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SPECTRAL ANALYSIS ON UPPER LIGHT CONE IN $\mathbf{R}^3$ AND THE RADON TRANSFORM

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Introduction. The upper light cone L in $\mathbf{R}^3$ is a homogeneous space of the 3-dimensional Lorentz group G. It may be identified with the space of horocycles in the upper hyperboloide H which is the symmetric space associated to G. There exists a duality between H and L (see [**5**] p. 144) and a general procedure leads to a generalized Radon transform:

$$R : \mathcal{D}(H) \longrightarrow \mathcal{D}(L)$$

and the dual Radon transform

$$B : \mathcal{E}(L) \longrightarrow \mathcal{E}(H).$$

These operations commute with the natural action of the group G. It was proved in [**12**] that the spectral analysis holds in the space $\mathcal{E}(H)$ with respect to the system of functions given as follows:

$$H \ni x \longrightarrow \langle x|p\rangle^{\mu}, \quad \rho \in L, \mu \in \mathbf{C}.$$

where $\langle\cdot|\cdot\rangle$ is the indefinite group invariant scalar product on $\mathbf{R}^3$.

In the present paper we study the problem of spectral analysis in the space $\mathcal{E}(L)$. One finds that the system of functions

$$L \ni p \longrightarrow \langle x|p\rangle^{\mu}, \quad x \in H$$

is not sufficient for obtaining the spectral analysis on L, although the duality between H and L can suggest it. The reason is that in $\mathcal{E}(L)$ there appear finite dimensional and discrete series of irreducible representations of G.

Nevertheless, we obtain a spectral analysis theorem for a larger class of elementary functions (Theorems 5.3 and 5.4). It permits us to characterize in terms of spectral analysis the G-invariant space ker$B$. In Section 6 we prove also a theorem of Pompeiu type, that is the necessary and sufficient condition for a compactly supported distribution on L to span by translations and linear combinations a dense subset in $\mathcal{E}(L)$.

1. The group and its homogeneous spaces. Throughout what follows G will denote the Lorentz group in three dimensions, that is the group of all linear mappings in $\mathbf{R}^3$ which preserve the form:

$$\langle x|y\rangle := -x_1y_1 - x_2y_2 + x_3y_3$$

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and the sign of x_3.

Let us distinguish in G the following three 1-parameter subgroups:

$$A := \left\{ a(t) | a(t) := \begin{pmatrix} 1 & 0 & 0 \\ 0 & \operatorname{ch} t & \operatorname{sh} t \\ 0 & \operatorname{sh} t & \operatorname{ch} t \end{pmatrix}, t \in \mathbf{R} \right\}$$

$$K := \left\{ k(\theta) | k(\theta) := \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}, \theta \in \mathbf{R} \right\}$$

$$N := \left\{ n(s) | n(s) := \begin{pmatrix} 1 & s & -s \\ -s & 1-\frac{s^2}{2} & \frac{s^2}{2} \\ -s & -\frac{s^2}{2} & 1+\frac{s^2}{2} \end{pmatrix}, s \in \mathbf{R} \right\}.$$

Every element $g \in G$ can be represented in a unique way as

$$g = g(\theta, t, s) = k(\theta)a(t)n(s)$$

(the Iwasawa decomposition).

The group N conserves in $\mathbf{R}^3$ the planes given by the equation $x_2 - x_3 = \langle x | p_0 \rangle = \text{const.}$, where $p_0 := (0, 1, 1)$. We denote also $x_0 := (0, 0, 1)$. The point x_0 is K-fixed and p_0 is N-fixed.

The orbit $H := Gx_0$ is the upper hyperboloide and as a homogeneous space is isomorphic to the quotient G/K. The upper light cone $L := Gp_0$ is in turn isomorphic to G/N.

For a given homogeneous space (G, Y) and a function f on Y we denote by L_g the translation

$$(1.2) \qquad L_g f(y) := f(g^{-1}y), \quad g \in G, y \in Y.$$

The representation $G \ni g \longrightarrow L_g$ will be referred to as the regular representation of G.

We are principally interested in the action of the group G in the spaces $\mathcal{E}(H)$, $\mathcal{E}(L)$ and $\mathcal{D}(H)$, $\mathcal{D}(L)$ of all smooth functions and all Schwartz test functions on H and L, respectively. The topology in $\mathcal{E}$ is the usual Fréchèt topology of the uniform convergence with all derivatives on compact sets. In the space $\mathcal{D}$ we have the topology of compact convergence with all derivatives. The dual space $\mathcal{D}'$ is the space of all distribution and $\mathcal{E}'$ can be identified with the space of compactly supported distributions. The group G acts continuously on $\mathcal{E}$ and $\mathcal{D}$ by means of the operators L_g and on the dual spaces $\mathcal{D}'$, $\mathcal{E}'$ by means of contragredient representation.

Let $T \in \mathcal{E}'(G)$ and $S \in \mathcal{D}'(Y)$. The convolution $T * S$ is the distribution on Y defined by the formula

$$(1.3) \qquad T * S(f) := T_g \otimes S_y(f(gy)), \quad f \in \mathcal{D}(Y).$$

The invariant measures on K, A, N respectively can be chosen in the following way:

$$(1.4)\qquad \int_K f\,dk := \frac{1}{2\pi}\int_0^{2\pi} f(k(\theta))d\theta;$$
$$\int_A \varphi da := \int_{\mathbf{R}} \varphi(a(t))dt$$
$$\int_N \varphi dn := \int_{\mathbf{R}} \varphi(n(s))ds.$$

In virtue of the Iwasawa decomposition the Haar measure on G can be defined as follows:

$$(1.5)\qquad \int_G f\,dg := \int_K \int_{\mathbf{R}} \int_N f(ka(t)n)e^t dkdtdn.$$

On the manifold H we can define the following G-invariant measure:

$$\int_H f\,dm := \int_K \int_N f(anx_0)dnda.$$

After determining a G-invariant measure dy on a homogeneous space y we are able to inject the space $\mathcal{E}(Y)$ into $\mathcal{D}'(Y)$ in such a way that the translations in $\mathcal{E}(Y)$ coincide with the contragredient action of G on $\mathcal{D}'(Y)$. Namely, given $f \in \mathcal{E}(Y)$ we put:

$$I(f)(\varphi) := \int_K f\varphi dy, \quad \varphi \in \mathcal{D}(Y).$$

With the aid of the Haar measure on K we can introduce the operator $P : \mathcal{E}(G) \longrightarrow \mathcal{E}(H)$ as follows

$$Pf(gK) := \int_K f(gk)dk.$$

The operator P is surjective and commutes with the regular representation. A distribution $T \in \mathcal{E}'(H)$ can be considered as a compactly supported distribution on G according to the formula:

$$\tilde{T}(f) := T(Pf), \quad f \in \mathcal{D}(G).$$

The space $\mathcal{E}'(H)$ becomes now a convolution algebra with respect to the operation defined by the formula:

$$T * S := \tilde{T} * S, \quad T, S \in \mathcal{E}'(H),$$

the right hand side being defined by (1.3).

