Canad. J. Math. Vol. **59** (5), 2007 pp. 917–942

Admissibility for a Class of Quasiregular Representations

Bradley N. Curry

Abstract. Given a semidirect product $G = N \rtimes H$ where N is nilpotent, connected, simply connected and normal in G and where H is a vector group for which $\mathrm{ad}(\mathfrak{h})$ is completely reducible and $\mathbf{R}$-split, let τ denote the quasiregular representation of G in $L^2(N)$. An element $\psi \in L^2(N)$ is said to be admissible if the wavelet transform $f \mapsto \langle f, \tau(\cdot)\psi\rangle$ defines an isometry from $L^2(N)$ into $L^2(G)$. In this paper we give an explicit construction of admissible vectors in the case where G is not unimodular and the stabilizers in H of its action on $\widehat{N}$ are almost everywhere trivial. In this situation we prove orthogonality relations and we construct an explicit decomposition of $L^2(G)$ into G-invariant, multiplicity-free subspaces each of which is the image of a wavelet transform . We also show that, with the assumption of (almost-everywhere) trivial stabilizers, non-unimodularity is necessary for the existence of admissible vectors.

Introduction

For the most general notion of continuous wavelet transform, we start with a separable, locally compact topological group G, and a unitary representation τ of G acting in the Hilbert space $\mathcal{H}_\tau$. Given a vector $\psi \in \mathcal{H}_\tau$, we have a linear mapping W_ψ from $\mathcal{H}_\tau$ into the space of bounded continuous functions on G defined by $W_\psi(f) = \langle f, \tau(\,\cdot\,)\psi\rangle$. In the event that W_ψ actually defines an isometry of $\mathcal{H}_\tau$ into $L^2(G)$, then we say that W_ψ is a continuous wavelet transform, and that ψ is admissible for τ. When G has Type I reduced dual, the two extreme cases — where τ is irreducible or where τ is the regular representation — are well understood [8, 11]. Most closely related to discrete wavelets is the case where G is a semidirect product $G = N \rtimes H$ with N normal and where τ is the quasiregular representation of G in $L^2(N)$. The simplest example of this case is the "$ax + b$" group $G = \mathbf{R} \rtimes \mathbf{R}_+^*$, where the quasiregular representation of G in $L^2(\mathbf{R})$ certainly does have admissible vectors, since it is the direct sum of two (square-integrable) irreducible representations. General semidirect products of the form $G = \mathbf{R}^n \rtimes H$, where H is a closed subgroup of $\mathrm{GL}(n, \mathbf{R})$, are studied in [13, 22]. There H is said to be admissible if the corresponding quasiregular representation has an admissible vector, and an (almost) characterization of all admissible H is proved.

It is natural then to consider the continuous wavelet transform for the quasiregular representation of $G = N \rtimes H$ when $\mathbf{R}^n$ is replaced by a locally compact, connected, unimodular group N. The paper [12] lays out the general theory under the assumption that both of the following conditions hold: (i) for a.e. λ belonging to the dual $\widehat{N}$, the stabilizer H_λ in H is compact, and (ii) $\widehat{N}$ has a co-null subset consisting of

Received by the editors November 3, 2004; revised March 16, 2005.
AMS subject classification: 22E27, 22E30.

finitely many open orbits. There are a number of important situations in which these assumptions hold (see for example [10]). Assumption (i) is certainly a natural one; in the case where $N = \mathbf{R}^n$, it is shown relatively easily in [13] that (i) is in fact a *necessary* condition for admissibility. The necessity of (i) in the case where N is not abelian remains an open question however, and seems to be quite difficult even in simple examples. On the other hand, easy examples and the general results of [13] show that (ii) is not necessary.

In this paper we consider the class of $G = N \rtimes H$ satisfying the following conditions:

(i) N is any connected, simply connected nilpotent Lie group,
(ii) H is a vector group acting on N in such a way that the Lie algebra $\mathrm{ad}(\mathfrak{h})$ is completely reducible and **R**-split.

The group G is exponential, meaning that the exponential map defined on its Lie algebra $\mathfrak{g}$ is a bijection onto G. The orbit method applies both to N and G, and the relationship between coadjoint orbits in the linear dual $\mathfrak{n}^*$ of $\mathfrak{n}$, and coadjoint orbits of G in $\mathfrak{g}^*$ is well understood. A great deal is also known about the spectral decomposition of the quasiregular representation in this context [14, 17]. In this paper we clarify the relationship between explicit orbital parametrizations in $\mathfrak{n}^*$ and $\mathfrak{g}^*$ as well. In Section 1 we recall the method of stratification by which the collective orbit structure can be described, applying this method both to $\mathfrak{n}^*$ and to $\mathfrak{g}^*$. With carefully chosen bases for $\mathfrak{n}$ and $\mathfrak{g}$, this procedure yields subsets Λ° of $\mathfrak{n}^*$ and Λ of $\mathfrak{g}^*$, which parametrize a.e. the duals $\widehat{N}$ and $\widehat{G}$ respectively, and such that if $p\colon \mathfrak{g}^* \to \mathfrak{n}^*$ is the restriction map, then $p(\Lambda)$ is explicitly described as a subset of Λ°. The action of H on $\widehat{N}$ is realized a.e. as an action of H on Λ°, and the Fourier transform of a function in $L^2(N)$ has domain Λ° by means of Pukanzsky's explicit version of the Plancherel formula. Thus the issues surrounding conditions (i) and (ii) above — the "size" of the stabilizers in H and the collective structure of the H-orbits in $\widehat{N}$ — can be addressed in concrete terms.

In Section 1 we show that there is a Zariski open subset Λ^1 of Λ° and a single vector subgroup H_0 of H such that $H_0 = H_\lambda$ holds for all $\lambda \in \Lambda^1$. Thus, in light of the preceding constructions, condition (i) is simplified: it just says that $H_0 = (1)$. Nevertheless, it is still an open question as to whether this is necessary for the existence of τ-admissible vectors. Therefore, for the purposes of this paper we make the assumption that condition (i) holds, and hence that $H_0 = (1)$. With this assumption in place, we describe the action of H on Λ^1 and obtain an explicit cross-section $\Sigma \subset \Lambda^1$ for the H-orbits in Λ^1. It is shown that $p|_\Lambda$ is a bijection onto Σ. A decomposition of τ is described in terms of an explicit measure on Σ. The observation is made that if N is not abelian, then the irreducible decomposition of τ has infinite multiplicity. In fact we construct an explicit, direct-sum decomposition of $L^2(N)$ into τ-invariant subspaces $L^2(N)^\beta$ that are pairwise isomorphic and multiplicity-free. In the case where $N = \mathbf{R}^n$, one has $L^2(N)^\beta = L^2(N)$.

By virtue of the results [13, Theorem 1.8] and [11, Theorem 0.2], we expect the existence of admissible vectors to be tied to the non-unimodularity of G, and this is shown to be precisely the case. Note that in this context, both H and N are unimodular, so G is non-unimodular if and only if the H-action on N is non-unimodular. First we prove a Caldéron condition for the admissibility with respect to the subrepresentations τ^β of τ acting in $L^2(N)^\beta$. The construction of τ^β-admissible vectors is now relatively easy when G is non-unimodular, and we use this construction, together with the relationship between Σ and Λ described above, to prove the following.

Theorem *Let $G = N \rtimes H$ where N is a connected, simply connected nilpotent Lie group and H is a vector group such that the Lie algebra* $\operatorname{ad}(\mathfrak{h})$ *is* $\mathbf{R}$*-split and completely reducible. Assume furthermore that for a.e. $\lambda \in \widehat{N}$, the stabilizer H_λ is trivial. Let τ be the quasiregular representation of G in $L^2(N)$. Then τ has an admissible vector if and only if G is not unimodular.*

Finally, in the case where admissible vectors exist, we generalize the methods of [18] to show that the wavelet transform yields an explicit direct-sum decomposition of the regular representation of G into pairwise isomorphic, multiplicity-free subrepresentations, each of which is isomorphic with τ^β.

1 Orbital Parameters in $\mathfrak{n}^*$ and in $\mathfrak{g}^*$

We begin by setting some notation. Let $\mathfrak{g}$ be a Lie algebra over $\mathbf{R}$ of the form $\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{h}$, where $\mathfrak{n}$ is nilpotent, $\mathfrak{n} \supset [\mathfrak{g}, \mathfrak{g}]$, and where $\mathfrak{h}$ is an abelian subalgebra of $\mathfrak{g}$ with $\operatorname{ad}(\mathfrak{h})$ completely reducible and $\mathbf{R}$-split. Let $G = N \rtimes H$ be the connected, simply connected Lie group with Lie algebra $\mathfrak{g}$. Let $\mathfrak{g}^*$ (resp., $\mathfrak{n}^*$) be the linear dual of $\mathfrak{g}$ (resp., $\mathfrak{n}$), and let $p\colon \mathfrak{g}^* \to \mathfrak{n}^*$ be the restriction mapping. For a subalgebra $\mathfrak{s}$ of $\mathfrak{g}$ let $\mathfrak{s}^\perp = \{\ell \in \mathfrak{g}^* \mid \ell|_{\mathfrak{s}} = 0\}$. We denote the coadjoint action of G on $\mathfrak{g}^*$ multiplicatively, as well as the coadjoint action of N on $\mathfrak{n}^*$ and the "restricted coadjoint action" of G on $\mathfrak{n}^*$. For any subset $\mathfrak{t}$ of $\mathfrak{g}$, if f is a linear functional defined on $[\mathfrak{g}, \mathfrak{t}]$, then set

$$\mathfrak{t}^f = \{Z \in \mathfrak{g} \mid f[Z, T] = 0 \text{ holds for every } T \in \mathfrak{t}\}.$$

If $\mathfrak{t}$ is an ideal in $\mathfrak{g}$, then $\mathfrak{t}^f$ is a subalgebra of $\mathfrak{g}$. Recall that for any $\ell \in \mathfrak{g}^*$, the Lie algebra $\mathfrak{g}(\ell)$ of its stabilizer $G(\ell)$ in G is $\mathfrak{g}^\ell$, and similarly for $f \in \mathfrak{n}^*$, the Lie algebra of its stabilizer $N(f)$ in N is $\mathfrak{n}(f) = \mathfrak{n}^f \cap \mathfrak{n}$.

Next we summarize some results concerning the classification and parametrization of coadjoint orbits [6, 7]. Let $\mathfrak{g}$ be any completely solvable Lie algebra, and choose any Jordan–Hölder sequence $(0) = \mathfrak{g}_0 \subset \mathfrak{g}_1 \subset \cdots \subset \mathfrak{g}_n = \mathfrak{g}$, with ordered basis $\{Z_1, Z_2, \ldots, Z_n\}$ so that $Z_j \in \mathfrak{g}_j - \mathfrak{g}_{j-1}$. Let δ_j be the character of G such that $\operatorname{Ad}(s)Z_j = \delta_j(s)Z_j \bmod \mathfrak{g}_{j-1}$, and let $\mathbf{d}\delta_j$ denote its differential.

(1) To each $\ell \in \mathfrak{g}^*$ there is associated an index set $\mathbf{e}(\ell) \subset \{1, 2, \ldots, n\}$, defined by $\mathbf{e}(\ell) = \{1 \le j \le n \mid \mathfrak{g}_j \not\subset \mathfrak{g}_{j-1} + \mathfrak{g}(\ell)\}$. For a subset $\mathbf{e}$ of $\{1, 2, \ldots, n\}$, the set $\Omega_{\mathbf{e}} = \{\ell \in \mathfrak{g}^* \mid \mathbf{e}(\ell) = \mathbf{e}\}$ is G-invariant. The $\Omega_{\mathbf{e}}$ are determined by polynomials as follows: to each index set $\mathbf{e}$ one associates the skew-symmetric matrix

$$M_{\mathbf{e}}(\ell) = \big[\ell[Z_i, Z_j]\big]_{i,j \in \mathbf{e}}.$$

Setting $Q_{\mathbf{e}}(\ell) = \det M_{\mathbf{e}}(\ell)$, one finds that there is a total ordering $\prec$ on the set $\mathcal{E} = \{\mathbf{e} \mid \Omega_{\mathbf{e}} \neq \varnothing\}$ such that $\Omega_{\mathbf{e}} = \{\ell \in \mathfrak{g}^* \mid Q_{\mathbf{e}'}(\ell) = 0 \text{ for all } \mathbf{e}' \prec \mathbf{e}, \text{ and } Q_{\mathbf{e}}(\ell) \neq 0\}$. We refer to the collection of non-empty $\Omega_{\mathbf{e}}$ as the coarse stratification of $\mathfrak{g}^*$, and to its elements as coarse layers.

https://doi.org/10.4153/CJM-2007-039-0 Published online by Cambridge University Press

(2) Let $\mathbf{e} \in \mathcal{E}$; then $|\mathbf{e}|$ is even, and we set $d = |\mathbf{e}|/2$. To each $\ell \in \Omega_{\mathbf{e}}$ there is associated a "polarizing sequence" of subalgebras

$$\mathfrak{g} = \mathfrak{p}_0(\ell) \supset \mathfrak{p}_1(\ell) \supset \cdots \supset \mathfrak{p}_d(\ell) = \mathfrak{p}(\ell),$$

and an index *sequence pair* $\mathbf{i}(\ell) = \{i_1 < i_2 < \cdots < i_d\}$ and $\mathbf{j}(\ell) = \{j_1, j_2, \dots, j_d\}$, having values in $\mathbf{e}(\ell)$, defined by the recursive equations:

$$i_k = \min\{1 \le j \le n \mid \mathfrak{g}_j \cap \mathfrak{p}_{k-1}(\ell) \not\subset \mathfrak{p}_{k-1}(\ell)^{\ell}\},$$

$$\mathfrak{p}_k(\ell) = \left(\mathfrak{p}_{k-1}(\ell) \cap \mathfrak{g}_{i_k}\right)^{\ell} \cap \mathfrak{p}_{k-1}(\ell),\ j_k = \min\{1 \le j \le n \mid \mathfrak{g}_j \cap \mathfrak{p}_{k-1}(\ell) \not\subset \mathfrak{p}_k(\ell)\}.$$

