{pmatrix} b_{11} & b_{12} & b_{13} \\ b_{21} & b_{22} & b_{23} \\ b_{31} & b_{32} & b_{33} \end{pmatrix}$$

be a 3×3 matrix over R. We first construct an idempotent in order to show 2-cleaness of B. Set

$$F = \begin{pmatrix} b_{11} - 1 & b_{22} - 1 & 3 - b_{11} - b_{22} \\ b_{11} - 1 & b_{22} - 1 & 3 - b_{11} - b_{22} \\ b_{11} - 1 & b_{22} - 1 & 3 - b_{11} - b_{22} \end{pmatrix}.$$

It may be directly verified that $F^2 = F$. Thus

$$B - F = \begin{pmatrix} 1 & b_{12} - b_{22} + 1 & b_{13} + b_{11} + b_{22} - 3 \\ b_{21} - b_{11} + 1 & 1 & b_{23} + b_{11} + b_{22} - 3 \\ b_{31} - b_{11} + 1 & b_{32} - b_{22} + 1 & b_{33} + b_{11} + b_{22} - 3 \end{pmatrix}.$$

We only need to show that $B - F$ is 2-good. Now there exist invertible matrices T, V and W such that

$$VT(B - F)W = \begin{pmatrix} c_1 & 0 & c_2 \\ c_3 & 1 & 0 \\ 0 & c_4 & c_5 \end{pmatrix} = \begin{pmatrix} 0 & 1 & c_2 \\ 0 & 0 & 1 \\ 1 & c_4 & c_5 \end{pmatrix} + \begin{pmatrix} c_1 & -1 & 0 \\ c_3 & 1 & -1 \\ -1 & 0 & 0 \end{pmatrix}$$

for an appropriate c_i $(i = 1, \dots, 5)$ and thus is a sum of two units. Hence B is 2-clean. This completes the proof. ■

Remark 4 (1) For the matrix ring $M_n(R)$, it is customary to write $GL_n(R)$ for $U(M_n(R))$. An elementary matrix is the result of an elementary row operation performed on the identity matrix. We denote by $E_n(R)$ the subgroup of $GL_n(R)$ generated by the elementary matrices, permutation matrices, and -1. Observing the decompositions of the 2×2 and 3×3 matrices above, we see that these matrices can be written as the sum of an idempotent matrix and two elements of $E_n(R)$.

(1) For any ring R, R can be embedded in the 2×2 matrix ring $M_2(R)$. That is, all rings can be embedded in a 2-clean ring by Lemma 3.

(2) We know that 2-clean rings contain clean rings and 2-good rings. However, the converse is not true. For example, the matrix ring $M_2(\mathbb{Z})$ is not clean since $\mathbb{Z}$ is not a exchange ring, and the matrix ring $M_2(\mathbb{Z}[x])$ is not 2-good (see [[15, Proposition 8]).

(3) It is well known that for a clean ring R, idempotents can be lifted modulo $J(R)$. However, a 2-clean ring does not have this property in general. Let $R = \mathbb{Z}_{(2)} \cap \mathbb{Z}_{(3)} = \{m/n \in \mathbb{Q} : m,\ n \in \mathbb{Z},\ 2 \nmid n \text{ and } 3 \nmid n\}$ and set $S = M_2(R)$. Then $J(S) = J(M_2(R)) = M_2(J(R)) = M_2(6R)$. Let $F = \left(\begin{smallmatrix} 3 & 0 \\ 6 & 3 \end{smallmatrix}\right)$. Then $F^2 - F \in J(S)$, but there is no idempotent E of S such that $F - E \in J(S)$ since non-trivial idempotents of S are only of form $\left(\begin{smallmatrix} a & b \\ c & 1-a \end{smallmatrix}\right)$ where $bc = a - a^2$ for $a, b, c \in R$. Thus S is 2-clean by Lemma 3, but there exists an idempotent which can not be lifted modulo $J(S)$.