The space $\mathcal{D}(H)$ constitutes a subalgebra of $\mathcal{E}'(H)$.

Every point $p \in L$ determines a subset $\xi_p \subset H$ called a horocycle and defined as:

$$\xi_p := \{x \in H | \langle x|p\rangle = 1\}.$$

Let us observe that $g\xi_p = \xi_{gp}$. In particular every horocycle is of the form $g\xi_{p_0}, g \in G$. Since the isotropy group of the point ξ_{p_0} is just N we can identify the space of all horocycles with $G/N = L$.

In a similar way there is a one-to-one correspondence between the points of H and subsets of L defined by

$$\check{x} := \{p \in L | \langle x|p\rangle = 1\} \quad \text{for } x \in H.$$

We observe that $\check{x}_0 = Kp_0$ and $(gx)^\vee = g\check{x}$. The sets $\check{x} \subset L$ will be referred to as circles in L.

2. The Fourier, the Radon and the dual Radon transforms. Let us consider on $H \times L$ the family of functions given by:

$$H \times L \ni (x,p) \longrightarrow \langle x|p\rangle^\mu, \quad \mu \in \mathbf{C}.$$

We shall use the notation:

$$e_{\mu,p}(x) := \langle x|p\rangle^\mu \quad \text{and}$$
$$e_{\mu,x}(p) := \langle x|p\rangle^\mu, \quad x \in H, p \in L.$$

The invariance of the form $\langle\cdot|\cdot\rangle$ leads to relations:

$$e_{\mu,gx} = L_g e_{\mu,x} \quad \text{and} \quad e_{\mu,gp} = L_g e_{\mu,p}. \tag{2.1}$$

In particular the functions e_{μ,p_0} are N-fixed and the functions e_{μ,x_0} are K-fixed. We have:

$$e_{\mu,x}(ka(t)p_0) = e_{\mu,x}(kp_0)e^{\mu t} \quad \text{and}$$
$$e_{\mu,p_0}(na(t)x_0) = e^{-\mu t}. \tag{2.2}$$

We shall refer to the functions $e_{\mu,x}$ on L and to $e_{\mu,p}$ on H as to the exponential functions on corresponding manifolds.

The Fourier transform of a function $f \in \mathcal{D}(H)$ is defined by the formula

$$\hat{f}(\lambda, p) := \int_H e_{-i\lambda-1/2,p}(x) f(x) dm(x), \quad \lambda \in \mathbf{C}, p \in L. \tag{2.3}$$

For $T \in \mathcal{E}'(H)$ we put:

(2.2) $$\hat{T}(\lambda, p) := T(e_{-i\lambda - 1/2, p}).$$

By the very definition one obtains:

(2.5) $$(L_g T)\hat{}(\lambda, \cdot) = L_g \hat{T}(\lambda, \cdot) \quad \text{and}$$
$$\hat{T}(\lambda, ka(t)p_0) = e^{-(i\lambda + 1/2)t}\hat{T}(\lambda, kp_0).$$

The Fourier transform originally defined on $C \times L$ is uniquely determined by its values on $\mathbf{C} \times Kp_0 = \mathbf{C} \times \mathbf{S}^1$. The restriction of the Fourier transformation to the space $\mathcal{E}'(K\backslash H)$ of K-invariant and compactly supported distributions is called the spherical Fourier tansform on H.

We shall write

$$\hat{T}(\lambda) := \hat{T}(\lambda, p_0).$$

THEOREM 1.1. [**5**] *The spherical Fourier Transform on H is an isomorphism of the convolution algebra $\mathcal{E}'(K\backslash H)$ onto the space A_s of all holomorphic functions φ on the complex plane satisfying the following conditions:*

$1°\varphi(-z) = \varphi(z)$,

$2°$ *There exist constants $R, m, r > O$ such that*

$$|\varphi(z)| \leqq R(1 + |z|)^m e^{r|\mathrm{Im} z|}.$$

We observe that the image of the spherical Fourier transform on H is just the subalgebra of all even elements in the algebra of the classical Fourier transforms of the elements of $\mathcal{E}'(\mathbf{R})$.

The Radon transform on H is the mapping which assigns to a function $f \in \mathcal{D}(H)$ the function $Rf \in \mathcal{D}(L)$ given by the formula:

(2.6) $$Rf(g\rho_0) := \int_N f(gnx_0)dn.$$

The value $Rf(gp_0)$ can be interpreted as the mean value of the function f on the horocycle $g\xi_{p_0}$.

The dual Radon transform maps $\mathcal{E}(L)$ into $\mathcal{E}(H)$ according to the formula:

(2.7) $$B\psi(gx_0) := \int_K \psi(gkp_0)dk.$$

The value $B\psi(gx_0)$ is the mean value of ψ on the circle gkp_0. Both operations commute with the translations acting on corresponding manifolds.

There is a simple relation between the Radon and the Fourier transforms on H. Let $f \in \mathcal{D}(H)$. We have

$$\begin{aligned}\hat{f}(\lambda, kp_0) &= \int_H f(x)\langle x|kp_0\rangle^{-i\lambda-1/2}dm \\ &= \int_H f(kx)\langle x|kp_0)\rangle^{-i\lambda-1/2}dm(x) \\ &= \int_{\mathbf{R}}\int_N f(ka(t)nx_0)\langle a(t)nx_0|p\rangle^{-i\lambda-1/2}dn\,dt \\ &= \int_{\mathbf{R}} e^{\langle i\lambda+1/2\rangle t}\int f(ka(t)nx_0)dn\,dt \\ &= \int_{\mathbf{R}} e^{\langle i\lambda+1/2\rangle t}Rf(ka(t)p_0)dt.\end{aligned}$$

The symmetric space Fourier transform of $f \in \mathcal{D}(H)$ is then equal to the classical Fourier transform of the function

$$\mathbf{R} \ni t \longrightarrow e^{t/2}Rf(ka(t)p_0).$$

Let us consider on $\mathcal{D}(L)$ the operator given by the formula

$$\tilde{\varphi}(\lambda, k) := \int_{\mathbf{R}} e^{\langle i\lambda+1/2\rangle t}\varphi(ka(t)p_0)dt.$$

Then we have

$$\hat{f}(\lambda, kp_0) = (Rf)\tilde{}(\lambda, k). \tag{2.8}$$

In particular for $f \in \mathcal{D}(K\backslash H)$ one obtains:

$$\hat{f}(\lambda) = (Rf)\tilde{}(\lambda, k).$$

PROPOSITION (1.2.) *Let $\psi \in \mathcal{D}(K\backslash L)$. Then $\psi \in R\mathcal{D}(K\backslash H)$ if and only if the function $\mathbf{R} \ni t \longrightarrow e^{t/2}\psi(a(t)p_0)$ is even.*

Proof. In virtue of the formula (2.8) and the inversion formula for the classical Fourier transformation we obtain for any $f \in \mathcal{D}(K\backslash H)$:

$$F(t) := e^{t/2}Rf(a(t)\rho_0) = \int_{\mathbf{R}} \hat{f}(\lambda)e^{-i\lambda t}d\lambda. \tag{2.9}$$

Taking into account that $\hat{f}$ is an even function (Theorem 1.1) we observe that F is even.