For each k, $i_k < j_k$, and $\mathbf{e}(\ell)$ is the disjoint union of the values of $\mathbf{i}(\ell)$ and $\mathbf{j}(\ell)$. Note that since $\mathbf{i}(\ell)$ must be increasing, it is determined by $\mathbf{e}(\ell)$ and $\mathbf{j}(\ell)$. For any splitting of $\mathbf{e}$ into such a sequence pair $(\mathbf{i}, \mathbf{j})$ we set $\Omega_{\mathbf{e},\mathbf{j}} = \{\ell \in \Omega_{\mathbf{e}} \mid \mathbf{j}(\ell) = \mathbf{j}\}$. These sets are also algebraic and G-invariant, and we refer to the collection of non-empty $\Omega_{\mathbf{e},\mathbf{j}}$ as the fine stratification of $\mathfrak{g}^*$. For $1 \le k \le d$, if we set

$$M_{\mathbf{e},k}(\ell) = \left[\ell[Z_i, Z_j]\right]_{i,j \in \{i_1, j_1, i_2, j_2, \dots, i_k, j_k\}},$$

let $\mathbf{Pf}_{\mathbf{e},k}(\ell)$ denote the Pfaffian of $M_{\mathbf{e},k}(\ell)$, and let $\mathbf{P}_{\mathbf{e},\mathbf{j}}(\ell) = \mathbf{Pf}_{\mathbf{e},1}(\ell)\mathbf{Pf}_{\mathbf{e},2}(\ell) \cdots \mathbf{Pf}_{\mathbf{e},d}(\ell)$, then there is a total ordering $\lll$ on the pairs $\mathbf{e}, \mathbf{j}$ such that

$$\Omega_{\mathbf{e},\mathbf{j}} = \{\ell \in \mathfrak{g}^* \mid \mathbf{P}_{\mathbf{e}',\mathbf{j}'}(\ell) = 0 \text{ for all } (\mathbf{e}', \mathbf{j}') \lll (\mathbf{e}, \mathbf{j}) \text{ and } \mathbf{P}_{\mathbf{e},\mathbf{j}}(\ell) \neq 0\}.$$

The following rational functions are naturally associated with the fine stratification. Fix $\ell \in \Omega$. Define $\rho_0(Z, \ell) = Z$; assume that $\rho_{k-1}(Z, \ell)$ is defined and set

$$\rho_k(Z,\ell) = \rho_{k-1}(Z,\ell) - \frac{\ell[\rho_{k-1}(Z,\ell), \rho_{k-1}(Z_{i_k},\ell)]}{\ell[\rho_{k-1}(Z_{j_k},\ell), \rho_{k-1}(Z_{i_k},\ell)]}\, \rho_{k-1}(Z_{j_k},\ell) - \frac{\ell[\rho_{k-1}(Z,\ell), \rho_{k-1}(Z_{j_k},\ell)]}{\ell[\rho_{k-1}(Z_{i_k},\ell), \rho_{k-1}(Z_{j_k},\ell)]} \rho_{k-1}(Z_{i_k},\ell).$$

Set $Y_k(\ell) = \rho_{k-1}(Z_{i_k}, \ell)$, and $X_k(\ell) = \rho_{k-1}(Z_{j_k}, \ell)$, $1 \le k \le d$; then it can be shown [2, Lemma 1.5] that for each $1 \le k \le d$,

$$\mathbf{Pf}_{\mathbf{e},k}(\ell) = \ell[Y_1(\ell), X_1(\ell)]\ell[Y_2(\ell), X_2(\ell)] \cdots \ell[Y_k(\ell), X_k(\ell)].$$

If we set

$$\mathfrak{m}_k(\ell) = \operatorname{span}\{Y_1(\ell), Y_2(\ell), \dots, Y_k(\ell), X_1(\ell), X_2(\ell), \dots, X_k(\ell)\},$$

then for each $\ell \in \Omega$, $\mathfrak{g} = \mathfrak{m}_k(\ell) \oplus \mathfrak{m}_k(\ell)^{\ell}$ and $\rho_k(Z, \ell)$ is the projection of Z into $\mathfrak{m}_k(\ell)^{\ell}$ parallel to $\mathfrak{m}_k(\ell)$. It follows that

$$\ell[\rho_k(Z,\ell), \rho_k(T,\ell)] = \ell[\rho_k(Z,\ell), T], \quad Z, T \in \mathfrak{g}, \ell \in \mathfrak{g}^*.$$

The functions $\rho_k(\,\cdot\,, \ell)$ have the additional properties:

(i) $\rho_k(\mathfrak{g}_j, \ell) \subset \mathfrak{g}_j, 1 \le j \le n, 0 \le k \le d,$
(ii) $\rho_k(\mathfrak{g}, \ell) \cap \mathfrak{g}_{i_{k+1}-1} \subset \mathfrak{g}(\ell), 0 \le k \le d-1.$

Finally, if α is an automorphism of $\mathfrak{g}$ such that $\alpha(\mathfrak{g}_j) = \mathfrak{g}_j$ holds for every j, then α^* leaves each fine layer invariant.

(3) Now fix a layer $\Omega_{\mathbf{e},\mathbf{j}}$ in the fine stratification. For each $\ell \in \Omega_{\mathbf{e},\mathbf{j}}$, define the "dilation set" $\varphi(\ell) = \{ j \in \mathbf{e} \mid \mathfrak{g}_{j-1}^{\ell} \cap \ker(\mathbf{d}\delta_j) = \mathfrak{g}_j^{\ell} \cap \ker(\mathbf{d}\delta_j)\}$. The index set $\varphi(\ell)$ identifies those directions in the orbit of ℓ where the coadjoint action of G "dilates" by the character δ_j^{-1}. The indices in $\varphi(\ell)$ are included in the values of the sequence $\mathbf{i}$ and are defined by $\varphi(\ell) = \{i_k \mid \mathbf{d}\delta_{i_k}(X_k(\ell)) \neq 0\}$. There are examples where $\varphi(\ell)$ is not constant on the fine layer. For each subset φ of the values of $\mathbf{i}$, the set $\Omega_{\mathbf{e},\mathbf{j},\varphi} = \{\ell \in \Omega_{\mathbf{e},\mathbf{j}} \mid \varphi(\ell) = \varphi\}$ is an algebraic subset of $\Omega_{\mathbf{e},\mathbf{j}}$, and we refer to this further refinement of the fine stratification as the ultra-fine stratification of $\mathfrak{g}^*$. The ultra-fine stratification also has an ordering for which the minimal layer is a Zariski open subset of the minimal fine layer.

(4) Now fix an ultra-fine layer $\Omega = \Omega_{\mathbf{e},\mathbf{j},\varphi}$, and for $\ell \in \Omega$, $j = i_k \in \varphi$, set

$$q_j(\ell) = \frac{\mathbf{d}\delta_j(X_k(\ell))}{\ell[X_k(\ell), Z_j]}.$$

Let $V = V_{\mathbf{e},\varphi} = \{\ell \in \mathfrak{g}^* \mid \text{ if } j \in \mathbf{e} - \varphi, \text{ then } \ell(Z_j) = 0\}$. Then the set

$$\Lambda = \Lambda_{\mathbf{e},\mathbf{j},\varphi} = \left\{ \ell \in V \cap \Omega \;\middle|\; \text{for every } j \in \varphi, |q_j(\ell)| = 1 \right\}$$

is a topological cross-section for the orbits in Ω. If $\mathfrak{g}$ is nilpotent, then the ultra-fine stratification coincides with the fine stratification and $\Lambda = V \cap \Omega$.

We now return to the case where $\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{h}$ as described above, and we apply the stratification procedure first to the nilpotent Lie algebra $\mathfrak{n}$. We fix once and for all an ordered basis $\{Z_1, Z_2, \dots, Z_n\}$ of $\mathfrak{n}$ for which the following hold for all $1 \le j \le n$:

(i) $\mathfrak{n}_j = \operatorname{span}\{Z_1, Z_2, \dots, Z_j\}$ is an ideal in $\mathfrak{g}$,
(ii) for each $A \in \mathfrak{h}$, Z_j is an eigenvector for $\operatorname{ad} A$.

Having chosen the basis $Z_1, Z_2, \dots, Z_n$ for $\mathfrak{n}$, let Ω° be the minimal (and hence Zariski open) fine layer in $\mathfrak{n}^*$, with Λ° its cross-section. Denote the objects referred to in (1)–(3) above by $\mathbf{e}^\circ, \mathbf{i}^\circ, \mathbf{j}^\circ$, and ρ_k°. For each $1 \le j \le n$, set $e_j = Z_j^* \in \mathfrak{n}^*$ and set $\gamma_j = -\mathbf{d}\delta_j$ so that $\operatorname{ad}^* A(e_j) = \gamma_j(A)e_j$, $A \in \mathfrak{h}$. For each $h \in H$, since $\operatorname{Ad}^*(h)(\Omega^\circ) = \Omega^\circ$ and the e_j are eigenvectors of $\operatorname{Ad}^*(h)$, we have that $\operatorname{Ad}^*(h)(\Lambda^\circ) = \Lambda^\circ$.

With this in mind, we choose a convenient basis for $\mathfrak{h}$. Set $c = n - 2d^\circ$, write $\{1, \dots, n\} - \mathbf{e}^\circ = \{u_1 < u_2 < \dots < u_c\}$, and set $\lambda_a = \ell(Z_{u_a}), 1 \le a \le c$. Then $\ell \to \lambda = (\lambda_1, \lambda_2, \dots, \lambda_c)$ identifies Λ° with a Zariski open subset of $\mathbf{R}^c$. We select a subset $\alpha_\nu, 1 \le \nu \le r$ of $\gamma_{u_a}, 1 \le a \le c$ as follows: $a_1 = \min\{1 \le a \le c \mid \gamma_{u_a} \neq 0\}$, $a_2 = \min\{1 \le a \le c \mid \gamma_{u_a} \text{ is not a multiple of } \gamma_{u_{a_1}}\}$, $a_3 = \min\{1 \le a \le c \mid \gamma_{u_a} \text{ is not in the span of } \gamma_{u_{a_1}}, \gamma_{u_{a_2}}\}$, and so on, until for some $r > 0$, every γ_j belongs to the span of $\{\gamma_{u_{a_\nu}} \mid 1 \le \nu \le r\}$. Set $\alpha_\nu = \gamma_{u_{a_\nu}}, 1 \le \nu \le r$. We shall refer to the set $\{\alpha_\nu \mid 1 \le \nu \le r\}$ as the minimal spanning set of roots with respect to the orbital cross-section Λ°. We shall use the notation $\mathfrak{h}_\nu = \bigcap_{w=1}^{\nu} \ker \alpha_w, 1 \le \nu \le r$. We now make an important observation: let $f \in \Lambda^\circ$; for each $j \in \mathbf{e}^\circ$, $f(Z_j) = 0$, and if $j \notin \mathbf{e}^\circ, 1 \le j \le n$, then $\mathfrak{h}_r \subset \ker \gamma_j$. It follows that $\mathfrak{h}_r \subset \mathfrak{n}^f$ holds for every $f \in \Lambda^\circ$.

Let $\{A_1, A_2, \dots A_r\} \subset \mathfrak{h}$ be a basis of $\mathfrak{h}$ mod $\mathfrak{h}_r$ that is dual to the minimal spanning set of roots, so that $\alpha_v(A_w) = 0$ or 1 according as $v \neq w$ or $v = w$. Choosing a basis $\{A_{r+1}, \dots, A_p\}$ for $\mathfrak{h}_r$, we fix from now on the ordered basis $\{A_1, A_2, \dots, A_p\}$ for $\mathfrak{h}$. With the ordered Jordan–Hölder basis $\{Z_1, Z_2, \dots, Z_n, A_p, A_{p-1}, \dots, A_1\}$ for $\mathfrak{g}$ in place, we apply the stratification procedure to $\mathfrak{g}^*$ as described above (of course, we could rename $Z_m = A_1, Z_{m-1} = A_2$, etc.). Let $\Omega = \Omega_{\mathbf{e},\mathbf{j}}$ be the minimal, Zariski open, fine layer in $\mathfrak{g}^*$. Write the defining index sequence pair as $\mathbf{i} = \{i_1 < i_2 < \cdots < i_d\}$, $\mathbf{j} = \{j_1, j_2, \dots, j_d\}$, so that $2d$ is the dimension of the coadjoint orbits in Ω. Set

$$K^\circ = \{1 \le k \le d \mid j_k \le n\} = \{k_1 < k_2 < \cdots < k_{d^\circ}\}.$$

Lemma 1.1 *One has $p(\Omega) \subset \Omega^\circ$, and the index sequence pair for Ω° is*

$$\mathbf{i}^\circ = \{i_{k_1} < i_{k_2} < \cdots < i_{k_{d^\circ}}\}, \quad \mathbf{j}^\circ = \{j_{k_1}, j_{k_2}, \dots, j_{k_{d^\circ}}\}.$$

Proof By [1, Lemma2.2], $p(\Omega)$ is contained in the layer $\Omega^\circ_{\mathbf{e}^\circ,\mathbf{j}^\circ}$ of N-orbits in $\mathfrak{n}^*$ whose index data is the above. At the same time we have that $p(\Omega)$ is open in $\mathfrak{n}^*$, and since Ω° is dense in $\mathfrak{n}^*$, it follows that $\Omega^\circ = \Omega^\circ_{\mathbf{e}^\circ,\mathbf{j}^\circ}$. ■

The next lemma is proved in [1, Lemma 4.2] and clarifies the relationship between the functions $\rho_k, 0 \le k \le d$ and $\rho^\circ_r, 1 \le r \le d^\circ$.

Lemma 1.2 *Fix $k = k_r \in K^\circ$, $\ell \in \Omega$, and set $f = p(\ell)$. Set*

$$\mathcal{Y}_{k_r}(\ell) = \operatorname{span}\{Y_h(\ell) \mid 1 \le h \le k_r - 1, h \notin K^\circ\}.$$

We have each of the following.

(i) $\mathcal{Y}_{k_r}(\ell) \subset \mathfrak{n}(f)$.
(ii) *For each $k_{r-1} < h < k_r$, $\rho_h(Z, \ell) = \rho^\circ_{r-1}(Z, f) \bmod \mathcal{Y}_{k_r}(\ell)$ holds for all $Z \in \mathfrak{n}$.*
(iii) *For any $Z \in \mathfrak{n}$, $\ell[Z, Y_{k_r}(\ell)] = f[Z, Y^\circ_r(f)]$ and $\ell[Z, X_{k_r}(\ell)] = f[Z, X^\circ_r(f)]$.*
(iv) $\rho_k(Z, \ell) = \rho^\circ_r(Z, f) \bmod \mathcal{Y}_{k_r}(\ell)$ *holds for all $Z \in \mathfrak{n}$.*

We now focus on the special properties of the stratification procedure on $\mathfrak{g}$ when applied to the elements $\ell \in p^{-1}(\Lambda^\circ)$.

Lemma 1.3 *Let $\ell \in \Omega$ such that $f = p(\ell) \in \Lambda^\circ$.*

(i) *One has $\rho_k(\mathfrak{h}, \ell) \subset \mathfrak{h}$, $1 \le k \le d$.*
(ii) *For each $j \in \mathbf{e}^\circ$, $A \in \mathfrak{h}$, one has $\ell[\rho_k(A, \ell), Z_j] = \ell[A, Z_j] = 0$, $1 \le k \le d$.*

Proof We proceed by induction on k; if $k = 0$, then $\rho_0(\,\cdot\,, \ell)$ is the identity map and both statements (i) and (ii) are clear. Suppose that $k \ge 1$ and that (i) and (ii) hold for $k - 1$.