Lemma 5 *Let R be a ring, $m,\ n \geq 1$ and $k \geq 2$. If the matrix rings $M_n(R)$ and $M_m(R)$ are both k-clean, then so is the matrix ring $M_{n+m}(R)$.*

Proof Let $A \in M_{n+m}(R)$ be a typical $(n + m) \times (n + m)$ matrix which we will write in the block decomposition form

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix},$$

where $A_{11} \in M_n(R),\ A_{22} \in M_m(R)$ and $A_{12},\ A_{22}$ are appropriately sized rectangular matrices. By hypothesis, there exist invertible $n \times n,\ m \times m$ matrices $U_1, U_2, \dots, U_k$

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and $V_1, V_2, \ldots, V_k$, and idempotent matrices E_1, E_2 such that $A_{11} = U_1 + U_2 + \cdots + U_k + E_1$ and $A_{22} = V_1 + V_2 + \cdots + V_k + E_2$. Thus the decomposition

$$\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} = \begin{pmatrix} U_1 & A_{12} \\ O & V_1 \end{pmatrix} + \begin{pmatrix} U_2 & O \\ A_{21} & V_2 \end{pmatrix} + \cdots + \begin{pmatrix} U_k & O \\ O & V_k \end{pmatrix} + \begin{pmatrix} E_1 & O \\ O & E_2 \end{pmatrix}$$

shows that A is k-clean. ∎

Corollary 6 *Let $k \geq 1$. If R is a k-clean ring, then so is the matrix ring $M_n(R)$ for any positive integer n.*

Proof For $k = 1$, it follows from [6, Corollary 1]. For that $k \geq 2$, it is clear by induction and by Lemma 5. ∎

Theorem 7 *Let R be a ring and let the free R-module F be (isomorphic to) the direct sum of $\alpha \geq 2$ copies of R where α is a cardinal number. Then the ring of endomorphisms E of F is 2-clean.*

Proof Assume first that $\alpha \geq 2$ is finite, so $E \cong M_\alpha(R)$. Then E is 2-clean for $\alpha = 2, 3$ by Lemma 3, and the values of $\alpha < \omega$ for which E is 2-clean are closed under addition by Lemma 5. So E is 2-clean for all finite α.

Assume now that α is infinite. Then $E \cong M_2(E)$ follows from $F \cong F \oplus F$, and so E is 2-clean by Lemma 3. ∎

4 Row and Column-Finite Matrix Rings

Let $B(R)$ be the ring of all $\omega \times \omega$ row and column-finite matrices over a ring R. In [13], O'Meara showed that $B(R)$ is an exchange ring for any regular R. However, a related question on clean rings is still open, even when R is a field. In this section, we will consider the 2-cleanness of $B(R)$.

Lemma 8 **(P. Nielsen)** *Let R be a ring and let*

$$X = \begin{pmatrix} U_1 & A_1 & & & \\ & U_2 & & & \\ & B_2 & U_3 & A_3 & \\ & & & U_4 & \\ & & & B_4 & \\ & & & & \ddots \end{pmatrix}, \text{ or } X = \begin{pmatrix} U_1 & & & & & \\ B_1 & U_2 & A_2 & & & \\ & & U_3 & & & \\ & & B_3 & U_4 & A_4 & \\ & & & & & \ddots \end{pmatrix},$$

where the U_i are square matrices of size $n_i \times n_i$, and with both A_i and B_i block matrices of size $n_i \times n_{i+1}$ and $n_{i+1} \times n_i$ respectively. If U_i is a unit for each $i \geq 1$, then X is a unit in $B(R)$.

Proof We will assume X has the first form, since the other is done similarly. By multiplying on the left or right by

$$\begin{pmatrix} U_1^{-1} & & \\ & U_2^{-1} & \\ & & \ddots \end{pmatrix}$$

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we may reduce the case that the $U_i = I_i$. In this case, the inverse to X is

$$\begin{pmatrix} I_1 & -A_1 & & & & \\ & I_2 & & & & \\ & -B_2 & I_3 & -A_3 & & \\ & & & I_4 & & \\ & & & -B_4 & I_5 & -A_5 \\ & & & & & & \ddots \end{pmatrix},$$

as required. ■

Theorem 9 *Let R be a ring. Then the $\omega \times \omega$ row and column-finite matrix ring $B(R)$ is 2-clean.*

Proof Given $X \in B(R)$. We can write

$$X = \begin{pmatrix} P_1 & A_1 & & \\ B_1 & P_2 & A_2 & \\ & B_2 & P_3 & \\ & & & \ddots \end{pmatrix}$$

with P_i square matrices of size $n_i \times n_i$, and with both A_i and B_i block matrices of size $n_i \times n_{i+1}$ and $n_{i+1} \times n_i$ respectively. Further, without loss of generality, we may assume $n_i > 1$ and so P_i is 2-clean by Lemma 3. Write $P_i = U_i + V_i + E_i$ where U_i and V_i are both units in $M_{n_i}(R)$, and E_i is an idempotent matrix. The following decomposition gives 2-cleanness: $X = U + V + E$, where

$$U = \begin{pmatrix} U_1 & A_1 & & & \\ & U_2 & & & \\ & B_2 & U_3 & A_3 & \\ & & & U_4 & \\ & & & B_4 & \\ & & & & \ddots \end{pmatrix}, \quad V = \begin{pmatrix} V_1 & & & & & \\ B_1 & V_2 & A_2 & & & \\ & & V_3 & & & \\ & & B_3 & V_4 & A_4 & \\ & & & & & \ddots \end{pmatrix},$$

and $E = \operatorname{diag}(E_1, E_2, E_3, E_4, \dots)$. By Lemma 8, U and V are both units in $B(R)$, and E is an idempotent matrix. ■