On the other hand if the function

$$t \longrightarrow e^{t/2}\psi(a(t)p_0)$$

is even its classic Fourier transform belongs to $\mathcal{D}(K\backslash H)\hat{}$, hence for some $\varphi \in \mathcal{D}(K\backslash H)$ we have:

$$\int_{\mathbf{R}} R\varphi(a(t)p_0)e^{(i\lambda+1/2)t}dt = \int_{\mathbf{R}} e^{t/2}\psi(a(t)\rho_0)e^{i\lambda t}dt.$$

By the injectivity of the classical Fourier transform we obtain $R\varphi = \psi$.

3. Some properties of invariant subspaces in $\mathcal{E}(L)$. The manifold L has an additional structure, namely that of the multiplication of elements by positive reals. This action obviously commutes with the group action. We denote:

$$(3.1) \qquad \tau(t)f(p) := f(e^t p).$$

For a given $\mu \in \mathbf{C}$ let us denote by $\mathcal{E}_\mu(L)$ the space of all elements of $\mathcal{E}(L)$ which satisfy:

$$(3.2) \qquad f(rp) = r^\mu f(p), \quad r > 0.$$

The space $\mathcal{E}_\mu$ is invariant with respect to the left regular representation of G. The elements of $\mathcal{E}_\mu(L)$ are uniquely determined by their values of the orbit

$$B := Kp_0 = \{p \in L | p_3 = 1\}.$$

For any $g \in G$ and $b \in B$ we have:

$$L_g f(b) = f(g^{-1}b) = f\left(\frac{g^{-1}b}{\langle x_0|g^{-1}b\rangle}\langle x_0|g^{-1}b\rangle\right)$$
$$= f(g^{-1}\cdot b)\langle gx_0|b\rangle^\mu = e_{\mu,b}(gx_0)f(g^{-1}\cdot b),$$

where

$$g \cdot b := \frac{gb}{\langle g^{-1}x|b\rangle}$$

defines an action of G on B. We introduce a representation of G on $\mathcal{E}(B)$:

$$U_g^\mu f(b) := \langle gx_0|b\rangle^\mu f(g^{-1}\cdot b).$$

Then we have

$$(3.3) \qquad L_g f|B = U_g^\mu(f|B) \quad \text{for } f \in \mathcal{E}_\mu(L).$$

Let

$$\int_B \varphi db := \int_K \varphi(kp_0)dk.$$

A direct calculation shows:

$$(3.4)\qquad \int_B (g \circ b)db = \int_B \varphi(b)\langle gx_0|b\rangle^{-1}db.$$

We denote

$$(\varphi|\psi) := \int_B \varphi(b)\psi(b)db.$$

Then we obtain by applying (3.4):

$$(3.5)\qquad (U_g^{\mu}\varphi|\psi) = (\varphi|U_g^{-\mu-1}\psi).$$

In particular

$$(U_g^{\mu-1/2}\varphi|\psi) = (\varphi|U_{g^{-1}}^{-\mu-1/2}\psi).$$

For an integral value $\mu = n$ the space $\mathcal{E}_n(L)$ contains an invariant subspace consisting of restrictions to L of all homogeneous polynomials of order n. In order to describe all invariant subspaces in $\mathcal{E}_\mu(L)$ we introduce the following family of functions on B:

$$(3.6)\qquad \tilde{e}_m(k(\theta)\rho_0) := e^{im\theta}.$$

PROPOSITION 3.1. 1° *If $\mu \in \mathbf{C}$ is not an integer then the representation U_μ is irreducible.*

2° *If $\mu = n \in \mathbf{N}\cup\{0\}$ then in the representation space $\mathcal{E}(B)$ there exist three invariant subspaces:*

$\tilde{F}_n^- :=$ *closed linear span of* $\{\tilde{e}_k|k \leqq n\}$,
$\tilde{F}_n^+ :=$ *closed linear span of* $\{\tilde{e}_k|k \geqq -n\}$ *and*
$\tilde{E}_n := \tilde{F}^- \cap \tilde{F}^+$.

3° *If $\mu = -n, n \in \mathbf{N}$ then in the representation space there exist three invariant subspaces:*

$\tilde{F}_{-n}^- :=$ *closed linear span of* $\{\tilde{e}_k|k \leqq -n\}$,
$\tilde{F}_{-n}^+ :=$ *closed linear span of* $\{\tilde{e}_k|k \geqq n\}$ *and*
$\tilde{E}_{-n} := \tilde{F}_{-n}^- + \tilde{F}_{-n}^+$.

The space $\tilde{F}_{-n}^{\pm}$ is just the annihilator of the space $\tilde{F}_{n-1}^{\mp}$ with respect to the form $(\cdot|\cdot)$. We shall denote by $F_n^{\pm}, n \in \mathbf{Z}$ the subspace of $\mathcal{E}_\mu$ of those elements whose restrictions to B belong to $\tilde{F}_n^{\pm}$; similarly $E_n := F_n^- \cap F_n^+$, $n \in \mathbf{N} \cup \{0\}$ and $E_{-n} := F_{-n}^- + F_{-n}^+$. By virtue of (3.3) all these spaces are G-invariant.

Let us introduce:

$$(3.7) \qquad e_m^{\mu}(k(\vartheta)a(t)\rho_0) := e^{im\vartheta + \mu t}.$$

We shall also consider the restrictions of functions on L to the orbit $A\rho_0$ which can be considered as a function on $\mathbf{R}$. Given $f \in \mathcal{E}(L)$ we write

$$(3.8) \qquad f^A(t) := f(a(t)p_0).$$

Then we have

$$(\tau(s)f)^A(t) = f^A(s+t).$$

Let $f \in \mathcal{E}(L)$. By the Fourier series of f we mean the decomposition:

$$(3.9) \qquad f(k(\theta)\rho) = \sum_{n=-\infty}^{\infty} e^{in\theta} f_n(p), \text{ where}$$

$$f_n(p) := \frac{1}{2\pi} \int_0^{2\pi} e^{-in\theta} f(k(\theta)p)d\theta.$$

The function f_n belongs to $\mathcal{E}(L)$ and satisfies

$$f_n(k(\theta)p) = e^{in\theta} f(p).$$

The series (3.9) converges in $\mathcal{E}(L)$ see [**9**]. In particular we have

$$f(k(\theta)a(t)p_0) = \sum_{n=-\infty}^{\infty} e^{in\theta} f_n^A(t).$$

For a given closed and K-invariant subspace $V \subset \mathcal{E}(L)$ we shall denote by V_n the subspace of V consisting of the functions f_n, $f \in V$ and by V_n^A the space of all functions f_n^A, $f \in V$. In the sequel we are studying the invariance properties of the spaces V_n^A for V being G-invariant.