To prove (i) for k, let $A \in \mathfrak{h}$. The assumption that (i) holds for $k - 1$ says that $\rho_{k-1}(A, \ell)$ belongs to $\mathfrak{h}$. Suppose first that $j_k > n$. Then the assumption that (i) and

(ii) hold for $k-1$ also gives $X_k(\ell) \in \mathfrak{h}$, and since $\mathfrak{h}$ is abelian, $[A, X_k(\ell)] = 0$. Thus

$$\begin{aligned}\rho_k(A,\ell) &= \rho_{k-1}(A,\ell) - \frac{\ell[A, Y_k(\ell)]}{\ell[X_k(\ell), Y_k(\ell)]} X_k(\ell) - \frac{\ell[A, X_k(\ell)]}{\ell[Y_k(\ell), X_k(\ell)]} Y_k(\ell) \\ &= \rho_{k-1}(A,\ell) - \frac{\ell[A, Y_k(\ell)]}{\ell[X_k(\ell), Y_k(\ell)]} X_k(\ell)\end{aligned}$$

belongs to $\mathfrak{h}$. On the other hand, if $j_k \le n$, then the assumption that (ii) holds for $k-1$ says that $\ell[A, X_k(\ell)] = \ell[A, Y_k(\ell)] = 0$, hence $\rho_k(A,\ell) = \rho_{k-1}(A,\ell)$ belongs to $\mathfrak{h}$ in this case. This completes the induction step for part (i).

As for (ii), let $j \in \mathbf{e}^\circ$ and let $A \in \mathfrak{h}$; we need only show that $\ell[A, \rho_k(Z_j,\ell)] = \ell[A, \rho_{k-1}(Z_j,\ell)]$. As before, we suppose first that $j_k > n$, so that we have $X_k(\ell) \in \mathfrak{h}$ and $\ell[A, X_k(\ell)] = 0$. The assumption that (ii) holds for $k-1$ now gives

$$\ell[Z_j, X_k(\ell)] = \ell[Z_j, \rho_{k-1}(Z_{j_k},\ell)] = 0.$$

Hence

$$\begin{aligned}\ell[A, \rho_k(Z_j,\ell)] &= \ell[A, \rho_{k-1}(Z_j,\ell)] - \frac{\ell[Z_j, Y_k(\ell)]\ell[A, X_k(\ell)]}{\ell[X_k(\ell), Y_k(\ell)]} \\ &\qquad - \frac{\ell[Z_j, X_k(\ell)]\ell[A, Y_k(\ell)]}{\ell[Y_k(\ell), X_k(\ell)]} \\ &= \ell[A, \rho_{k-1}(Z_j,\ell)].\end{aligned}$$

For the case $j_k \le n$, the assumption that (ii) holds for $k-1$ immediately gives $\ell[A, X_k(\ell)] = \ell[A, Y_k(\ell)] = 0$, whence $\ell[A, \rho_k(Z_j,\ell)] = \ell[A, \rho_{k-1}(Z_j,\ell)]$. This completes the proof. ■

Lemma 1.4 *Let $\ell \in \Omega$ such that $f = p(\ell) \in \Lambda^\circ$. Assume that $\{1, 2, \dots, d\} - K^\circ$ is non-empty, and write $\{1, 2, \dots, d\} - K^\circ = \{h_1 < h_2 < \cdots\}$. Choose an index $h_\nu \in \{1, 2, \dots, d\} - K^\circ$.*

(i) *For $0 \le k < h_\nu$, one has $\rho_k(A_\nu, \ell) = A_\nu$.*
(ii) *One has $\nu \le r$ and $\{i_{h_1} < i_{h_2} < \cdots < i_{h_\nu}\} = \{u_{a_1} < u_{a_2} < \cdots < u_{a_\nu}\}$.*
(iii) *One has $\{j_{h_1} = m, j_{h_2} = m-1, \dots, j_{h_\nu} = m - \nu + 1\}$.*

Proof Suppose that $\nu = 1$; we repeat the argument for Lemma 1.3(i) with the additional fact that the case $j_k > n$ cannot occur here, as $h_1 = \min\{1 \le k \le d \mid j_k > n\}$. It follows immediately that $\rho_k(A_1, \ell) = A_1$, $1 \le k < h_1$.

Now set $u = u_{a_1}$, $i = i_{h_1}$ and $j = j_{h_1}$; we show that $u = i$. First we claim that $u \le i$. To see this, note that by definition of u, $\ell[\mathfrak{h}, \mathfrak{g}_{u-1}] = 0$. If $u > i$ were true, then $\mathfrak{h} \subset \mathfrak{g}_i^\ell$ and $\mathfrak{g}_i \subset \mathfrak{h}^\ell$. The first of these inclusions implies that

$$\mathfrak{p}_{h_1-1}(\ell) = \mathfrak{p}_{h_1-1}(\ell) \cap \mathfrak{n} + \mathfrak{h}.$$

Since $i \notin \mathbf{i}^\circ$, $\mathfrak{g}_i \cap \mathfrak{p}_{h_1-1}(\ell) \subset (\mathfrak{p}_{h_1-1}(\ell) \cap \mathfrak{n})^\ell$. This together with the second inclusion above gives

$$\begin{aligned}\mathfrak{g}_i \cap \mathfrak{p}_{h_1-1}(\ell) &\subset (\mathfrak{p}_{h_1-1}(\ell) \cap \mathfrak{n})^\ell \cap \mathfrak{h}^\ell \\ &\subset (\mathfrak{p}_{h_1-1}(\ell) \cap \mathfrak{n} + \mathfrak{h})^\ell \\ &= (\mathfrak{p}_{h_1-1}(\ell))^\ell,\end{aligned}$$

contradicting the definition of $i = i_{h_1}$. Thus the claim is proved. In light of this and the fact that $(\mathbf{e} - \mathbf{e}^\circ) \cap \{1, 2, \dots, n\} = \mathbf{i} - \mathbf{i}^\circ$, it remains to show that $u \in \mathbf{e}$. Suppose then that $u \notin \mathbf{e}$; then for any $\ell \in \Omega$, we have $Z_u = T(\ell) + W(\ell)$ where $T(\ell) \in \mathfrak{g}(\ell)$ and $W(\ell) \in \mathfrak{g}_{u-1}$. But, again since $\ell[\mathfrak{h}, \mathfrak{g}_{u-1}] = 0$, it follows that

$$\ell(Z_u) = \gamma_u(A_1)\ell(Z_u) = \ell[A_1, Z_u] = \ell[A_1, T(\ell)] + \ell[A_1, W(\ell)] = 0$$

holds for all $\ell \in \Omega$, which is impossible since Ω is dense in $\mathfrak{g}^*$.

Next we show that $j = m$. Observe that $\mathfrak{g}_{m-1} = \mathfrak{n} + \mathfrak{h}_1$ and $\mathfrak{h}_1 \subset \mathfrak{p}_{h_1}(\ell) \subset \mathfrak{p}_{h_1-1}(\ell)$. On the other hand, since $i \in \mathbf{i} - \mathbf{i}^\circ$, we have $j > n$ and $\mathfrak{p}_{h_1-1}(\ell) \cap \mathfrak{n} \subset \mathfrak{p}_{h_1}(\ell)$. It follows that $\mathfrak{p}_{h_1-1}(\ell) \cap \mathfrak{g}_{m-1} = \mathfrak{p}_{h_1-1}(\ell) \cap \mathfrak{n} + \mathfrak{h}_1 \subset \mathfrak{p}_{h_1}(\ell)$, which means that $j = m$.

Now suppose that $\nu > 1$ and that the proposition holds for $1 \le w \le \nu - 1$. To prove part (i) for ν, let $0 \le k < h_\nu$. We proceed by induction on k, the statement being clear when $k = 0$. If $k \in K^\circ$, then by Lemma 1.3 we have $\ell[A_\nu, X_k(\ell)] = \ell[A_\nu, Y_k(\ell)] = 0$, and hence $\rho_k(A_\nu, \ell) = \rho_{k-1}(A_\nu, \ell)$. If $k \notin K^\circ$, say $k = h_w$, then by our induction hypothesis, $i_k = u_{a_w}$, $j_k = m - w + 1$, and $X_k(\ell) = A_w$. Hence $\ell[A_\nu, X_k(\ell)] = \ell[A_\nu, A_w] = 0$ and

$$\ell[A_\nu, Y_k(\ell)] = \ell[\rho_{k-1}(A_\nu, \ell), Z_{i_k}] = \ell[A_\nu, Z_{i_k}] = 0.$$

So $\rho_k(A_\nu, \ell) = \rho_{k-1}(A_\nu, \ell)$ in this case also. Now by induction on k, part (i) is true for ν.

As for part (ii), set $u = u_{a_\nu}$, $i = i_{h_\nu}$, and $j = j_{h_\nu}$. Then $[\mathfrak{h}_{\nu-1}, \mathfrak{g}_{u-1}] = (0)$. Imitating the argument above for the case $\nu = 1$, we see that the assumption that $u > i$ leads to the inclusions $\mathfrak{h}_{\nu-1} \subset \mathfrak{g}_i^\ell$ and $\mathfrak{g}_i \subset \mathfrak{h}_{\nu-1}^\ell$. In the same way as when $\nu = 1$, we claim that $\mathfrak{p}_{h_\nu-1}(\ell) = \mathfrak{p}_{h_\nu-1}(\ell) \cap \mathfrak{n} + \mathfrak{h}_{\nu-1}$. To see this, note that $\mathfrak{h}_{\nu-1} \subset \mathfrak{p}_{h_\nu-1}(\ell)$, so obviously $\mathfrak{p}_{h_\nu-1}(\ell) \supset \mathfrak{p}_{h_\nu-1}(\ell) \cap \mathfrak{n} + \mathfrak{h}_{\nu-1}$. Counting dimensions gives equality:

$$\begin{aligned}\dim(\mathfrak{p}_{h_\nu-1}(\ell)) &= m - h_\nu + 1, \\ \dim(\mathfrak{p}_{h_\nu-1}(\ell) \cap \mathfrak{n}) &= n - |\{i_k \in \mathbf{i}^\circ \mid k \le h_\nu - 1\}| \\ &= n - \{h_\nu - 1 - (\nu - 1)\} \\ &= n - (h_\nu - \nu),\end{aligned}$$

so

$$\begin{aligned}\dim((\mathfrak{p}_{h_\nu-1}(\ell) \cap \mathfrak{n}) + \mathfrak{h}_{\nu-1}) &= n - (h_\nu - \nu) + p - \nu + 1 = m - h_\nu + 1 \\ &= \dim(\mathfrak{p}_{h_\nu-1}(\ell)).\end{aligned}$$

Now we follow verbatim the same line of reasoning as in the case $v = 1$ to arrive at a contradiction, thereby concluding that $u \leq i$. Since by induction we already have $i_{h_w} = u_{a_w}$, $1 \leq w \leq v-1$, we get $i_{h_w} < u$ for $1 \leq w \leq v-1$. Now, arguing as in the case $v = 1$, we find that it remains to show that $u \in \mathbf{e}$. But again, the argument for this is identical to the case $v = 1$: if $u \notin \mathbf{e}$, then we find that $\ell(Z_u) = \ell[A_v, Z_u] = 0$ holds for all $\ell \in \Omega$, etc.

Finally we show that $j = m - v + 1$. As in the case $v = 1$, $\mathfrak{g}_{m-v+1} = \mathfrak{n} + \mathfrak{h}_{v-1}$ and $\mathfrak{h}_v \subset \mathfrak{p}_{h_v}(\ell) \subset \mathfrak{p}_{h_v-1}(\ell)$. Also $i \in \mathbf{i} - \mathbf{i}^\circ$, so $j > n$ and $\mathfrak{p}_{h_v-1}(\ell) \cap \mathfrak{n} \subset \mathfrak{p}_{h_v}(\ell)$. It follows that $\mathfrak{p}_{h_v-1}(\ell) \cap \mathfrak{g}_{m-v} = \mathfrak{p}_{h_v-1}(\ell) \cap \mathfrak{n} + \mathfrak{h}_v \subset \mathfrak{p}_{h_v}(\ell)$. Since we already have $j_{h_w} = m - w + 1$ for $1 \leq w \leq v-1$, $j = m - v + 1$ follows. This completes the proof. ■

Lemma 1.5 *Let $d - d^\circ < w \leq p$. Then for each $\ell \in \Lambda^\circ$ and $0 \leq k \leq d$, one has $\rho_k(A_w, \ell) = A_w$.*

Proof As usual we proceed by induction on k, the case $k = 0$ being clear. Suppose that $k \geq 1$ and that the lemma holds for $k - 1$. If $k \in K^\circ$, then Lemma 1.3 gives $\ell[A_w, X_k(\ell)] = \ell[A_w, Y_k(\ell)] = 0$. If $k = h_v \in \{1, 2, \dots, d\} - K^\circ$, then Lemma 1.4 gives $X_k(\ell) = A_v$ and $Y_k(\ell) = \rho_{k-1}(Z_{u_{a_v}}, \ell)$, so that in this case also $\ell[A_w, X_k(\ell)] = \ell[A_w, Y_k(\ell)] = 0$. In either case then, we have $\rho_k(A_w, \ell) = \rho_{k-1}(A_w, \ell)$. ■

Proposition 1.6 *Let $\mathfrak{g}$ be a completely solvable Lie algebra of the form $\mathfrak{g} = \mathfrak{n} \oplus \mathfrak{h}$, where $\mathfrak{n}$ is a nilpotent ideal and $\mathfrak{h}$ is an abelian subalgebra such that $\mathrm{ad}(\mathfrak{h})$ is completely reducible. Let $\{Z_1, Z_2, \dots, Z_n, A_p, A_{p-1}, \dots, A_2, A_1\}$ be an ordered Jordan–Hölder basis of $\mathfrak{g}$ with the following properties.*

(a) *$\{Z_1, Z_2, \dots, Z_n\}$ is a basis of $\mathfrak{n}$ with respect to which $\mathrm{ad}(\mathfrak{h})$ is diagonalized.*
(b) *$\{A_1, A_2, \dots, A_r\}$ is dual to the minimal spanning set of roots and $\{A_{r+1}, \dots, A_p\}$ is a basis for $\mathfrak{h}_r$ where $\mathfrak{h}_r$ is defined as above.*

Let $\Omega = \Omega_{\mathbf{e},\mathbf{j}}$ be the minimal fine layer in $\mathfrak{g}^*$ and $\Omega^\circ = \Omega_{\mathbf{e}^\circ,\mathbf{j}^\circ}$ the minimal fine layer in $\mathfrak{n}^*$, with respect to the bases chosen above. Write K° and $\{1, 2, \dots, d\} - K^\circ = \{h_1 < h_2 < \cdots < h_{d-d^\circ}\}$ as above. Let $\mathbf{Pf}_{\mathbf{e}^\circ,w}$, $1 \leq w \leq d^\circ$ be the Pfaffian polynomials that define Ω°. Then one has the following.