Remark 10 From the proof of Theorem 9, we may consider row and column-finite matrix rings over a 2-good ring similarly. In fact, we obtain that if R is 2-good then so is the row and column-finite matrix ring $B(R)$, and that for any ring R the row and column-finite matrix ring $B(R)$ is 3-good.

5 2-Clean Group Rings

Given a group G and a ring R, denote the group ring by RG. In this section, we consider the group ring RC_n where R is a local ring and C_n is a cyclic group of order n. Some results of Xiao and Tong [19] are extended.

Theorem 11 *Let R be a local ring with $\overline{R} = R/J(R)$ and let C_n be a cyclic group of order n. If $\operatorname{char}\overline{R} \neq 2$, then RC_n is 2-good.*

Proof If $\operatorname{char}\overline{R} = 0$ or $(\operatorname{char}\overline{R}, n) = 1$, then $\overline{n}$ and $\overline{2}$ are invertible in $\overline{R}$. Note that $\overline{R}$ is a division ring, then $\overline{R}C_n$ is semisimple from $n \cdot \overline{1} = \overline{n} \in U(\overline{R})$, and so $\overline{R}C_n$ is clean. This implies that $\overline{R}C_n$ is 2-good by [1, Proposition 10]. We know that if G is locally finite then $J(R)G \subseteq J(RG)$ by [18]. Clearly, $J(R)C_n \subseteq J(RC_n)$, and then $\overline{R}C_n \cong RC_n/J(R)C_n \twoheadrightarrow RC_n/J(RC_n)$. So the factor ring $RC_n/J(RC_n)$ is 2-good since 2-good rings are closed under factor rings. By [15, Proposition 3], RC_n is also 2-good. If $n = mp^k$ where $\operatorname{char}\overline{R} = p \neq 2$, $k \geq 1$, and $(m, p) = 1$. Then $C_n \cong C_{p^k} \times C_m$, and so $RC_n \cong (RC_{p^k})C_m$. By [11, Theorem], RC_{p^k} is also a local ring and $\operatorname{char} RC_{p^k} = p$. The rest is proved similarly as above since $(p, m) = 1$. Thus we complete the proof. ■

By Theorem 11, we obtain the following corollary immediately.

Corollary 12 *Let R be a local ring with $\overline{R} = R/J(R)$ and let C_n be a cyclic group of order n. If $\operatorname{char}\overline{R} \neq 2$, then RC_n is 2-clean.*

Corollary 13 *([19, Theorem 2.3]) If C_3 is a cyclic group of order 3, then the group ring $\mathbb{Z}_{(p)}C_3$ is 2-clean for any prime number $p \neq 2$.*

Remark 14 The group ring RC_n which satisfies the conditions of Theorem 11 need not be clean. In [6], Han and Nicholson showed that the group ring $\mathbb{Z}_{(7)}C_3$ is not clean where $\mathbb{Z}_{(7)} = \{m/n \in \mathbb{Q} : 7 \nmid n\}$.

Let $C_m = \{1, g, g^2, \dots, g^{m-1}\}$ with $g^m = 1$ where m is odd. Set $S = \{1, 2, \dots, m-1\}$. Define $\sigma \colon S \longrightarrow S$ by $i \longmapsto 2i (\bmod m)$. It is checked easily that σ is a permutation of $\{1, 2, \dots, m-1\}$. Let F be a field with $\operatorname{char} F = 2$ and let $e = e_0 + e_1 g + \cdots + e_{m-1} g^{m-1} \in FC_m$ be an idempotent. Note that $2 = 0$ and $g^m = 1$, so $e^2 = e_0^2 + e_{\sigma(1)} g^{\sigma(1)} + \cdots + e_{\sigma(m-1)} g^{\sigma(m-1)}$. Suppose that σ is a cyclic permutation. Then we have $e_0^2 = e_0$ and $e_1^2 = e_1 = e_2 = \cdots = e_{m-1}$, and so idempotents of FC_m are $0,\ 1,\ 1 + g + \cdots + g^{m-1},\ g + g^2 + \cdots + g^{m-1}$.