PROPOSITION 3.2. *Let $V \subset \mathcal{E}(L)$ be a closed and G-invariant subspace. Assume that $f \in V_0^A, \varphi \in \mathcal{D}(\mathbf{R})$ and suppose that $e^{t/2}\varphi$ is even. Then $f * \varphi \in V_0^A$.*

Proof. Let $u \in \mathcal{D}(K \backslash G / K)$ and $\psi \in V_0 = V \cap \mathcal{E}(K \backslash L)$. The function

$$L \ni p \longrightarrow \int_G u(g)\psi(gp)dg$$

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also belongs to the space V_0 and then the function

$$\mathbf{R} \ni t \longrightarrow \int_G u(g)\psi(ga(t)p_0)dg$$

is an element of V_0^A.

Representing the Haar measure on G in the form (1.5) we obtain:

$$\begin{aligned} I(t) &:= \int_G u(g)\psi(ga(t)p_0)dg \\ &= \int_K \int_{\mathbf{R}} \int_N u(ka(s)n)\psi(ka(s)na(t)p_0)e^s dkdsdn \\ &= \int_{\mathbf{R}} \int_N u(a(s)n)dn\psi(a(s)a(t)p_0)e^s ds \\ &= \int_{\mathbf{R}} Ru(a(s)p_0)e^s\psi(a(s-t)p_0)ds. \end{aligned}$$

According to Proposition 1.2 the function

$$s \longrightarrow e^{s/2}Ru(a(s)p_0)$$

is even hence

$$e^s Ru(a(s)p_0) = Ru(a(-s)p_0).$$

We obtain

$$I(t) = \int_{\mathbf{R}} Ru(a(-s)p_0)\psi(a(s-t)p_0)ds = f * \varphi(t)$$

where

$$f(t) := \psi(a(t)p_0) = \psi^A(t) \quad \text{and} \quad \varphi(s) := Ru(a(s)p_0).$$

The proof follows.

COROLLARY 3.3. *Let* $f \in V_0^A$ *and let* $T \in \mathcal{E}'(\mathbf{R})$ *be such that* $e^{t/2}T$ *is even. Then* $T * f \in V_0^A$.

Proof. Proposition 3.2 proves the statement for the distributions given by the test functions. By the density of $\mathcal{D}(\mathbf{R})$ in $\mathcal{E}'(\mathbf{R})$ and the continuity of the mapping

$$\mathcal{E}'(\mathbf{R}) \times \mathcal{E}(\mathbf{R}) \ni (T, f) \longrightarrow T * f \in \mathcal{E}(R)$$

the proof follows.

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Let us denote by $\mathcal{E}_s'(\mathbf{R})$ the algebra of all even elements of $\mathcal{E}(\mathbf{R})$.

COROLLARY 3.4. *If V is closed and G-invariant then the space $e^{t/2}V_0^A$ of all functions of the form $e^{t/2}f$, $f \in V_0^A$ is a $\mathcal{E}_s'(\mathbf{R})$-convolution module.*

Now, we are going to deduce the same invariance property for functions which not necessarily belong to V_0 but satisfy a special condition of analicity.
Let us define

$$D_0\varphi := \varphi'(0), \quad \varphi \in \mathcal{E}(\mathbf{R}).$$

Definition. A function $f \in \mathcal{E}(L)$ will be called *A-analytic* if

$$\tau(t)f(gp_0) = \sum_{m=0}^{\infty} \frac{t^m}{m!} D_0^m (l_{g^{-1}} f)^A \tag{3.10}$$

and the series converges in $\mathcal{E}(\mathbf{R})$ for all $t \in \mathbf{R}$.

Given $f \in \mathcal{E}(L)$ we denote by $V(f)$ the closed linear span of all functions $L_g f, g \in G$.
We are going to prove:

PROPOSITION 3.5. *Let $f \in \mathcal{E}(L)$ be A-analytic. Then the function*

$$\psi := e^{t/2}\tau(t)f + e^{-t/2}\tau(-t)f$$

belongs to $V(f)$ for all $t \in \mathbf{R}$.

Proof. As proved in [**5**] every differential operator D on L which commutes with the regular representation is of the form:

$$Df(gp_0) = P(D_0)(L_{g^{-1}}f)^A, \tag{3.11}$$

where $P(\cdot)$ is a polynomial with constant coefficients. Let us denote by $\mathfrak{g}, \mathfrak{a}, \mathfrak{n}$ the Lie algebras of the groups G, A, N respectively. By $\mathfrak{U}(\mathfrak{g})$ we denote the enveloping algebra of $\mathfrak{g}$ and by $Z(\mathfrak{g})$ its center. The differential of the regular representation of G on $\mathcal{E}(L)$ give us a representation of $\mathfrak{U}(\mathfrak{g})$ on $\mathcal{E}(L)$ by means of differential operators. This action of $\mathfrak{U}(\mathfrak{g})$ conserves closed G-invariant subspaces. In particular the elements of the center $Z(\mathfrak{g})$ define differential operators on $\mathcal{E}(L)$ commuting with the regular representation, thus being of the form (3.11). The determination of the corresponding polynomial P can be done with the aid of the Harish-Chandra isomorphism ([**9**] Lemma 2.3.4). If $Z \in Z(\mathfrak{g})$ we have

$$Zf(gp_0) = L_{g^{-1}}Zf(p_0) = Z(L_{g^{-1}}f)(p_0). \tag{3.12}$$

On the other hand $Z = Y + Q$, where $Y \in \mathfrak{U}(\mathfrak{a})$ and Q belongs to the ideal $\mathfrak{U}(\mathfrak{g})\mathfrak{n}$. Since the elements of the ideal vanish on $\mathcal{E}(L)(= \mathcal{E}(G/N))$ we obtain $Zf = Yf$.