(i) *$d - d^\circ = r$, and the increasing sequence $\{i_{h_1} < i_{h_2} < \cdots < i_{h_r}\}$ is precisely the sequence $\{u_{a_1} < u_{a_2} < \cdots < u_{a_r}\}$ corresponding to the minimal spanning set of roots.*
(ii) *$j_{h_v} = m - v + 1$, $1 \leq v \leq r$.*
(iii) *Let $\ell \in \Omega \cap p^{-1}(\Lambda^\circ)$ with $f = p(\ell)$. For each $1 \leq k \leq d$, let*

$$v_0 = \max\{1 \leq v \leq r \mid h_v \leq k\},$$
$$w_0 = \max\{1 \leq w \leq d^\circ \mid k_w \leq k\}.$$

Then

$$\mathbf{Pf}_{\mathbf{e},k}(\ell) = \prod_{v=1}^{v_0} \ell(Z_{i_{h_v}})\, \mathbf{Pf}_{\mathbf{e}^\circ,w_0}(f).$$

https://doi.org/10.4153/CJM-2007-039-0 Published online by Cambridge University Press

(iv) *For every $\ell \in \Omega$, the dilation set $\varphi(\ell)$ is precisely the set $\{i_k \mid k \notin K^\circ\} = \{i_{h_\nu} \mid 1 \leq \nu \leq r\}$ and hence the minimal fine layer in $\mathfrak{g}^*$ coincides with the minimal ultra-fine layer.*

Proof It follows from Lemma 1.4 that the sequence $\{i_{h_1} < i_{h_2} < \cdots < i_{h_{d-d^\circ}}\}$ coincides with the first $d - d^\circ$ terms of the sequence $\{u_{a_1} < u_{a_2} < \cdots < u_{a_r}\}$. Now if $d-d^\circ < w \leq r$, then Lemma 1.5 implies that $A_w \in \mathfrak{g}(\ell)$ holds for all $\ell \in p^{-1}(\Lambda^\circ)$. But this means that $f(Z_{u_{a_w}}) = f[A_w, Z_{u_{a_w}}] = 0$ holds for all $f \in \Lambda^\circ$. Since Λ° is a dense open subset of $V = \operatorname{span}\{e_{u_1}, e_{u_2}, \dots, e_{u_c}\}$, this is impossible. Thus part (i) is proved. Part (ii) now follows from Lemma 1.4.

For part (iii) we compute using Lemma 1.2, Lemma 1.4, and the properties of ρ_k:

$$\begin{aligned}
&\ell[Y_1(\ell), X_1(\ell)]\ell[Y_2(\ell), X_2(\ell)]\cdots\ell[Y_k(\ell), X_k(\ell)]\\
&\qquad = \ell[Z_{i_1}, Z_{j_1}]\ell[Z_{i_2}, \rho_1(Z_{j_2}, \ell)]\cdots\ell[Z_{i_k}, \rho_{k-1}(Z_{j_k}, \ell)]\\
&\qquad = \prod_{\nu=1}^{\nu_0} \ell[Z_{i_{h_\nu}}, \rho_{h_\nu-1}(Z_{j_{h_\nu}}, \ell)] \prod_{w=1}^{w_0} \ell[Z_{i_{k_w}}, \rho_{k_w-1}(Z_{j_{k_w}}, \ell)]\\
&\qquad = \prod_{\nu=1}^{\nu_0} \ell[Z_{i_{h_\nu}}, A_\nu] \prod_{w=1}^{w_0} f[Z_{i_{k_w}}, \rho^\circ_{w-1}(Z_{j_{k_w}}, f)]\\
&\qquad = \Bigl(\prod_{\nu=1}^{\nu_0} \ell(Z_{i_{h_\nu}})\Bigr)\mathbf{Pf}_{\mathbf{e}^\circ, w_0}(f).
\end{aligned}$$

Finally for part (iv), Lemma 1.4, part (i) shows that for $k = h_\nu \notin K^\circ$, we have $X_k(\ell) = A_\nu$, hence $\varphi(\ell) = \{i_k \in \mathbf{i} \mid \mathbf{d}\delta_{i_k}(X_k(\ell)) \neq 0\} = \{i_k \in \mathbf{i} \mid k \notin K^\circ\}$ holds for each $\ell \in \Omega$. ■

Corollary 1.7 *Let $\ell \in \Omega \cap p^{-1}(\Lambda^\circ)$ with $f = p(\ell)$. Then one has*

$$\dim(\mathfrak{h}/\mathfrak{h} \cap \mathfrak{g}(\ell)) = \frac{1}{2}\bigl(\dim(\mathfrak{g}/\mathfrak{g}(\ell)) - \dim(\mathfrak{n}/\mathfrak{n}(f))\bigr).$$

Proof This amounts to showing that $\mathfrak{h}_r = \bigcap_{\nu=1}^r \ker\alpha_\nu = \mathfrak{h} \cap \mathfrak{g}(\ell)$ holds for each $\ell \in \Omega \cap p^{-1}(\Lambda^\circ)$. It is already clear that for such ℓ, $\mathfrak{h}_r \subset \mathfrak{h} \cap \mathfrak{g}(\ell)$. On the other hand, if $A \in \mathfrak{h} \cap \mathfrak{g}(\ell)$, then for each $1 \leq \nu \leq r$, $\alpha_\nu(A)\ell(Z_{i_{h_\nu}}) = -\ell[A, Z_{i_{h_\nu}}] = 0$. From Proposition 1.6(iii), we have $\ell(Z_{i_{h_\nu}}) \neq 0$, hence $A \in \ker\alpha_\nu$, and the equation above is proved. Now

$$\dim \mathfrak{h}/\mathfrak{h} \cap \mathfrak{g}(\ell)) = r = d - d^\circ = \frac{1}{2}\bigl(\dim(\mathfrak{g}/\mathfrak{g}(\ell)) - \dim(\mathfrak{n}/\mathfrak{n}(f))\bigr). \qquad ■$$

Corollary 1.8 *With the hypothesis of Proposition* 1.6*, we have*

$$p(\Omega) \cap \Lambda^\circ = \{f \in \Lambda^\circ \mid f(Z_{i_{h_\nu}}) \neq 0, \text{ holds for all } 1 \leq \nu \leq r\}$$

and

$$p(\Lambda) = \{f \in \Lambda^\circ \mid |f(Z_{i_{h_\nu}})| = 1, \text{ holds for all } 1 \leq \nu \leq r\}.$$

https://doi.org/10.4153/CJM-2007-039-0 Published online by Cambridge University Press

Proof Recall that $\Omega = \{\ell \in \mathfrak{g}^* \mid \mathbf{Pf}_{\mathbf{e},\mathbf{j}}(\ell) \neq 0\}$, and that

$$\Omega^\circ = \{f \in \mathfrak{n}^* \mid \mathbf{Pf}_{\mathbf{e}^\circ,\mathbf{j}^\circ}(f) \neq 0\}.$$

By Proposition 1.6 part (iii), if $f = p(\ell) \in \Lambda^\circ$, then $\mathbf{Pf}_{\mathbf{e},\mathbf{j}}(\ell) = R(f)\mathbf{Pf}_{\mathbf{e}^\circ,\mathbf{j}^\circ}(f)$ where $R(f)$ is a product of the factors $f(Z_{i_{h_\nu}}), 1 \leq \nu \leq r$. These observations mean that $f \in p(\Omega) \cap \Lambda^\circ$ if and only if $f \in \Lambda^\circ$ and $R(f) \neq 0$. The first equation above follows.

As for the second, set

$$V = \{\ell \in \mathfrak{g}^* \mid \ell(Z_j) = 0 \text{ for all } j \in \mathbf{e} - \varphi\},$$
$$V^\circ = \{f \in \mathfrak{n}^* \mid f(Z_j) = 0 \text{ for all } j \in \mathbf{e}^\circ\}.$$

Observe that, by virtue of preceding results, we have $p(V) = V^\circ$ and $p(\Omega) \cap V^\circ = p(\Omega) \cap \Lambda^\circ$. Now from Proposition 1.6(iv) and the definition of the cross-section Λ, we have

$$\Lambda = \{\ell \in V \cap \Omega \mid |q_{i_{h_\nu}}(\ell)| = 1, \text{ holds for all } 1 \leq \nu \leq r\}.$$

Let $f \in p(\Lambda)$, $f = p(\ell)$ for some $\ell \in \Lambda$. Then $f \in p(V) = V^\circ$, and $f \in p(\Omega) \subset \Omega^\circ$, so $f \in \Lambda^\circ$. But now an examination of the definition of q_j together with the observation that $X_{h_\nu}(\ell) = A_\nu$ gives $q_{i_{h_\nu}}(\ell)^{-1} = \ell(Z_{i_{h_\nu}})$. Hence f belongs to the right-hand side of the above equation.

On the other hand, let $f \in \Lambda^\circ$ with $|f(Z_{i_{h_\nu}})| = 1, 1 \leq \nu \leq r$. Let $\ell \in p^{-1}(f) \cap V$. By definition of Ω, we have $\ell \in \Omega \cap V$, and $|\ell(Z_{i_{h_\nu}})| = 1, 1 \leq \nu \leq r$. Hence $\ell \in \Lambda$ and $f \in p(\Lambda)$. ■

2 The Wavelet Transform

In this section, we apply the algebraic constructions of Section 1 in order to address the question of admissibility. Denote by $\mathrm{Irr}(N)$ the Borel space of irreducible unitary representations of N, and by $\widehat{N}$ the Borel space of unitary equivalence classes in $\mathrm{Irr}(N)$. Let $\kappa^\circ \colon \mathfrak{n}^*/N \to \widehat{N}$ be the canonical Kirillov correspondence. With the constructions of Section 1 in place, we associate to each linear functional $f \in \mathfrak{n}^*$ a specific irreducible representation π_f whose equivalence class is $\kappa^\circ(Nf)$, as follows. First of all, the basis $\{Z_1, Z_2, \ldots, Z_n\}$ provides us with global coordinates on N via the exponential mapping, and Lebesgue measure becomes Haar measure on N: $d(\exp X) = dX, X \in \mathfrak{n}$. We denote this measure by dx. Next, we partition $\mathfrak{n}^*$ by the fine stratification, and let $\Omega^\circ = \Omega_{\mathbf{e}^\circ,\mathbf{j}^\circ}$ be the fine layer containing f. Then $\mathfrak{p}(f) = \sum_j \mathfrak{n}_j^f \cap \mathfrak{n}_j = \mathfrak{p}_d(f)$ is a subalgebra of $\mathfrak{n}$ with the property that $\mathfrak{p}(f)^f = \mathfrak{p}(f)$. Rearranging the sequence $\mathbf{j}^\circ$ in *increasing* order $\{j_1 < j_2 < \cdots < j_d\}$, we have that

$$(s_1, s_2, \ldots, s_d) \mapsto \exp(s_d Z_{j_d}) \exp(s_{d-1} Z_{j_{d-1}}) \cdots \exp(s_1 Z_{j_1}) P(f)$$

is a global chart for $N/P(f)$, and Lebesgue measure on $\mathbf{R}^d$ is thereby carried to an invariant measure on $N/P(f)$. Let χ_f be the unitary character on $P(f) = \exp \mathfrak{p}(f)$ whose differential is if. Then the unitary representation π_f, induced from $P(f)$ to N by χ_f, is irreducible. Denoting by $[\pi_f]$ its equivalence class in $\widehat{N}$, one has

$\kappa^\circ(Nf) = [\pi_f]$. We denote the Hilbert space in which π_f acts by $\mathcal{H}_f$. Note that the map $J_f\colon \mathcal{H}_f \to L^2(\mathbf{R}^d)$ defined by

$$J_f\psi(s) = \psi(\exp(s_1 Z_{j_1})\exp(s_2 Z_{j_2})\cdots\exp(s_d Z_{j_d}))$$

is an isometric isomorphism.

An algorithm for determination of the Plancherel measure class and the Plancherel formula for nilpotent groups in terms of the orbit method is given in [20]. A similar result for the class of exponential solvable groups is proved in [4], and it is this version, specialized to the nilpotent case, that we use here.

The procedure is implemented as follows. Recall that we have a cross-section Λ° for the coadjoint orbits in Ω° and that $\Lambda^\circ = \Omega^\circ \cap V^\circ$ where $V^\circ = \{f \in \mathfrak{n}^* \mid f(Z_j) = 0$ holds for all $j \in \mathbf{e}^\circ\}$. Let Ω be the minimal fine layer in $\mathfrak{g}^*$, and set $\Lambda^1 = \Lambda^\circ \cap p(\Omega)$. Recall that we have written $\{1, 2, \ldots, n\} - \mathbf{e}^\circ = \{u_1 < u_2 < \cdots < u_c\}$, where $c = n - 2d$. Via the identification $f \to (f(Z_{u_1}), f(Z_{u_2}), \ldots, f(Z_{u_c}))$, we regard Λ^1 not only as a subset of $\mathfrak{n}^*$, but also as a (dense open) subset of $\mathbf{R}^c$, and we shall henceforth use the notation $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_c)$ for elements of Λ^1. Accordingly, we shall write π_λ for the irreducible representation corresponding to λ as constructed above; note that each of the Hilbert spaces $\mathcal{H}_\lambda$ is isomorphic with $L^2(\mathbf{R}^d)$ via the map J_λ. Also, with $\mathbf{Pf} = \mathbf{Pf}_{\mathbf{e}^\circ, d^\circ}$ the Pfaffian polynomial on $\mathfrak{n}^*$ as defined in Section 1, we shall write $\mathbf{Pf}(\lambda)$, $\lambda \in \Lambda^1$. At the same time we let $d\lambda$ denote Lebesgue measure on Λ^1. We describe the Fourier transform and Plancherel formula in these terms. For each $\lambda \in \Lambda^1$ and $\psi \in L^1(N) \cap L^2(N)$, set $F(\psi)(\lambda) = \int_N \psi(x)\pi_\lambda(x)\,dx$. Then $F(\psi)(\lambda)$ belongs to the space $\mathcal{H}_\lambda \otimes \overline{\mathcal{H}}_\lambda$ of Hilbert–Schmidt operators on $\mathcal{H}_\lambda$. Now let μ be the Borel measure on Λ^1 defined by

$$d\mu(\lambda) = \frac{1}{(2\pi)^{n+d}} |\mathbf{Pf}(\lambda)|\, d\lambda.$$

Then $\{\mathcal{H}_\lambda \otimes \overline{\mathcal{H}}_\lambda\}_{\lambda \in \Lambda^1}$ is a measurable field of Hilbert spaces and we set

$$\mathbf{H} = \int_{\Lambda^1}^{\oplus} \mathcal{H}_\lambda \otimes \overline{\mathcal{H}}_\lambda\, d\mu(\lambda).$$

Now $\lambda \to \pi_\lambda$ is a Borel function from Λ^1 to $\mathrm{Irr}(N)$, $F(\psi)$ belongs to $\mathbf{H}$, and the map $F\colon L^1(N) \cap L^2(N) \to \mathbf{H}$ as defined above extends to all of $L^2(N)$ as a unitary isomorphism.

With the Fourier transform on N in place, we turn to the quasiregular representation of G in $L^2(N)$. From now on we shall use the letter f to refer to elements of $L^2(N)$. Let $\delta\colon H \to \mathbf{R}_+^*$ be the character $\delta(h) = \delta_1(h)\delta_2(h)\cdots\delta_n(h)$, and let G have the Haar measure $d\nu_G(xh) = dx\delta(h)^{-1}d\nu_H(h)$. Define the unitary representation $\tau\colon G \to \mathcal{U}(L^2(N))$ as follows. For $f \in L^2(N)$, set

$$(\tau(h)f)(x_0) = f(h^{-1}x_0h)\delta(h)^{-1/2}, \quad h \in H$$

$$(\tau(x)f)(x_0) = f(x^{-1}x_0), \quad x \in N.$$

Recall that τ is isomorphic with the representation of G induced from H by the trivial character. Fix $\psi \in L^2(N)$ and for each $f \in L^2(N)$, denote by $m_{f,\psi}$ the bounded continuous function on G defined by $m_{f,\psi}(s) = \langle f, \tau(s)\psi\rangle_{L^2(N)}$, $s \in G$.