Theorem 15 *Let R be a local ring with $\operatorname{char}\overline{R} = 2$ and let C_n be a cyclic group of order n. Write $n = m \cdot 2^k$ $(k \geq 0)$ where $(m, 2) = 1$. If $\overline{R}$ is a field and the permutation σ of $\{1, 2, m-1\}$ induced by multiplication by 2 modulo m is cyclic, then the group ring RC_n is semiperfect.*

Proof Suppose $k \geq 1$. Then $C_n \cong C_{2^k} \times C_m$, and so $RC_n \cong (RC_{2^k})C_m$. By [11, Theorem] , RC_{2^k} is local. Since $\overline{R}$ is a field and $\overline{R}C_{2^k} \twoheadrightarrow \overline{RC_{2^k}}$ is a ring epimorphism, $\overline{RC_{2^k}}$ is a field and $\operatorname{char}\overline{RC_{2^k}} = \operatorname{char}\overline{R} = 2$. Hence we may assume $n = m$. Note that $\overline{R}C_m$ is semisimple since $(m, 2) = 1$ and $J(R)C_m \subseteq J(RC_m)$, so $J(R)C_m = J(RC_m)$. This shows that $\overline{RC_m} \cong \overline{R}C_m$ with $\operatorname{char}\overline{R} = 2$. Since $\overline{R}$ is a field and σ is a cyclic permutation of $\{1, 2, \dots, m-1\}$, $\overline{R}C_m$ has only four idempotents, and so all idempotents in $\overline{RC_m}$ are $\overline{0},\ \overline{1},\ \overline{1} + \overline{g} + \cdots + \overline{g}^{m-1},\ \overline{g} + \overline{g}^2 + \cdots + \overline{g}^{m-1}$. However in RC_m the elements

$$f_1 = 0,\ f_2 = 1,\ f_3 = m^{-1}(1 + g + \cdots + g^{m-1}),$$
$$f_4 = m^{-1}((m-1) - g - g^2 - \cdots - g^{m-1})$$

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are idempotents such that

$$\overline{f}_1 = \overline{0},\ \ \overline{f}_2 = \overline{1},\ \ \overline{f}_3 = \overline{1} + \overline{g} + \cdots + \overline{g}^{m-1},\ \ \overline{f}_4 = \overline{g} + \overline{g}^2 + \cdots + \overline{g}^{m-1}.$$

This shows that RC_m is semiperfect. ■

The following result is immediate from Theorem 15 and [1, Theorem 9].

Corollary 16 *Let R be a local ring with $\operatorname{char} \overline{R} = 2$ and let C_n be a cyclic group of order n. Write $n = m \cdot 2^k$ $(k \geq 0)$ where $(m,\ 2) = 1$. If $\overline{R}$ is a field and the permutation σ of $\{1, 2, m-1\}$ induced by multiplication by 2 modulo m, is cyclic, then the group ring RC_n is clean.*

Corollary 17 *([19, Theorem 3.2]) If C_3 is a cyclic group of order 3, then the group ring $\mathbb{Z}_{(2)}C_3$ is clean.*

Remark 18 The requirement that σ be cyclic in Theorem 15 cannot be removed. In fact, it is determined only by m whether the permutation σ of $\{1, 2, \ldots, m-1\}$ is cyclic. We calculate that σ is cyclic in the case $m = 3, 5, 11, 13, \ldots$. However, for $m = 7$ or 9, σ is not cyclic. Here, $\mathbb{Z}_{(2)}C_7$ is not semiperfect. In fact, in $\mathbb{Z}_2[X]$, $X^7 - \overline{1} = (X + \overline{1})(X^3 + X - \overline{1})(X^3 + X^2 + \overline{1})$. But in $\mathbb{Z}_{(2)}[X]$, $X^7 - 1 = (X-1)(X^6 + X^5 + X^4 + X^3 + X^2 + X + 1)$ and $X^6 + X^5 + X^4 + X^3 + X^2 + X + 1$ is irreducible. So $\mathbb{Z}_{(2)}C_7$ is not semiperfect by [18, Theorem 5.8]. Note that $\overline{\mathbb{Z}_{(2)}C_7}$ is semisimple, hence idempotents cannot be lifted modulo $J(\mathbb{Z}_{(2)}C_7)$, and so $\mathbb{Z}_{(2)}C_7$ is not clean.

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