Let us identify $\mathfrak{a} \cong \mathbf{R}$ by chosing d/dt as a base in $\mathfrak{a}$. The element Y considered as a polynomial on $\mathfrak{a}^* \cong \mathbf{R}$ is mapped into a symmetric polynomial under the application γ defined by:

$$\gamma p(x) := \rho\left(x - \frac{1}{2}\right), \quad x \in \mathbf{R} \quad ([\mathbf{9}]\text{ p. }168).$$

Let us consider $Y_0 := \gamma^{-1}(x^2)$ and let $Z_0 = Y_0 + Q \in Z(\mathfrak{g})$. Then

$$Y_0 = \left(\frac{d}{dt} + \frac{1}{2}\right)^2 =: X^2.$$

The differential operator

$$X = \frac{d}{dt} + \frac{1}{2}$$

assigns to a function $\varphi \in \mathcal{E}(\mathbf{R})$ the element

$$e^{-t/2}\frac{d}{dt}e^{t/2}\varphi.$$

In virtue of (3.12) and the last statement we obtain

LEMMA 3.6. *For any $f \in \mathcal{E}(L)$ the function*

$$\begin{aligned}(3.13) \qquad L \ni gp_0 \longrightarrow Z_0 f(gp_0) = Y_0 f(gp_0) &= X^2(L_{g^{-1}}f)^A \\ &= e^{t/2}\frac{d^2}{dt^2}e^{t/2}(L_{g^{-1}}f)^A(0)\end{aligned}$$

belongs to $V(f)$.

Let us note that, assuming the convergence of the series in question we can write for $\varphi \in \mathcal{E}(\mathbf{R})$:

$$\begin{aligned}(3.14) \qquad &\left(2\sum_{k=0}\frac{t^{2k}}{(2k)!}e^{-t/2}\frac{d^{2k}}{dt^{2k}}e^{t/2}\varphi\right)(0) \\ &= e^{-t/2}\left(\sum_{k=0}\frac{t^k}{k!}D_0^k + \sum_{k=0}\frac{(-t)^k}{k!}D_0^k\right)e^{t/2}\varphi \\ &= e^{-t/2}\left((\tau(t) + \tau(-t))e^{t/2}\varphi\right)(0) \\ &= e^{t/2}\varphi(t) + e^{-t/2}\varphi(-t).\end{aligned}$$

If we apply the formula to the function $(L_{g^{-1}} f)^A$, (the convergence of the series is assured by the supposition of f being A-analytic) the right hand side becomes the function φ and the left hand side as a function of the variable $g p_0$ belongs to $V(f)$ in virtue of the Lemma 3.6. The proof follows.

As a corollary we obtain

THEOREM 3.7. *Let* $f \in \mathcal{E}(L)$ *be an* A*-analytic function. Then the space* $(e^{t/2}V(f))^A \subset \mathcal{E}(\mathbf{R})$ *is an* $\mathcal{E}_s'(\mathbf{R})$*-convolution module.*

Proof. If $T \in \mathcal{E}'(\mathbf{R})$ is symmetric then

$$T_t(e^{t/2} f(ga(t)p_0) + e^{-t/2} f(ga(-t)\rho_0)) = 2T(e^{t/2}(L_{g^{-1}} f)^A).$$

As a function of $\rho = g\rho_0$ this element belongs to $V(f)$ according to Proposition 3.5. The function

$$R \ni s \longrightarrow T_t(e^{t/2} f(a(s+t)\rho_0)) = e^{-s/2} T * (e^{t/2} f)^A(s)$$

belongs then to $V(f)^A$. This ends the proof.

Corollary 3.4 and Theorem 3.7 suggest the following

Conjecture. If $V \subset \mathcal{E}(L)$ is closed and G-invariant then $e^{t/2}V^A$ is an $\mathcal{E}_s'(\mathbf{R})$-convolution module.

4. Spectral analysis for $\mathcal{E}_s'(\mathbf{R})$-modules. The spectral analysis theorem of L. Schwartz describes translation invariant closed subspaces in $\mathcal{E}(\mathbf{R})$ as the spaces generated by functions of the form $x^m e^{i\lambda x}, \lambda \in \mathbf{C}, m \in \mathbf{N}$. A closed subspace $V \subset \mathcal{E}(\mathbf{R})$ is translation invariant if and only if it is an $\mathcal{E}'(\mathbf{R})$ convolution module.

Let us consider

$$(4.1) \qquad V^{\perp} := \{T \in \mathcal{E}'(\mathbf{R}) | \langle T, \varphi \rangle = 0, \varphi \in V\}.$$

The annihilator space $V^{\perp}$ is an ideal in the convolution algebra $\mathcal{E}'(\mathbf{R})$. By the Hahn-Banach theorem

$$(4.2) \qquad V = (V^{\perp})^{\perp} := \{\varphi \in \mathcal{E}(\mathbf{R}) | \langle T, \varphi \rangle = 0, T \in V\}.$$

The problem of describing the closed $\mathcal{E}'(\mathbf{R})$-modules reduces to the description of ideals of $\mathcal{E}'(\mathbf{R})$.

In this section we shall denote by $\hat{f}$ the classical Fourier transform.

The space $J := (V^{\perp})\hat{}$ forms an ideal in the algebra $A = (\mathcal{E}'(\mathbf{R}))\hat{}$. The theorem of L. Schwartz characterizes the ideals of A in terms of its spectrum. Given an ideal $J \subset A$ we denote:

$$\mathrm{Sp}\, J := \{\lambda \in \mathbf{C} | \psi(\lambda) = 0, \psi \in J\}.$$

If $\lambda \in \operatorname{Sp} J$ we denote by $m(\lambda)$ the multiplicity of λ, that is the maximal natural number such that all elements of J have in λ zero of order $\leqq m(\lambda)$.

THEOREM 4.1. [**8**] (Spectral synthesis theorem) *Each ideal $J \subset A$ is uniquely determined by the set of pairs $(\lambda, m(\lambda)), \lambda \in \operatorname{Sp} J$.*

If particular the theorem states that $V \neq 0$ if and only if $\operatorname{Sp} J \neq 0$ for $J := (V^{\perp})\hat{}$, V being a closed $\mathcal{E}'(\mathbf{R})$-module.

A simple calculation proves that $\lambda \in \operatorname{Sp} J$ with multiplicity $\geqq m(\lambda)$ if and only if the function $x^m e^{i\lambda x}$ belongs to V for every $m \leqq m(\lambda)$. The spectrum of V is defined as the spectrum of $(V^{\perp})\hat{}$. In this way one obtains the version of the Schwartz theorem mentioned at the beginning of the section:

THEOREM 4.1$'$. *Every translation invariant and closed subspace $V \subset \mathcal{E}(\mathbf{R})$ is generated by all functions of the form*

$$x^m e^{i\lambda x}, \quad m \in \mathbf{N} \cup \{0\}, \quad \lambda \in \mathbf{C}$$

contained in V. If $V \neq \mathcal{E}(\mathbf{R})$ then $\operatorname{Sp} J$ is a countable set without accumulation points.