Recall that ψ is admissible for τ if $m_{f,\psi}$ is square-integrable for each $f \in L^2(N)$ and $\|m_{f,\psi}\|_{L^2(G)} = \|f\|_{L^2(N)}$. Following [12], we search for admissible vectors by means of the Fourier transform on $L^2(N)$. For $f \in L^2(N)$ set $\widehat{f}(\lambda) = F(f)(\lambda)$, $\lambda \in \Lambda^1$ and let $\widehat{\tau}(s) = F \circ \tau(s) \circ F^{-1}$, $s \in G$. The representation $\widehat{\tau}$ is described in terms of the usual action of H on $\widehat{N}$. Specifically, for $\pi \in \operatorname{Irr}(N)$ and $h \in H$, set $(h \cdot \pi)(x) = \pi(h^{-1}xh)$, $x \in N$. For each $h \in H$, the representation $h \cdot \pi_f$ is equivalent to π_{hf} via the intertwining operator $C(h, f)\colon \mathcal{H}_f \to \mathcal{H}_{hf}$ defined by

$$(C(h, f)\phi)(x) = \phi(h^{-1}xh)\delta_{\mathfrak{j}^\circ}(h)^{-1/2}, \quad \phi \in \mathcal{H}_f,$$

where $\delta_{\mathfrak{j}^\circ}(h) = \prod_{j\in\mathfrak{j}^\circ} \delta_j(h)$. Passing to the quotient $\widehat{N}$ and applying the orbit method, one sees that the stabilizer $H_{[\pi_f]}$ of $[\pi_f]$ in H coincides with the analytic subgroup $\{h \in H \mid hf \in Nf\} = \exp(\mathfrak{h} \cap (\mathfrak{n} + \mathfrak{n}^f))$. For $\lambda \in \Lambda^1$, since the action of H is already diagonalized, we have $h\lambda \in \Lambda^1$, and since Λ^1 is an orbital cross-section, we have that $H_{[\pi_\lambda]} = H_\lambda = \exp(\mathfrak{h} \cap \mathfrak{g}^\lambda) = \exp(\mathfrak{h}_r) = H_r$ holds for each $\lambda \in \Lambda^1$. For $h \in H$ and $\lambda \in \Lambda^1$, let $D(h, \lambda)\colon \mathcal{B}(\mathcal{H}_\lambda) \to \mathcal{B}(\mathcal{H}_{h\lambda})$ be defined by

$$D(h, \lambda)(T) = C(h, \lambda) \circ T \circ C(h, \lambda)^{-1}.$$

Adapting the result [12, Proposition 2.1] to the present context, we have the following description of $\widehat{\tau}$ in terms of the preceding orbital parameters for the Fourier transform.

Proposition 2.1 *Let $f \in L^2(N)$, $h \in H$, $x \in N$, $\lambda \in \Lambda^1$. One has*

(i) $(\widehat{\tau}(h)\widehat{f})(\lambda) = D(h, h^{-1}\lambda)(\widehat{f}(h^{-1}\lambda))\delta(h)^{1/2}$;

(ii) $(\widehat{\tau}(x)\widehat{f})(\lambda) = \pi_\lambda(x) \circ \widehat{f}(\lambda)$.

We observe that [12, Proposition 2.2] also restates in the same way.

Proposition 2.2 *For each $h \in H$, $d\mu(h\lambda) = \delta(h)^{-1}d\mu(\lambda)$.*

An easy calculation shows that for each $x \in N$ and $h \in H$, one has

$$m_{f,\psi}(xh) = (f * (\tau(h)\psi)^*)(x)$$

where $\psi^*(x) = \overline{\psi}(x^{-1})$. We then apply the Fourier transform:

(2.1)

$$\begin{aligned}
&\int_G |m_{f,\psi}|^2\, d\nu_G \\
&\quad = \int_H \int_N |(f * (\tau(h)\psi)^*)(x)|^2\, dx\, \delta(h)^{-1} d\nu_H(h) \\
&\quad = \int_H \int_{\Lambda^1} \|\widehat{f}(\lambda) \circ (\widehat{\tau}(h)\widehat{\psi})(\lambda)^*\|_{HS}^2\, d\mu(\lambda)\delta(h)^{-1} d\nu_H(h) \\
&\quad = \int_{\Lambda^1} \Big(\int_H \|\widehat{f}(\lambda) \circ C(h, h^{-1}\lambda)\widehat{\psi}(h^{-1}\lambda)^* C(h, h^{-1}\lambda)^{-1}\|_{HS}^2\, d\nu_H(h) \Big)\, d\mu(\lambda).
\end{aligned}$$

https://doi.org/10.4153/CJM-2007-039-0 Published online by Cambridge University Press

If N is abelian, so that the Fourier transform is scalar-valued, then

$$\left\| \widehat{f}(\lambda) \circ C(h, h^{-1}\lambda)\widehat{\psi}(h^{-1}\lambda)^* C(h, h^{-1}\lambda)^{-1} \right\|_{HS}^2 = |\widehat{f}(\lambda)|^2 \, |\widehat{\psi}(h^{-1}\lambda)|^2$$

and it becomes apparent from (2.1) that a necessary condition for τ-admissibility is that H_λ be compact for μ-a.e. λ. Note that in the context of this paper that means simply that $\mathfrak{h}_r = (0)$. Now for the class of groups considered here, it is reasonable to expect that the condition $\mathfrak{h}_r = (0)$ is necessary for the existence of τ-admissible vectors even when N is not abelian, but that question remains open. Therefore, for the remainder of this paper, we shall just make the assumption that $\mathfrak{h}_r = (0)$. We observe that, if N is not abelian, then this means that the irreducible decomposition of τ will have infinite multiplicity: we have $r = \dim(H) = \dim H\lambda$ holds for all $\lambda \in \Lambda^1$ and (since $\mathfrak{h}_r = (0)$), it follows that the generic dimension of H-orbits in $\mathfrak{h}^\perp$ is r. Now Corollary 1.7 says that $r = d - d^\circ$, where $2d$ is the generic dimension of G orbits (that meet $\mathfrak{h}^\perp$) and $2d^\circ$ is the generic dimension of N orbits in $\mathfrak{n}^*$. Combining these observations with the results of [15, 16], we have that τ has finite multiplicity if and only if $r = d$, if and only if N is abelian.

Recall also that in the case where N is abelian, (2.1) is the starting point for proving the Caldéron condition for admissibility (for quite general groups H): ψ is admissible for τ if and only if $\int_H |\psi(h^{-1}\lambda)|^2 \, d\nu_H(h) = 1$ holds for μ-a.e. λ [22, Theorem 2.1]. We shall see below that this result can be generalized to the case where N is not abelian: we shall write τ as a direct sum of multiplicity-free subrepresentations τ^β so that a Caldéron condition for τ^β-admissibility holds.

We begin by describing the action of H on Λ^1 explicitly. Recall that we have chosen the ordered basis $\{A_\nu \mid 1 \le \nu \le r\}$ for $\mathfrak{h}$ in conjuction with a sequence $\{1 \le u_{a_1} < u_{a_2} < \cdots < u_{a_r} \le n\}$ of indices corresponding to a minimal spanning set of roots, as defined in Section 1. In particular for each $1 \le \nu, w \le r$, $\gamma_{u_{a_\nu}}(A_w) = \delta_{\nu w}$, and if $a < a_w$, $\gamma_{u_a}(A_w) = 0$. Write

$$Q(t, \lambda) = \exp(t_1 A_1) \exp(t_2 A_2) \cdots \exp(t_r A_r)\lambda, \quad t \in \mathbf{R}^r, \lambda \in \Lambda^1.$$

Then for each $\lambda \in \Lambda^1$, $t \to Q(t, \lambda)$ is a diffeomorphism of $\mathbf{R}^r$ with $H\lambda$. The following notation will be helpful in the descriptions that follow: for each $1 \le a \le c$, if $a < a_1$, set $h^a = 1 \in H$, and for $a \ge a_1$, let $h^a(t) = \exp(t_1 A_1) \exp(t_2 A_2) \cdots \exp(t_{w(a)} A_{w(a)})$ where $w(a) = \max\{1 \le w \le r \mid a_w \le a\}$.

For each $1 \le a \le c$ we see that $Q_a(t, \lambda) = \delta(h^a(t))^{-1}\lambda_a$. More explicitly, if we set

$$\gamma_{a,\nu} = \gamma_{u_a}(A_\nu), \quad 1 \le a \le c, 1 \le \nu \le r,$$

then for $a = a_\nu$ we have $\delta(h^a(t))^{-1} = e^{t_\nu}$, while if $a \ne a_\nu$, $1 \le \nu \le r$, then

$$\delta(h^a(t))^{-1} = e^{\gamma_{a,1}t_1 + \gamma_{a,2}t_2 + \cdots + \gamma_{a,w(a)}t_{w(a)}}.$$

Hence

$$Q_a(t, \lambda) = \begin{cases} e^{t_\nu}\lambda_a & \text{if } a = a_\nu, \\ e^{\gamma_{a,1}t_1 + \gamma_{a,2}t_2 + \cdots + \gamma_{a,w(a)}t_{w(a)}}\lambda_a & \text{if } a \ne a_\nu. \end{cases}$$

https://doi.org/10.4153/CJM-2007-039-0 Published online by Cambridge University Press

For $1 \le \nu \le r$, set

$$z_\nu = e^{t_\nu}|\lambda_{a_\nu}|, \quad \epsilon_\nu = \text{ sign}(\lambda_{a_\nu}).$$

Making these substitutions into the function Q, we obtain a function $P(z, \lambda)$ each coordinate of which has the form

$$P_a(z,\lambda) = \begin{cases} z_\nu \epsilon_\nu & \text{if } a = a_\nu, \\ \Big(\frac{z_1}{|\lambda_{a_1}|}\Big)^{\gamma_{a,1}} \Big(\frac{z_2}{|\lambda_{a_2}|}\Big)^{\gamma_{a,2}} \cdots \Big(\frac{z_{w(a)}}{|\lambda_{a_{w(a)}}|}\Big)^{\gamma_{a,w(a)}} \lambda_a & \text{if } a \neq a_\nu. \end{cases}$$

The function P is easily seen to have the following properties.

(i) For each $\lambda \in \Lambda^1$, $P(\,\cdot\,, \lambda)$ maps $(\mathbf{R}_+^*)^r$ diffeomorphically onto $H\lambda$.

(ii) For each fixed $(z_1, z_2, \dots, z_r) \in (\mathbf{R}_+^*)^r$, $P(z_1, z_2, \dots, z_r, \,\cdot\,)$ maps Λ^1 into Λ^1 and is H-invariant.

We set $\Sigma = \{P(1, 1, \dots, 1, \lambda) \mid \lambda \in \Lambda^1\}$; it is easily seen that Σ is a submanifold of Λ^1 having dimension $c - r$, and that Σ meets the H-orbit of λ at the single point $P(1, 1, \dots, 1, \lambda)$. In fact, we have the following.

Lemma 2.3 *Let Λ be the cross-section in Ω for the G-orbits in Ω. If $\mathfrak{h}_r = (0)$, then $p|_\Lambda$ is a bijection of Λ onto Σ.*

Proof By part (b) of Proposition 1.6 and our assumption that $\mathfrak{h}_r = (0)$, we have $\Lambda \subset \mathfrak{h}^\perp = \{\ell \in \mathfrak{g}^* \mid \ell(\mathfrak{h}) = \{0\}\}$, and hence $p|_\Lambda$ is a bijection. By Corollary 1.8, we have $p(\Lambda) = \{\lambda \in \Lambda^1 \mid |\lambda_{a_\nu}| = 1, 1 \le \nu \le r\}$. An examination of the map $P(z, \lambda)$ above shows that $\Sigma = P(1, \lambda) \subset p(\Lambda)$ and that for each $\ell \in \Lambda$ with $\lambda = p(\ell)$, $\lambda = P(1, \lambda) \in \Sigma$. This completes the proof. ■

For each $\epsilon \in \{-1, 1\}^r$, set $\Lambda^1_\epsilon = \{\lambda \in \Lambda^1 \mid \text{ sign}(\lambda_{a_\nu}) = \epsilon_\nu, 1 \le \nu \le r\}$ and $\Sigma_\epsilon = \Sigma \cap \Lambda^1_\epsilon$. In the event that $r = c$, then for each ϵ, Σ_ϵ is the single point $(\epsilon_1, \epsilon_2, \dots, \epsilon_c)$. In this case we let $d\sigma$ be the counting measure on Σ, multiplied by $1/(2\pi)^{n+d}$. Otherwise write $\{1, 2, \dots, c\} - \{a_\nu \mid 1 \le \nu \le r\} = \{b_1 < b_2 < \cdots < b_q\}$; set $\sigma_w = \lambda_{b_w}, 1 \le w \le q$. Then each set Σ_ϵ is identified with an open subset of $\mathbf{R}^q$, and we thereby transfer Lebesgue measure to each Σ_ϵ. The resulting measure on Σ, including the multiple $1/(2\pi)^{n+d}$, will be denoted by $d\sigma = d\sigma_1 d\sigma_2 \cdots d\sigma_q$. At the same time we identify H with $(\mathbf{R}_+^*)^r$, so that

$$\exp(t_1 A_1)\exp(t_2 A_2)\cdots\exp(t_r A_r) = (e^{t_1}, e^{t_2}, \dots, e^{t_r}) = (z_1, z_2, \dots, z_r).$$

The natural Haar measure on H is then

$$d\nu_H(z_1, z_2, \dots, z_r) = \frac{dz_1 dz_2 \cdots dz_r}{z_1 z_2 \cdots z_r}.$$

By virtue of this identification and by restricting λ to Σ, the function $P(z, \lambda)$ yields a map from $H \times \Sigma$ to Λ^1. We claim that $P(z, \sigma) = z\sigma$. Observe that for $a \neq a_\nu, 1 \le \nu \le r$, we have

$$\delta_{u_a}(z)^{-1} = z_1^{\gamma_{a,1}} z_2^{\gamma_{a,2}} \cdots z_{w(a)}^{\gamma_{a,w(a)}}, \quad z = (z_1, z_2, \dots z_r) \in H.$$

https://doi.org/10.4153/CJM-2007-039-0 Published online by Cambridge University Press

Since $|\lambda_{a_v}| = 1$ for $\lambda \in \Sigma$, $P(z,\sigma)$ is defined coordinate-wise on $H \times \Sigma_\epsilon$ by

$$P_a(z,\sigma) = \begin{cases} z_v\epsilon_v & \text{if } a = a_v, \\ \delta_{u_{b_w}}(z)^{-1}\sigma_w & \text{if } a = b_w. \end{cases}$$

The claim follows. It is clear that P is a diffeomorphism and that for any non-negative measurable function ϕ on Λ^1,

$$\int_{\Lambda^1} \phi(\lambda)\, d\lambda = \int_\Sigma \int_H \phi(z\sigma)\delta_{u_{b_1}}(z)^{-1}\delta_{u_{b_2}}(z)^{-1}\cdots\delta_{u_{b_q}}(z)^{-1}\, dz d\sigma.$$

From now on we identify Λ^1 with $H \times \Sigma$ as above. Now set

$$\delta_{\mathbf{e}^\circ}(z) = \prod_{j \in \mathbf{e}^\circ} \delta_j(z), \quad z \in H.$$

Lemma 2.4 *For each $\lambda = z\sigma \in \Lambda^1$, one has $\mathbf{Pf}(z\sigma) = \delta_{\mathbf{e}^\circ}(z)^{-1}\mathbf{Pf}(\sigma)$. Moreover, the formula*

$$\int_{\Lambda^1} \phi(\lambda)\, \mu(\lambda) = \int_\Sigma \int_H \phi(z\sigma)\delta(z)^{-1}\, d\nu_H(z)|\mathbf{Pf}(\sigma)| d\sigma,$$

holds for any non-negative measurable function ϕ on Λ^1.