The space $\mathcal{E}_s'(\mathbf{R})$ of all even distributions forms a subalgebra of $\mathcal{E}'(\mathbf{R})$ whose Fourier transforms is the algebra A_s of all even elements in A. The spectral synthesis theorem for A_s was proved in [**4**].

THEOREM 4.2. *Let J be an ideal in A_s and let I be the ideal generated in A by J. Then $J = I \cap A_s$. In particular J is uniquely determined by the set of pairs $(\lambda, m(\lambda)), \lambda \in \operatorname{Sp} J = \operatorname{Sp} I$.*

This theorem permits us to describe the $\mathcal{E}_s'(\mathbf{R})$-modules in $\mathcal{E}_s(\mathbf{R})$. Let $V \subset \mathcal{E}_s(\mathbf{R})$ be such a module. We define

$$(4.3) \qquad V^{\pm} := (V^{\perp}) \cap \mathcal{E}_s'(\mathbf{R}) \quad \text{and} \quad (V^{\pm})^{\pm} := (V^{\pm})^{\perp} \cap \mathcal{E}_s(\mathbf{R}).$$

Again, by the Hahn-Banach theorem we have

$$(4.4) \qquad V = (V^{\pm})^{\pm}.$$

Let $T \in V^{\pm}$ and let λ_0 be zero of order m of the transform $\hat{T}$, that is let

$$\frac{d^k}{d\lambda^k}\hat{T}(\lambda_0) = 0 \quad \text{for all } k \leqq m.$$

Then we have

$$0 = \frac{d^k}{d\lambda^k} T_x(e^{i\lambda x})|_{\lambda=\lambda_0} = T_x((ix)^k e^{i\lambda_0 x}) \quad \text{for all } k \leqq m,$$

and by the symmetry the same holds for $-\lambda_0$. In virtue of (4.4) we deduce that $\lambda \in \mathrm{Sp}(V^{\perp})\hat{}$ with the multiplicity $m(\lambda)$ if and only if the symmetric parts of the functions $x^k e^{i\lambda x}$ and $x^k e^{-i\lambda x}$ belong to V for all $k \leqq m(\lambda)$.

By applying Theorem 4.2 we get

THEOREM 4.2′. *Each closed $\mathcal{E}_s'(\mathbf{R})$-module $V \subset \mathcal{E}_s(\mathbf{R})$ is generated by all functions of the form*

$$x^k \operatorname{ch} \lambda x, \lambda \in \mathrm{Sp}\, V, k\text{-even and less or equal to } m(\lambda) \text{ and}$$

$$x^k \operatorname{sh} \lambda x, \lambda \in V, k\text{-odd and less or equal to } m(\lambda).$$

$\mathrm{Sp}\, V$ is countable and has no accumulation points.

The last result solves the problem of spectral synthesis in the space $\mathcal{E}(K\backslash H)$ (see [**1**] or [**11**]). In order to obtain at least the spectral analysis in the space $\mathcal{E}(L)$ we need information about the $\mathcal{E}_s'$-modules in the whole space $\mathcal{E}(\mathbf{R})$.

In the sequel we denote by A_a, $\mathcal{E}_a(\mathbf{R})$, $\mathcal{E}_a'(\mathbf{R})$ the subspaces of all alternating elements in A, $\mathcal{E}(\mathbf{R})$ and $\mathcal{E}'(\mathbf{R})$ respectively. The spaces $\mathcal{E}_s(\mathbf{R})$ and $\mathcal{E}_a(\mathbf{R})$ are $\mathcal{E}_s'$-submodules of $\mathcal{E}(\mathbf{R})$ which contain no exponentials. We can suppose that on studying $\mathcal{E}_s'$-modules we should rather consider the functions of the form

$$ax^k e^{\lambda x} + bx^m e^{-\lambda x}$$

as elementary functions on $\mathbf{R}$.

At the beginning let us assume that $V \subset \mathcal{E}_a(\mathbf{R})$ is a closed $\mathcal{E}_s'(\mathbf{R})$-module. The operator of derivation $D : f \longrightarrow f'$ is continuous, commutes with the convolution and maps $\mathcal{E}_s(\mathbf{R})$ onto $\mathcal{E}_a(\mathbf{R})$ with kernel consisting of constant functions. The space

$$W := (D^{-1}V) \cap \mathcal{E}_s(\mathbf{R})$$

is closed and is a $\mathcal{E}_s'(\mathbf{R})$-submodule of $\mathcal{E}_s(\mathbf{R})$ hence is generated by the functions

$$x^{2k} \operatorname{ch} \lambda x, \quad \mathrm{Sp}\, W, 2k \leqq m(\lambda) \text{ and}$$
$$x^{2k+1} \operatorname{sh} \lambda x, \quad \mathrm{Sp}\, W, 2k+1 \leqq m(\lambda).$$

The module V is then generated by the derivatives of the above functions. If $\lambda \in \mathrm{Sp}\, W$ and $\lambda \neq 0$ the multiplicity of λ in V understood as the highest power of the variable x appearing as a factor in this set of generating functions is equal to max $(0, m(\lambda))$. The multiplicity of $\lambda = 0$ in V is equal to $\max(0, m(0))$.

We have in this way a counterpart of Theorem 4.2′:

PROPOSITION 4.3. *If $V \subset \mathcal{E}_a(\mathbf{R})$ is a closed $\mathcal{E}_s(\mathbf{R})$-module then V is generated by all functions of the form*

$$x^2 \operatorname{sh} \lambda x \quad \text{and} \quad x^{2k-1} \operatorname{ch} x$$

contained in V. If $V \neq \mathcal{E}_a(\mathbf{R})$ then the set of λ's for which some of above functions appear in V is countable and has no accumulation points.

Passing to the general case, let V be a nontrivial, closed $\mathcal{E}_s'$-module in $\mathcal{E}(\mathbf{R})$ and $V^\perp \subset \mathcal{E}_s'(\mathbf{R})$ its annihilator which is also an $\mathcal{E}_s'$-module. The space $J := (V^\perp)\hat{} \subset A$ is then an A_s-module. We shall consider now the possible relations of J to A_s and A_a.

Case 1. If $J \subset A_s$ then by Theorem 4.2 it is determined uniquely by $\operatorname{Sp} J$ and the multiplicities. The elements of $V^\perp$ annihilate the whole space $\mathcal{E}_a(\mathbf{R})$ as well as the functions:

$$x^k \operatorname{ch} \lambda x, \lambda \in \operatorname{Sp} V, k + 2m \leqq m(\lambda)$$
$$x^k \operatorname{sh} \lambda x, \lambda \in \operatorname{Sp} V, k = 2m + 1 \leqq m(\lambda).$$

In this case $V = \mathcal{E}_a(\mathbf{R}) + (V^\perp)^\perp = \mathcal{E}_a(\mathbf{R}) + (V \cap \mathcal{E}_s(\mathbf{R}))$.