Proof Fix $\lambda = z\sigma \in \Lambda^1$, let $\ell_\sigma \in \mathfrak{g}^*$ such that $p(\ell_\sigma) = \sigma$ and set $\ell = z\ell_\sigma \in \mathfrak{g}^*$. Let $\delta_{\mathbf{e}} = \prod_{j\in\mathbf{e}} \delta_j$, where $\mathbf{e}$ is the jump set corresponding to the minimal layer in $\mathfrak{g}^*$. By [2, Lemma 1.6], $\mathbf{Pf}_{\mathbf{e},\mathbf{j}}(\ell) = \delta_{\mathbf{e}}(z)^{-1}\mathbf{Pf}_{\mathbf{e},\mathbf{j}}(\ell_\sigma)$. But part (a) of Proposition 1.6, together with our choice of basis of $\mathfrak{h}$ dual to the minimal spanning set of roots, insures that

$$\delta_{\mathbf{e}}(z)^{-1} = z_1 z_2 \cdots z_r \delta_{\mathbf{e}^\circ}(z)^{-1}.$$

On the other hand, observing that $p(\ell) = z\sigma$, part (c) of Proposition 1.6 gives

$$\mathbf{Pf}_{\mathbf{e},\mathbf{j}}(\ell) = \prod_{v=1}^{r} \ell(Z_{u_{a_r}})\mathbf{Pf}_{\mathbf{e}^\circ,\mathbf{j}^\circ}(z\sigma) = z_1 z_2 \cdots z_r \mathbf{Pf}_{\mathbf{e}^\circ,\mathbf{j}^\circ}(z\sigma).$$

Similarly $\mathbf{Pf}_{\mathbf{e},\mathbf{j}}(\ell_\sigma) = \mathbf{Pf}_{\mathbf{e}^\circ,\mathbf{j}^\circ}(\sigma)$, and hence

$$\begin{aligned} z_1 z_2 \cdots z_r \mathbf{Pf}_{\mathbf{e}^\circ,\mathbf{j}^\circ}(z\sigma) = \mathbf{Pf}_{\mathbf{e},\mathbf{j}}(\ell) &= z_1 z_2 \cdots z_r \delta_{\mathbf{e}^\circ}(z)^{-1}\mathbf{Pf}_{\mathbf{e},\mathbf{j}}(\ell_\sigma) \\ &= z_1 z_2 \cdots z_r \delta_{\mathbf{e}^\circ}(z)^{-1}\mathbf{Pf}_{\mathbf{e}^\circ,\mathbf{j}^\circ}(\sigma). \end{aligned}$$

The first part of the lemma is proved.

As for the second part, write $\delta_{u_b}(z) = \prod_{w=1}^{q} \delta_{u_{b_w}}(z)$; again by virtue of our choice of basis for $\mathfrak{h}$, we have

$$\delta(z)^{-1} = z_1 z_2 \cdots z_r\, \delta_{\mathbf{e}^\circ}(z)^{-1}\delta_{u_b}(z)^{-1}.$$

Hence

$$d\mu(\lambda) = |\mathbf{Pf}(z\sigma)|\delta_{u_b}(z)^{-1}dzd\sigma = z_1z_2\cdots z_r\delta_{\mathbf{e}^\circ}(z)^{-1}\delta_{u_b}(z)^{-1}d\nu_H(z)|\mathbf{Pf}(\sigma)|d\sigma$$
$$= \delta(z)^{-1}d\nu_H(z)|\mathbf{Pf}(\sigma)|\, d\sigma. \qquad \blacksquare$$

Fix an orthonormal basis $\{e^\beta \mid \beta \in B\}$ for $L^2(\mathbf{R}^d)$, (where B is some index set) and for each $\lambda = z\sigma \in H\sigma$, set $e^\beta_\lambda = C(z,\sigma)J_\sigma^{-1}e^\beta$, so that $\{e^\beta_\lambda\}_\beta$ is an orthonormal basis of $\mathcal{H}_\lambda$. For each $\lambda \in \Lambda^1$ and each basis index β, we have the subspace $\mathcal{H}_\lambda \otimes e^\beta_\lambda = \{T \in \mathcal{H}_\lambda \otimes \overline{\mathcal{H}}_\lambda \mid \text{Image}\,(T^*) \subset \mathbf{C}\overline{e^\beta_\lambda}\}$. Recall that $\mathcal{H}_\lambda \otimes e^\beta_\lambda$ is the set of maps of the form $v \mapsto \langle v, e^\beta_\lambda\rangle w$ where $w \in \mathcal{H}_\lambda$, and the obvious map $\mathcal{H}_\lambda \to \mathcal{H}_\lambda \otimes e^\beta_\lambda$ is an isometric isomorphism. For each basis index β, set $\mathbf{H}^\beta = \int^\oplus_{\Lambda^1} \mathcal{H}_\lambda \otimes e^\beta_\lambda\, d\mu(\lambda)$, so that $\mathbf{H} = \oplus_\beta \mathbf{H}^\beta$. Setting $\mathbf{K} = \int^\oplus_{\Lambda^1} \mathcal{H}_\lambda\, d\mu(\lambda)$, we have an obvious isometric isomorphism of $\mathbf{K}$ onto each $\mathbf{H}^\beta$: $w = \{w(\lambda)\}_{\lambda\in\Lambda^1} \in \mathbf{K}$ corresponds to the element

$$\{w(\lambda) \otimes e^\beta_\lambda\}_{\lambda\in\Lambda^1} \in \mathbf{H}^\beta.$$

For any element $g = \{g(\lambda)\}_{\lambda\in\Lambda^1} = \{w(\lambda) \otimes e^\beta_\lambda\}_{\lambda\in\Lambda^1}$ of $\mathbf{H}^\beta$, one calculates that $(\widehat{\tau}(x)g)(\lambda) = \pi_\lambda(x)w(\lambda) \otimes e^\beta_\lambda$ for $x \in N$ and

$$(\widehat{\tau}(z)g)(\lambda) = C(z, z^{-1}\lambda)w(z^{-1}\lambda) \otimes e^\beta_\lambda\ \ \delta(z)^{1/2}, \quad z \in H.$$

Thus the subspace $\mathbf{H}^\beta$ of $\mathbf{H}$ is $\widehat{\tau}$-invariant, and its inverse Fourier image $L^2(N)^\beta = F^{-1}(\mathbf{H}^\beta)$ is τ-invariant. Accordingly, we write $\widehat{\tau} = \bigoplus_\beta \widehat{\tau}^\beta$ and $\tau = \bigoplus_\beta \tau^\beta$. Now for each basis index β, the preceding decomposition of the Plancherel measure μ gives a direct integral decomposition of $\mathbf{H}^\beta$:

$$\mathbf{H}^\beta \cong \int^\oplus_\Sigma \mathbf{H}^\beta_\sigma |\mathbf{Pf}(\sigma)|\, d\sigma, \tag{2.2}$$

where $\mathbf{H}^\beta_\sigma = \int^\oplus_H \mathcal{H}_{z\sigma} \otimes e^\beta_{z\sigma}\delta(z)^{-1}\, d\nu_H(z)$.

For the moment, fix $\sigma \in \Sigma$ and a basis index β. Define $\widehat{\tau}^\beta_\sigma \colon G \to \mathcal{U}(\mathbf{H}^\beta_\sigma)$ by the same formula as in Proposition 2.1 above: for $g = \{g(z)\}_{z\in H} \in \mathbf{H}^\beta_\sigma$ and $z_0 \in H$ define

(i) $\quad (\widehat{\tau}^\beta_\sigma(z)g(z_0) = D(z, z^{-1}z_0\sigma)(g(z^{-1}z_0))\delta(z)^{1/2}, \quad z \in H;$
(ii) $\quad (\widehat{\tau}^\beta_\sigma(x)g)(z_0) = \pi_{z_0\sigma}(x) \circ g(z_0), \quad x \in N.$

Proposition 2.5 *For each $\sigma \in \Sigma$ and for each β, $\widehat{\tau}^\beta_\sigma$ is unitarily isomorphic with $\tilde{\pi}_\sigma = \mathrm{ind}^G_N(\pi_\sigma)$ (and hence is irreducible.)*

Proof Fix $\sigma \in \Sigma$ and let $\mathcal{L}$ be the Hilbert space of $\tilde{\pi}_\sigma$. For $w \in \mathcal{L}$, $\lambda = z\sigma \in \Lambda^1$, set $(Tw)(z) = C(z,\sigma)(w(z)) \otimes e^\beta_{z\sigma}\delta(z)^{1/2}$. Then

$$\int_H \|(Tw)(z)\|^2_{HS}\delta(z)^{-1}\, d\nu_H(z) = \int_H \|C(z,\sigma)w(z) \otimes e^\beta_{z\sigma}\|^2_{HS}\, d\nu_H(z)$$
$$= \int_H \|w(z)\|^2_{\mathcal{H}_\sigma}\, d\nu_H(z) = \|w\|^2_{\mathcal{L}}.$$

https://doi.org/10.4153/CJM-2007-039-0 Published online by Cambridge University Press

Hence T is a linear isometry from $\mathcal{L}$ into $\mathbf{H}_\sigma^\beta$. It is easily seen that T is invertible. We compute that

$$\begin{aligned}\widehat{\tau}_\sigma^\beta(z)(Tw)(z_0) &= \widehat{\tau}_\sigma^\beta(z)\big(C(z_0,\sigma)w(z_0)\otimes e^\beta_{z_0\sigma}\delta(z_0)^{1/2}\big)\\ &= C(z,z^{-1}z_0\sigma)C(z^{-1}z_0,\sigma)w(z^{-1}z_0)\otimes e^\beta_{z_0\sigma}\delta(z^{-1}z_0)^{1/2}\delta(z)^{1/2}\\ &= C(z_0,\sigma)w(z^{-1}z_0)\otimes e^\beta_{z_0\sigma}\delta(z_0)^{1/2}\\ &= T(\tilde{\pi}_\sigma(z)w)(z_0).\end{aligned}$$ ■

It follows that the natural isomorphism (2.2) intertwines the representation $\widehat{\tau}^\beta$ with the direct integral of the representations $\widehat{\tau}_\sigma^\beta$. To sum up the preceding, we have shown that the Fourier transform, together with the decomposition of the Plancherel measure μ, implements a natural decomposition of τ into unitary irreducibles:

$$\tau \cong \bigoplus_\beta \int_\Sigma^\oplus \widehat{\tau}_\sigma^\beta |\mathbf{Pf}(\sigma)|\,d\sigma.$$

Now fix an index β, and for $f \in L^2(N)^\beta$, write $\widehat{f}(\lambda) = w_f(\lambda)\otimes e^\beta_\lambda$, where $w_f \in \mathbf{K}$. Note that for each $\lambda \in \Lambda^1$, $\|\widehat{f}(\lambda)\|_{HS} = \|w_f(\lambda)\|_{\mathcal{H}_\lambda}$. In the sequel we shall often drop the cumbersome subscripts on norms indicating the Hilbert space, relying on context and other notation to affect the appropriate distinctions.

Fix $\psi \in L^2(N)^\beta$ and set $u = w_\psi$ so that $\widehat{\psi}(\lambda) = u(\lambda)\otimes e^\beta_\lambda$. One calculates that for each $\lambda \in \Lambda^1$ and $z \in H$,

$$\begin{aligned}\|\widehat{f}(\lambda)\circ(\widehat{\tau}(z)\widehat{\psi})(\lambda)^*\|^2 &= \|w_f(\lambda)\|^2\|u(z^{-1}\lambda)\|^2\,\delta(z)\\ &= \|\widehat{f}(\lambda)\|^2\|\widehat{\psi}(z^{-1}\lambda)\|^2\delta(z).\end{aligned} \tag{2.3}$$

Define $\Delta_\psi\colon \Lambda^1 \to [0,+\infty)$ by

$$\Delta_\psi(\lambda) = \int_H \|\widehat{\psi}(z^{-1}\lambda)\|^2\,d\nu_H(z) = \int_H \|\widehat{\psi}(z\sigma)\|^2\,d\nu_H(z).$$

Note that Δ_ψ is constant on H-orbits in Λ^1. Combining the equations (2.1) and (2.3), we get

$$\begin{aligned}\int_G |m_{f,\psi}|^2\,d\nu_G &= \int_H\int_{\Lambda^1} \|w_f(\lambda)\|^2\|u(z^{-1}\lambda)\|^2\,d\mu(\lambda)\,d\nu_H(z)\\ &= \int_H\int_{\Lambda^1} \|\widehat{f}(\lambda)\|^2\|\widehat{\psi}(z^{-1}\lambda)\|^2\,d\mu(\lambda)d\nu_H(z)\\ &= \int_{\Lambda^1} \|\widehat{f}(\lambda)\|^2\Big(\int_H \|\widehat{\psi}(z^{-1}\lambda)\|^2\,d\nu_H(z)\Big)\,d\mu(\lambda)\\ &= \int_{\Lambda^1} \|\widehat{f}(\lambda)\|^2\,\Delta_\psi(\lambda)\,d\mu(\lambda).\end{aligned}$$

So it is clear that if $\Delta_\psi(\lambda) = 1$ holds μ-a.e., then $m_{f,\psi}$ belongs to $L^2(G)$ and $\|m_{f,\psi}\| = \|f\|$, that is, ψ is admissible for τ^β. An easy adaptation of the argument in [22, Theorem 2.1] shows that the converse is true.

Proposition 2.6 *Let $\psi \in L^2(N)^\beta$. Then ψ is admissible for τ^β if and only if $\Delta_\psi(\lambda) = 1$ holds for μ-a.e. $\lambda \in \Lambda^1$.*

Proof The proof is already halfway done; to complete it, suppose that $\|m_{f,\psi}\| = \|f\|$ holds for all $f \in L^2(N)^\beta$. Fix $\lambda_0 \in \Lambda^1$. For $r > 0$, let $B_r(\lambda_0)$ be the ball about λ_0 of radius r, let $\chi_{B_r(\lambda_0)}$ be the characteristic function of the set $B_r(\lambda_0)$, and let $f = f_{\lambda_o,r} \in L^2(N)^\beta$ be defined by $\widehat{f}(\lambda) = \mu(B_r(\lambda_0))^{-1/2}\chi_{B_r(\lambda_0)}e^\beta_\lambda \otimes e^\beta_\lambda$. Then $\|f\|^2 = 1$, so from our assumption and the above calculation, we have

$$1 = \int_{\Lambda^1} \|\widehat{f}(\lambda)\|^2 \Delta_\psi(\lambda)\, d\mu(\lambda) = \frac{1}{\mu(B_r(\lambda_0))} \int_{B_r(\lambda_0)} \Delta_\psi(\lambda)\, d\mu(\lambda).$$

The result now follows from standard differentiability results. ∎

Remark 2.7 Let $\psi \in L^2(N)^\beta$ be admissible for τ^β and let $f \in L^2(N)^\beta$. Write $\widehat{\psi}(\lambda) = u(\lambda) \otimes e^\beta_\lambda$ and $\widehat{f}(\lambda) = w_f(\lambda) \otimes e^\beta_\lambda$ as above. Then

$$\widehat{\tau}(xz)\widehat{\psi}(\lambda) = \pi_\lambda(x) \circ C(z, z^{-1}\lambda)u(z^{-1}\lambda) \otimes e^\beta_\lambda\, \delta(z)^{1/2}$$

and

$$\begin{aligned} W_\psi(f)(xz) &= \langle \widehat{f}, \widehat{\tau}(xz)\widehat{\psi}\rangle \\ &= \int_{\Lambda^1} \langle \widehat{f}(\lambda), \widehat{\tau}(xz)\widehat{\psi}(\lambda)\rangle\, d\mu(\lambda) \\ &= \int_{\Lambda^1} \langle w_f(\lambda), \pi_\lambda(x) \circ C(z, z^{-1}\lambda)u(z^{-1}\lambda)\rangle\, d\mu(\lambda)\, \delta(z)^{1/2}. \end{aligned}$$

Hence if $L^\beta \colon \mathbf{K} \to \mathbf{H}^\beta$ is the canonical isomorphism and $\widehat{\psi'} = L^{\beta'} \circ (L^\beta)^{-1}\widehat{\psi}$, then $W_{\psi'} \circ L^{\beta'} = W_\psi \circ L^\beta$.

We now show how to construct admissible vectors for τ^β: suppose that G is not unimodular and that η is a unit vector in $L^2(H, \nu_H)$ which also happens to belong to $L^2(H, \delta^{-1}\nu_H)$. Since $\delta \neq 1$, we have $\delta(0, 0, \dots, z_v, \dots, 0) \neq 1$ for some v, $1 \le v \le r$. Write $\delta(0, 0, \dots, z_v, \dots, 0) = z_v^p,\ p \neq 0$. Assume that $q = \dim(\Sigma) > 0$, and for each $\epsilon \in \{-1, 1\}^r$, let s_ϵ be the identification map from Σ_ϵ onto an open subset of $\mathbf{R}^q$.

We choose a measurable function $\tilde{u} \colon \mathbf{R}^q \to (0, \infty)$ such that for any polynomial function $P(t)$ on $\mathbf{R}^q$, we have $\int_{\mathbf{R}^q} \tilde{u}(t)^p |P(t)|\, dt < \infty$. Define $u \colon \Sigma \to (0, \infty)$ by $u(\sigma) = \tilde{u}(s_\epsilon(\sigma)), \sigma \in \Sigma_\epsilon$. Then we have $\int_\Sigma u(\sigma)^p |\mathbf{Pf}(\sigma)|\, d\sigma < \infty$. Now for each pair of basis indices α and β, define $\psi = \psi^{\alpha,\beta}_{\eta,u} \in L^2(N)^\beta$ by

$$\widehat{\psi}(z\sigma) = \eta(z_1, z_2, \dots, z_{v-1}, z_v u(\sigma), z_{v+1}, \dots, z_r)e^\alpha_{z\sigma} \otimes e^\beta_{z\sigma}. \tag{2.4}$$

With the identification $\lambda = z\sigma$, it will be helpful to abuse notation slightly by writing

$$\eta(\lambda) = \eta(z\sigma) = \eta(z_1, z_2, \dots, z_{\nu-1}, z_\nu u(\sigma), z_{\nu+1}, \dots, z_r),$$

so that $\widehat{\psi}(\lambda) = \eta(\lambda) e_\lambda^\alpha \otimes e_\lambda^\beta$. Now we have that

$$\int_H \|\widehat{\psi}(z\sigma)\|^2 \, d\nu_H(z) = \int_H |\eta(z)|^2 \, d\nu_H(z) = 1$$

holds for all $\sigma \in \Sigma$, and the calculation

$$\begin{aligned}
\int_N |\psi(x)|^2 \, dx &= \int_{\Lambda^1} \|\widehat{\psi}(\lambda)\|^2 \, d\mu(\lambda) \\
&= \int_\Sigma \int_H \|\widehat{\psi}(z\sigma)\|^2 \delta(z)^{-1} \, d\nu_H(z) |\mathbf{Pf}(\sigma)| d\sigma \\
&= \int_\Sigma \int_H |\eta(z_1, z_2, \dots, z_{\nu-1}, z_\nu u(\sigma), z_{\nu+1}, \dots, z_r))|^2 \delta(z)^{-1} \, d\nu_H(z) |\mathbf{Pf}(\sigma)| d\sigma \\
&= \int_\Sigma \int_H |\eta(z)|^2 (\delta(z_1)\delta(z_1) \cdots \delta(z_{\nu-1}) \delta(u(\sigma)^{-1} z_\nu) \delta(z_{\nu+1}) \cdots \delta(z_r))^{-1} \\
&\qquad \times d\nu_H(z) |\mathbf{Pf}(\sigma)| d\sigma \\
&= \int_H |\eta(z)|^2 \delta(z)^{-1} \, d\nu_H(z) \int_\Sigma u(\sigma)^p |\mathbf{Pf}(\sigma)| \, d\sigma < \infty
\end{aligned}$$

shows that $\psi \in L^2(N)^\beta$. Hence by Proposition 2.6, ψ is admissible for τ^β.

Next, suppose that $\psi = \psi_{\eta,u}^{\alpha,\beta}$ and $\psi' = \psi_{\eta',u}^{\alpha',\beta'}$ are two such admissible vectors. For $f \in L^2(N)^\beta$ and $f' \in L^2(N)^{\beta'}$ we compute that

$$\begin{aligned}
\langle m_{f,\psi}(s), m_{f,\psi'}(s) \rangle_{L^2(G)} &= \int_G m_{f,\psi}(s) \overline{m_{f',\psi'}(s)} \, d\nu_G(s) \\
&= \int_H \int_N (f * (\tau(z)\psi)^*)(x) \overline{(f' * (\tau(z)\psi')^*)(x)} \, dx \, \delta(z)^{-1} d\nu_H(z) \\
&= \int_H \int_{\Lambda^1} \langle \widehat{f}(\lambda) \circ (\tau(z)\psi)\widehat{\ }(\lambda)^*, \widehat{f}'(\lambda) \circ (\tau(z)\psi')\widehat{\ }(\lambda)^* \rangle_{HS} \\
&\qquad \times d\mu(\lambda) \delta(z)^{-1} d\nu_H(z) \\
&= \int_H \int_{\Lambda^1} \operatorname{Trace}\big(\widehat{f}'(\lambda)^* \circ \widehat{f}(\lambda) \circ (\tau(z)\psi)\widehat{\ }(\lambda)^* \circ (\tau(z)\psi')\widehat{\ }(\lambda) \big) \\
&\qquad \times d\mu(\lambda) \, \delta(z)^{-1} d\nu_H(z).
\end{aligned}$$

Now one checks that

$$\begin{aligned}
&\widehat{f}'(\lambda)^* \circ \widehat{f}(\lambda) \circ (\tau(z)\psi)\widehat{\ }(\lambda)^* \circ (\tau(z)\psi')\widehat{\ }(\lambda) \\
&\qquad = \langle w_f(\lambda), w_{f'}(\lambda) \rangle \overline{\eta(z^{-1}\lambda)} \eta'(z^{-1}\lambda) \delta(z) e_\lambda^{\beta'} \otimes e_\lambda^{\beta'} \cdot \delta_{\alpha,\alpha'},
\end{aligned}$$

https://doi.org/10.4153/CJM-2007-039-0 Published online by Cambridge University Press

where $\delta_{\alpha,\alpha'} = 1$ or 0 according as $\alpha = \alpha'$ or $\alpha \neq \alpha'$. Apply this with the decomposition of μ and we get

$$
\begin{aligned}
&\langle m_{f,\psi}(s), m_{f',\psi'}(s)\rangle_{L^2(G)} \\
&\quad = \int_H \int_{\Lambda^1} \langle w_f(\lambda), w_{f'}(\lambda)\rangle\, \overline{\eta(z^{-1}\lambda)}\eta'(z^{-1}\lambda)\, d\mu(\lambda) d\nu_H(z) \cdot \delta_{\alpha,\alpha'} \\
&\quad = \int_H \int_\Sigma \int_H \langle w_f(z'\sigma), w_{f'}(z'\sigma)\rangle \overline{\eta(z^{-1}z'\sigma)}\eta'(z^{-1}z'\sigma)\delta(z')^{-1} \\
&\qquad\quad \times d\nu(z')|\mathbf{Pf}(\sigma)| d\sigma d\nu(z) \cdot \delta_{\alpha,\alpha'} \\
&\quad = \int_\Sigma \int_H \langle w_f(z'\sigma), w_{f'}(z'\sigma)\rangle \Big(\int_H \overline{\eta(z^{-1}z'\sigma)}\eta'(z^{-1}z'\sigma)\, d\nu(z)\Big)\, \delta(z')^{-1} \\
&\qquad\quad \times d\nu(z')|\mathbf{Pf}(\sigma)| d\sigma \cdot \delta_{\alpha,\alpha'} \\
&\quad = \int_{\Lambda^1} \langle w_f(\lambda), w_{f'}(\lambda)\rangle\, d\mu(\lambda) \overline{\langle \eta, \eta'\rangle} \cdot \delta_{\alpha,\alpha'},
\end{aligned}
$$

which means we have the orthogonality relation

$$
\langle W_\psi(f), W_{\psi'}(f')\rangle_{L^2(G)} = \langle w_f, w_{f'}\rangle_{\mathbf{K}} \overline{\langle \eta, \eta'\rangle}_{L^2(H,\nu)} \cdot \delta_{\alpha,\alpha'}. \tag{2.5}
$$

In particular, this shows that if $\alpha \neq \alpha'$, then the images of W_ψ and $W_{\psi'}$ are orthogonal in $L^2(G)$. We are now ready to prove the main result.

Theorem 2.8 *Let $G = N \rtimes H$ where N is a connected, simply connected nilpotent Lie group, and where H is a vector group such that the Lie algebra* $\mathrm{ad}(\mathfrak{h})$ *is* $\mathbf{R}$*-split and completely reducible, and such that $H_{[\pi]} = (1)$ holds for almost every $[\pi] \in \widehat{N}$. Let τ be the quasiregular representation of G in $L^2(N)$. Then τ has an admissible vector if and only if G is not unimodular.*

Proof Suppose first that G is not unimodular. We need to construct an admissible vector for τ. To do this, we fix a Jordan–Hölder basis of G satisfying the conditions of Section 1, and with all notations from Section 1, we conclude that $\mathfrak{h}_r = (0)$. Recalling the structure of the Fourier transform on $L^2(N)$ developed in the preceding, and in particular the decomposition $L^2(N) = \bigoplus_\beta L^2(N)^\beta$, we then execute the construction given above for τ^β-admissible vectors: let η be a unit vector in $L^2(H, \nu_H)$ that also belongs to $L^2(H, \delta^{-1}\nu_H)$ and let v, $1 \leq v \leq r$, such that $\delta(0, 0, \dots, z_v, \dots, 0) \neq 1$. Write $\delta(0, 0, \dots, z_v, \dots, 0) = z_v^p,\ p \neq 0$, and assume that $q = \dim(\Sigma) > 0$. We omit the proof in the case where $q = 0$; in that case each τ^β is a finite direct sum of irreducible, square-integrable representaitons, and the proof is a simplification of what follows. For each $\epsilon \in \{-1, 1\}^r$, recall that s_ϵ is the identification map from Σ_ϵ onto an open subset of $\mathbf{R}^q$.

Now for each basis index β, we choose a measurable function $\tilde{u}^\beta \colon \mathbf{R}^q \to (0, \infty)$ such that for any polynomial function $P(t)$ on $\mathbf{R}^q$, we have

$$
\sum_\beta \int_{\mathbf{R}^q} \tilde{u}^\beta(t)^p |P(t)|\, dt < \infty.
$$

Define $u^\beta : \Sigma \to (0,\infty)$ by $u^\beta(\sigma) = \tilde{u}^\beta(s_\epsilon(\sigma))$, $\sigma \in \Sigma_\epsilon$, so that we have

$$\sum_\beta \int_\Sigma u^\beta(\sigma)^p |\mathbf{Pf}(\sigma)|\, d\sigma < \infty.$$

Let ψ^β denote the function $\psi^{\beta,\beta}_{\eta,u^\beta}$ as defined above, so that

$$\widehat{\psi}^\beta(z\sigma) = \eta(z_1, z_2, \dots, z_{v-1}, z_v u^\beta(\sigma), z_{v+1}, \dots, z_r) e^\beta_{z\sigma} \otimes e^\beta_{z\sigma}.$$

Then each ψ^β is admissible for τ^β and the images of W_{ψ^β} are pairwise orthogonal. Set $\psi = \sum_\beta \psi^\beta$. Then ψ belongs to $L^2(N)$: for each β,

$$\int_N |\psi^\beta(x)|^2\, dx = \int_H |\,\eta(z)|^2 \delta(z)^{-1}\, d\nu(z) \int_\Sigma u^\beta(\sigma)^p |\mathbf{Pf}(\sigma)|\, d\sigma,$$

so $\sum_\beta \|\psi^\beta\|^2 < \infty$. For any $f \in L^2(N)$,

$$W_\psi(f) = \langle f, \tau(\cdot)\psi\rangle = \sum_\beta \langle f^\beta, \tau(\,\cdot\,)\psi^\beta\rangle = \sum_\beta W_{\psi^\beta}(f^\beta)$$

and $\sum_\beta \|W_{\psi^\beta}(f^\beta)\|^2 = \sum_\beta \|f^\beta\|^2 = \|f\|^2$. Thus $W_\psi(f) \in L^2(G)$, and $\|W_\psi(f)\| = \|f\|$ holds for all $f \in L^2(N)$.