Case 2. Assume $J \subset A_a$. Then we have $V^\perp \subset \mathcal{E}_a'(\mathbf{R})$ and

$$V = (V^\perp)^\perp = \mathcal{E}_s(\mathbf{R}) + (V \cap \mathcal{E}_a(\mathbf{R})).$$

Applying Proposition 4.3 to the module $V \cap \mathcal{E}_a(\mathbf{R})$ we get:

The module V is generated by $\mathcal{E}_s(\mathbf{R})$ and by all functions of the form $x^{2k-1} \operatorname{ch} \lambda x$ and $x^{2k} \operatorname{sh} \lambda x$ contained in V.

From now we can assume that neither $(J \subset \mathcal{A}_s)$ nor $J \subset A_a$. Let us consider the subspaces:

$$W_s := \{f \in A_s | \exists \varphi \in J \quad f(\lambda) = \varphi(\lambda) + \varphi(-\lambda)\},$$
$$W_a := \{f \in A_a | \exists \varphi \in J \quad f(\lambda) = \varphi(\lambda) - \varphi(-\lambda)\}.$$

The space W_s is an ideal in A_s and W_a is an A_s- submodule of A_a. According to our assumption $W_s \neq 0$ and $W_a \neq 0$.

Case 3. Suppose $W_s \neq A_s$. According to Theorem 4.2 $\operatorname{Sp} W_s$ contains at least one point, say $\lambda_0 \in \mathbf{C}$. Then for any $T \in V^\perp$ one has

$$0 = \hat{T}(\lambda_0) + \hat{T}(-\lambda_0) = T_x(\operatorname{ch} \lambda_0 x).$$

It means that the function $\operatorname{ch} \lambda_0 x$ belongs to V.

Case 4. Let $W_a \neq A_a$. There exist an ideal $I \subset A_s$ such that

$$W_a := \{\psi | \psi(\lambda) = \lambda\varphi(\lambda), \varphi \in I\}.$$

Since I is nontrivial we can choose $\lambda_0 \in \operatorname{Sp} I$. Supposing $\lambda_0 \neq 0$ we obtain for every $T \in V^\perp$:

$$0 = \frac{1}{\lambda_0}(\hat{T}(\lambda_0) - \hat{T}(-\lambda_0)) = \frac{2}{\lambda_0} T_x(\operatorname{sh} \lambda_0 x)$$

which implies that $\operatorname{sh} \lambda_0 x \in V$.

If $\operatorname{Sp} I = \{0\}$ and $m(0) = 2k > 0$ then

$$W_a = \{\lambda^{2k+1}\varphi | \varphi \in A_s\}.$$

This being the case we have for all $T \in V^{\perp}$:

$$0 = \frac{d^{2k-1}}{d\lambda^{2k-1}}(\hat{T}(\lambda) - \hat{T}(-\lambda))|_{\lambda=0} = 2T_x((ix)^{2k-1})$$

which means that $x^{2k-1} \in V$.

(Note that in cases 3 and 4 if zero is the unique element of the spectrum then V is generated by polynomials.)

Case 5. It remains to consider the case $W_a = A_a$ and $W_s = A_s$. Let us define a relation R in $A_s \times A_a$ by the formula:

$$xRy \quad \text{if} \quad x + y \in J.$$

If xRy and xRy' then

$$y - y' \in J \cap A_a =: J_a.$$

We can lift the relation to $A_s \times (A_a/J_a)$ by putting

$$x\tilde{R}[y] \quad \text{if} \quad xRy.$$

This relation is an application commuting with the multiplication by elements of $A_s : \tilde{R}(fx) = f\tilde{R}x$.

Since the algebra A_s contains the unity we obtain:

$$\tilde{R}x = \tilde{R}(x \cdot 1) = x\tilde{R}(1).$$

Let us fix a representant y_0 of the class $\tilde{R}(1)$. Then one has

$$J = \{x + xy_0 + J_a | x \in A_s\}.$$

If in particular $y = x + xy_0 + j, j \in J_a$ then

$$x(\lambda) = \frac{1}{2}(y(\lambda) + y(-\lambda)).$$

Denoting $\check{y}(\lambda) := y(-\lambda)$ we have for every $y \in J$:

$$y - \frac{1}{2}(y + \check{y})(y_0 + 1) = (1 + y_0)y + (1 - y_0)\check{y} \in J_a.$$

Case 5A. If $J_a = A_a$ then $J + A$ which corresponds to the case $V = 0$ which was excluded.

Case 5B. Suppose $J_a = 0$. This being the case the elements of J are of the form $y = x(1 + y_0), x \in A_s$. If $1 + y_0$ is everywhere distinct from zero then the holomorphic and alternating function y_0 does not take the values 1 and -1. In virtue of the Picard theorem y_0 is constant hence zero and we obtain $J = A_s$ once again.

Case 5C. Finally we suppose $A_a \neq J_a \neq 0$. Let I be the ideal in A_s such that all elements of J_a are of the form $y(\lambda) = \lambda f(\lambda)$, $f \in I$. For any $\lambda_0 \in \operatorname{Sp} I$ and $T \in V^{\perp}$ we obtain:

$$\begin{aligned}0 =&(1 + y_0(\lambda_0))\hat{T}(\lambda_0) + (1 - y_0(\lambda_0))\hat{T}(-\lambda_0)\\ =&T_x((1 - y_0(\lambda_0))e^{i\lambda_0 x}) + (1 + y_0(\lambda_0))e^{-\lambda_0 x)}.\end{aligned}$$

This implies that the function

$$ae^{i\lambda_0 x} + be^{-i\lambda_0 x}$$

belongs to V for $a = 1 - y_0(\lambda_0)$ and $b = 1 + y_0(\lambda_0)$. The function is nontrivial for each λ_0.

All cases can be summarized in the following way:

THEOREM 4.4 *If V is a nontrivial closed $\mathcal{E}_s(\mathbf{R})$-module in $\mathcal{E}(\mathbf{R})$ then there exists $\lambda \in \mathbf{C}$ such that the function*

$$ae^{i\lambda x} + be^{-i\lambda x}$$

is nontrivial and belongs to V, or the identity function $f(x) = x$ belongs to V. If the module V does not contain any function of the first kind then V is finite-dimensional and is generated by polynomials.

5. Spectral analysis theorems for $\mathcal{E}(L)$. In this section we apply Theorems 4.1–4 to obtain results about spectral analysis and in several cases even spectral synthesis in the space $\mathcal{E}(L)$.