On the other hand, suppose that $\psi \in L^2(N)$ is admissible for τ, and fix any basis index β. Then ψ^β is admissible for τ^β, so by Proposition 2.6, $\Delta_{\psi^\beta}(\lambda) = 1$ a.e. on Λ^1, and hence $\Delta_{\psi^\beta}(\sigma) = 1$ a.e. on Σ. Now if G is unimodular, then $\delta(z) = 1$ for all $z \in H$, so by Lemma 2.4,

$$\begin{aligned}\int_\Sigma |\mathbf{Pf}(\sigma)|\, d\sigma &= \int_\Sigma \Delta_{\psi^\beta}(\sigma)|\mathbf{Pf}(\sigma)|\, d\sigma = \int_\Sigma \int_H \|\widehat{\psi}(z\sigma)\|^2\, d\nu(z)|\mathbf{Pf}(\sigma)|d\sigma \\ &= \int_{\Lambda^1} \|\widehat{\psi}(\lambda)\|^2\, d\mu(\lambda) = \|\psi\|^2 < \infty.\end{aligned}$$

This is possible only if $d\sigma$ is a finite measure. But by Lemma 2.3, Σ is diffeomorphic with the cross-section Λ for G-orbits in Ω, and it is known [4, Corollary 2.2.2] that $d\sigma$ can only be finite when $q = 0$ and Σ is a finite set. By Lemma 2.3, this means that the regular representation of the unimodular group G decomposes into a finite sum of irreducible (square integrable) representations. It is well known (see for example [11, Proposition 0.4]) that this can only happen when G is discrete. ■

Next we show that $L^2(G)$ can be decomposed by means of the wavelet transforms on each $L^2(N)^\beta$.

Lemma 2.9 *There is an orthonormal basis $\{\eta_j\}$ for $L^2(H,\nu_H)$, each element of which also belongs to $L^2(H,\delta^{-1}\nu_H)$.*

https://doi.org/10.4153/CJM-2007-039-0 Published online by Cambridge University Press

Proof Write $\delta(z)^{-1}d\nu_H(z) = z_1^{p_1}z_2^{p_2}\cdots z_r^{p_r}\,dz_1dz_2\cdots dz_r$ where $p_w \in \mathbf{R}$, $1 \le w \le r$, and choose $\nu \ge 0$ such that $\nu \ge -\min(p_1, p_2, \dots, p_r)$. For $j = (j_1, j_2, \dots, j_r) \in \{0, 1, \dots\}^r$, set

$$\eta_j(z) = \prod_{w=1}^{r}\left(e^{-z_w}(2z_w)^{\frac{\nu+1}{2}}\,L_{j_w}^{(\nu)}(2z_w)\,c_{\nu,j_w}^{-1/2}\right),\ z = (z_1, z_2, \dots, z_r) \in H,$$

where $L_l^{(\nu)}(s), l = 0, 1, \dots$ is the Laguerre polynomial

$$L_l^{(\nu)}(s) = \frac{1}{l!}e^s s^{-\nu}\left(\frac{d}{ds}\right)^l (e^{-s}s^{l+\nu}), 0 < s < \infty$$

and

$$c_{\nu,l} = \int_0^\infty e^{-s}s^\nu L_l^{(\nu)}(s)^2\,ds.$$

As in [18] we see that $\{\eta_j\}_{j\in\{0,1,2,\dots\}^r}$ is an orthonormal basis of $L^2(H, \nu_H)$. Also, since $\nu + p_w \ge 0$, $1 \le w \le r$, we have

$$\int_H |\eta_j(z)|^2\delta(z)^{-1}\,d\nu_H(z) = \prod_{w=1}^{r} c_{\nu,j_w}^{-1}\int_0^\infty e^{-2z_w}(2z_w)^{\nu+p_w}L_j^{(\nu)}(2z_w)^2 2^{1-p_w}\,dz_w < \infty.$$

∎

Assume that G is not unimodular, and that $q = \dim(\Sigma) > 0$. Let $\{\eta_j\}$ be the basis of $L^2(H, \nu)$ as in Lemma 2.9, and let $u\colon \Sigma \to (0, \infty)$ a measurable function such that $\int_\Sigma u(\sigma)^p|\mathbf{Pf}(\sigma)|d\sigma$, where p is chosen appropriately as above. Fix a basis index β_0, set

$$W_{j,u}^\alpha = W_{\psi_{\eta_j,u}^{\alpha,\beta_0}}, \alpha \in B, j \in \{0, 1, 2, \dots\}^r,$$

and set

$$\mathbf{J}_j^\alpha = W_{j,u}^\alpha(L^2(N)^{\beta_0}).$$

From (2.5) we see that the subspaces $\mathbf{J}_j^\alpha$ are pairwise orthogonal in $L^2(G)$ and that each is isomorphic with $\mathbf{K}$.

Theorem 2.10 *We have*

$$L^2(G) = \bigoplus_{\substack{\alpha\in B\\ j\in\{0,1,2,\dots\}^r}} \mathbf{J}_j^\alpha.$$

Proof We must show that $L^2(G)$ is contained in the direct sum. Let $Y \in L^2(G)$ and for $z \in H$ set $Y_z(x) = Y(xz), x \in N$. We have

$$\begin{aligned}\|Y\|^2 &= \int_H\left(\int_N |Y_z(x)|^2\,dx\right)\delta(z)^{-1}\,d\nu(z)\\ &= \int_H\left(\int_{\Lambda^1} \|\widehat{Y}_z(\lambda)\|^2\,d\mu(\lambda)\right)\delta(z)^{-1}\,d\nu(z).\end{aligned}$$

Since $\|\widehat{Y}_z(\lambda)\|^2 = \sum_{\alpha,\beta} |\langle \widehat{Y}_z(\lambda)e_\lambda^\alpha, e_\lambda^\beta\rangle|^2$, then for each pair of indices α and β,

$$\int_H |\langle \widehat{Y}_z(\lambda)e_\lambda^\alpha, e_\lambda^\beta\rangle|^2 \delta(z)^{-1}\, d\nu(z) < \infty$$

holds for μ-a.e. λ. Let $y_\lambda^{\alpha,\beta}$ denote the mapping $z \mapsto \langle \widehat{Y}_z(\lambda)e_\lambda^\alpha, e_\lambda^\beta\rangle \delta(z)^{-1/2}$; then there is a co-null subset Λ_0^1 of Λ^1, such that $y_\lambda^{\alpha,\beta} \in L^2(H, \nu_H)$ holds for each $\lambda \in \Lambda_0^1$, $\alpha, \beta \in B$. Now for each $\lambda \in \Lambda_0^1$, write $\lambda = z_\lambda \sigma$ and set $\eta_{j,\lambda}(z) = \eta_j(z^{-1}z_\lambda), z \in H$. Observe that $\{\eta_{j,\lambda} \mid j \in \{0, 1, 2, \dots\}^r\}$ is an orthonormal basis of $L^2(H, \nu_H)$. Hence for each $\lambda \in \Lambda_0^1$, $\alpha, \beta \in B$, we have complex numbers $\{a_j(\lambda, \alpha, \beta) \mid j \in \{0, 1, 2, \dots\}^r\}$ such that

$$y_\lambda^{\alpha,\beta} = \sum_{j \in \{0,1,2,\dots\}^r} a_j(\lambda, \alpha, \beta)\eta_{j,\lambda}.$$

This means that $\widehat{Y}_z(\lambda)\delta(z)^{-1/2} = \sum_{\alpha,\beta}\sum_j a_j(\lambda, \alpha, \beta)\eta_{j,\lambda}(z)e_\lambda^\beta \otimes e_\lambda^\alpha$. Now for each $\alpha \in B$, $j \in \{0, 1, 2, \dots\}^r$, set $g_j^\alpha(\lambda) = \sum_\beta a_j(\lambda, \alpha, \beta)e_\lambda^\beta \otimes e_\lambda^{\beta_0}$. We claim that $g_j^\alpha \in \mathbf{H}^{\beta_0}$ for all α. To see this we observe that

$$\begin{aligned}
\|Y\|^2 &= \int_H \int_{\Lambda^1} \|\widehat{Y}_z(\lambda)\|^2\, d\mu(\lambda)\delta(z)^{-1} d\nu(z) \\
&= \int_{\Lambda^1} \sum_{\alpha,\beta} \int_H |\langle \delta(z)^{-1/2}\widehat{Y}_z(\lambda)e_\lambda^\alpha, e_\lambda^\beta\rangle|^2\, d\nu(z)\, d\mu(\lambda) \\
&= \int_{\Lambda^1} \sum_{\alpha,\beta} \|y_\lambda^{\alpha,\beta}\|^2\, d\mu(\lambda) \\
&= \int_{\Lambda^1} \sum_{\alpha,\beta}\sum_j |a_j(\lambda, \alpha, \beta)|^2\, d\mu(\lambda) \\
&\geq \int_{\Lambda^1} \sum_\beta |a_j(\lambda, \alpha, \beta)|^2\, d\mu(\lambda) \\
&= \|g_j^\alpha\|^2.
\end{aligned}$$

Denote by f_j^α the inverse Fourier transform of g_j^α, and set $\psi = \psi_{\eta_j,u}^{\alpha,\beta_0}$. Then for a.e. $z \in H$, $(W_{j,u}^\alpha(f_j^\alpha))_z = (f_j^\alpha * (\tau(z)\psi)^*)$ belongs to $L^2(N)$, and for such z,

$$\begin{aligned}
(W_{j,u}^\alpha(f_j^\alpha)_z)\widehat{\ }(\lambda) &= g_j^\alpha(\lambda) \circ (\eta_j(z^{-1}z_\lambda)e_\lambda^\alpha \otimes e_\lambda^{\beta_0})^*\ \delta(z)^{1/2} \\
&= \delta(z)^{1/2}\Big(\sum_\beta a_j(\lambda, \alpha, \beta)e_\lambda^\beta \otimes e_\lambda^{\beta_0}\Big) \circ \eta_j(z^{-1}z_\lambda)e_\lambda^{\beta_0} \otimes e_\lambda^\alpha \\
&= \delta(z)^{1/2} \sum_\beta a_j(\lambda, \alpha, \beta)\eta_j(z^{-1}z_\lambda)e_\lambda^\beta \otimes e_\lambda^\alpha.
\end{aligned}$$

https://doi.org/10.4153/CJM-2007-039-0 Published online by Cambridge University Press

Summing over all α and j, we find

$$\sum_{\alpha,j} (W^{\alpha}_{j,u}(f^{\alpha}_j)_z)\widehat{\ }(\lambda) = \delta(z)^{1/2} \sum_{\alpha,\beta} \Big(\sum_j a_j(\lambda,\alpha,\beta)\eta_j(z^{-1}z_\lambda) \Big) e^{\beta}_{\lambda} \otimes e^{\alpha}_{\lambda}$$
$$= \sum_{\alpha,\beta} \langle \widehat{Y}_z(\lambda) e^{\alpha}_{\lambda}, e^{\beta}_{\lambda} \rangle \, e^{\beta}_{\lambda} \otimes e^{\alpha}_{\lambda}$$
$$= \widehat{Y}_z(\lambda).$$

Taking the inverse Fourier transform we obtain $Y_z = \sum_{\alpha,j} W^{\alpha}_{j,u}(f^{\alpha}_j)_z$ for a.e. $z \in H$, and hence

$$Y = \sum_{\alpha,j} W^{\alpha}_{j,u}(f^{\alpha}_j). \qquad \blacksquare$$

References

[1] D. Arnal, M. BenAmmar, B. Currey, and B. Dali, *Explicit construction of canonical coordinates for completely solvable Lie groups.* J. Lie Theory **15**(2005), 521–560.

[2] B. Currey, *A continuous trace composition sequence for $C^*(G)$ where G is an exponential solvable Lie group.* Math. Nachr. **159**(1992), 189–212.

[3] ———, *An explicit Plancherel formula for completely solvable Lie groups.* Michigan Math. J. **38**(1991), no. 1, 75–87.

[4] ———, *Explicit orbital parameters and the Plancherel measure for exponential Lie groups.* Pacific J. Math. (to appear).

[5] ———, *Smooth decomposition of finite multiplicity monomial representations for a class of completely solvable homogeneous spaces.* Pacific J. Math. **170**(1995), no. 2, 429–460.

[6] ———, *The structure of the space of coadjoint orbits of an exponential solvable Lie group* . Trans. Amer. Math. Soc. **332**(1992), no. 1, 241–269.

[7] B. Currey and R. Penney, *The structure of the space of co-adjoint orbits of a completely solvable Lie group* . Mich. Math. J. **36**(1989), no. 2, 309–320.

[8] M. Duflo and C. Moore, *On the regular representation of a nonunimodular locally compact group.* J. Functional Analysis **21**(1976), no. 2, 209–243.

[9] M. Duflo and M. Raïs, *Sur l'analyse harmonique sur les groupes de Lie résolubles.* Ann. Sci. École Norm. Sup. **9**(1976), no. 1, 107–144.

[10] R. Fabec, and G. Ólafsson, *The continuous wavelet transform and symmetric spaces.* Acta Applicandae Math. (to appear)

[11] H. Führ, *Admissible vectors for the regular representation.* Proc. Amer. Math. Soc. **130**(2002), no. 10, 2959–2970 (electronic).

[12] H. Ishi, *Wavelet transforms for semidirect product groups.* J. Fourier Anal. Appl. **12**(2006), no. 1, 37–52.

[13] R. S. Laugesen, N. Weaver, G. Weiss, and E. N. Wilson, *A characterization of the higher dimensional groups associated with continuous wavelets.* J. Geom. Anal. **12**(2002), no. 1, 89–102.

[14] R. Lipsman, *Harmonic analysis on exponential solvable homogeneous spaces: the algebraic or symmetric cases.* Pacific. J. Math. **140**(1989), no. 1, 117–147.

[15] ———, *Induced representations of completely solvable Lie groups.* Ann. Scuola Norm. Sup. Pisa Cl. Sci. **17**(1990), no. 1, 127–164.

[16] ———, *The multiplicity function on exponential and completely solvable homogeneous spaces.* Geom. Dedicata **39**(1991), no. 2, 155–161.

[17] R. Lipsman and J. Wolf, *Canonical semi-invariants and the Plancherel formula for parabolic groups.* Trans. Amer. Math. Soc. **269**(1982), no. 1, 111–131.

[18] H. Liu and L. Peng, *Admissible wavelets associated with the Heisenberg group.* Pacific J. Math. **180**(1997), no. 1, 101–123.

[19] N. V. Pedersen, *On the infinitesimal kernel of irreducible representations of nilpotent Lie groups.* Bull. Soc. Math. France **112**(1984), no. 4, 423–467.

[20] L. Pukánszky, *On the characters and the Plancherel formula of nilpotent Lie groups.* J. Functional Analysis **1**(1967), 255–280.

[21] ———, *On the unitary representations of exponential Lie groups.* J. Functional Analysis **2**(1968), 73–113.
[22] G. Weiss and E. N. Wilson, *The mathematial theory of wavelets.* In: Twentieth Century Harmonic Analysis – A Celebration, NATO Sci. Ser. II Math. Phys. Chem. 33, Kluwer Acad. Publ., Dordrecht, 2001, pp. 329–366.

Department of Mathematics and Computer Science
Saint Louis University
Saint Louis, MO 63103
U.S.A.
e-mail: curreybn@slu.edu