First let us distinguish a particular invariant subspace $\mathcal{N} \subset \mathcal{E}(L)$. Let

$$\mathcal{N} := \{f \in \mathcal{E}(L) | \int_K f(gkp)dk = 0, \quad g \in G, \quad p \in L\}.$$

This is the maximal invariant subspace of $\mathcal{E}(L)$ which does not contain any K-fixed elements. Let us note that in the case of the manifold H such a subspace is trivial. This fact is the reason of principal differences between the analysis on H and L. By the very definition it's clear that $\mathcal{N}$ is closed, G-invariant and moreover invariant with respect to the operators $\tau(t), t \in \mathbf{R}$. Since the regular representation of G commutes with τ the subspaces $\mathcal{N}_q \subset \mathcal{N}$ are also invariant. By Theorem 4.1′ we obtain:

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LEMMA 5.1. *The space* $\mathcal{N}_q^A$ *is uniquely determined by its spectrum* $\operatorname{Sp}\mathcal{N}_q^A$ *and the corresponding multiplicity function* $m_q(\lambda)$. $\mathcal{N}_q^A$ *is linear generated by the functions of the form*

$$t^m e^{i\lambda t}, \quad \lambda \in \operatorname{Sp}\mathcal{N}_q^A, \quad m \leqq m_q(\lambda).$$

According to the lemma the functions $e_q^{i\lambda}$ belong to $\mathcal{N}$ for each $\lambda \in \operatorname{Sp}\mathcal{N}_q^A$. Now, the integral equation defining $\mathcal{N}$ can be represented in the following way:

$$0 = \int_K L_{g^{-1}} e_q^{i\lambda}(kp_0)dk = (U^{i\lambda}_{g^{-1}}\tilde{e}_q|\tilde{e}_0) = (\tilde{e}_q|U^{1i\lambda-1}_{g^{-1}}\tilde{e}_0)$$

for all $g \in G$. The space generated by the vectors

$$U_g^{-i\lambda-1}e_0, \quad g \in G$$

differs from the whole space if and only if $-i\lambda - 1 = n \in \mathbf{N}$ (Proposition 3.1).

Thus $\mathcal{N}_q \cap \mathcal{E}_{i\lambda} \neq 0$ if and only if $i\lambda = -n - 1$ and in this case $n + 1 \leqq |q|$ according to Proposition 3.1.3. We have obtained

PROPOSITION 5.2. *The space* $\mathcal{N}_q, q \in \mathbf{Z}$ *is finite-dimensional and linearly generated by the functions of the form:*

$$\psi_{q,n,s}(k(\theta)a(t)p_0) := t^s e^{-nt+iq\theta} \quad \textit{with}$$
$$n \in \mathbf{N}, \quad n \leqq |q| \quad \textit{and} \quad s \leqq m_q(-n).$$

In particular this means that the representation of G in $\mathcal{N}$ is admissible and the character $e^{iq\theta}$ occurs in $\mathcal{N}$ with finite multiplicity which is less or equal to

$$\sum_{n=1}^{q} m_q(-n).$$

The calculation of the multiplicities $m_q(\lambda)$ $\quad q \in \mathbf{Z}$ is an open problem.

Now, consider a closed and G-invariant subspace $V \subset \mathcal{N}$. According to Lemma 3.6 the finite-dimensional spaces $e^{t/2}\mathcal{N}_q^A$ are invariant with respect to the operator d^2/dt^2. The eigenvectors of this operator in the space $e^{t/2}\mathcal{N}_q^A$ are the functions

$$e^{(-n+1/2)t}, \quad n \leqq |q|$$

and the elements of the Jordan base of the operator are of the form

$$P(t)e^{(n+1/2)t},$$

where P is a polynomial of order less than or equal to $m_q(-n)$. This leads to

PROPOSITION 5.3. *Every closed and G-invariant subspace $V \subset \mathcal{N}$ is linearly generated by combinations of the functions $\psi_{q,n,s}$ contained in V.*

Now, our purpose is to prove a spectral analysis theorem in $\mathcal{E}(L)$ hence, after obtaining even the spectral synthesis for subspaces of $\mathcal{N}$ it is sufficient to consider invariant subspaces in $\mathcal{E}(L)$ whose intersection with $\mathcal{N}$ are trivial. If $V \neq 0$ but $\mathcal{N} \cap V = 0$ we have $V_0 \neq 0$. In virtue of Corollary 3.4 the space $V := e^{t/2}V_0^A$ is an $\mathcal{E}_s'(\mathbf{R})$-module. By Theorem 4.4 the space V contains for some λ the function

$$ae^{i\lambda t} + be^{-1\lambda t} \quad \text{with} \quad |a|^2 + |b|^2 \neq 0,$$

or the function $f(t) = t$.

The space V_0 then contains the element

$$\psi(k(\theta)a(t)\rho_0) := ae^{(i\lambda-1/2)t} + be^{-(i\lambda+1/2)t}$$

or the function

$$\varphi(k(\theta)a(t)\rho_0) := te^{-i/2}.$$

The first function can be written as

$$ae_{i\lambda-1/2,x_0} + be_{-(i\lambda+1/2),x_0}$$

and the second as

$$e^{\langle 1\rangle}_{-1/2,x_0} := \left.\frac{\partial}{\partial\lambda}\right|_{\lambda=-1/2} e_{\lambda,x_0}.$$

Taking into account the G-invariance of V we obtain:

THEOREM 5.3. *If $V \subset \mathcal{E}(L)$ is nontrivial, closed and G-invariant but $V \cap \mathcal{N} = 0$ then either*

$$e^{\langle 1\rangle}_{-1/2,x} \in V \quad \textit{for all } x \in H,$$

or there exist $\lambda, a, b, \in \mathbf{C}$ such that $|a|^2 + |b|^2 > 0$ and for all $x \in H$ the function

$$ae_{i\lambda-1/2,x} + be_{-(i\lambda+1/2),x}$$

belongs to V.

THEOREM 5.4. *Each G-invariant and closed subspace $V \subset \mathcal{E}(L)$ such that $V \cap \mathcal{N} \neq 0$ contains for some $n \in \mathbf{N}$ at least one of the functions*

$$\psi^{\pm}(k(\theta)a(t)\rho_0) = e^{\pm in\theta - nt}.$$

Proof. According to Proposition 5.2 we have in $V \cap \mathcal{N}$ a function of the form

$$\psi_{q,n,0}(k(\theta)a(t)\rho_0) = e^{iq\theta - nt}$$

such that $n \leqq |q|$. This element belongs to F_{-n}^+ or F_{-n}^- depending on the sign of q. Both spaces are carrier spaces of irreducible representations of G hence together with the above element the whole space F_{-n}^+ or F_{-n}^- belongs to V. In particular ψ_n^+ or ψ_n^- belongs to V.

Theorems 5.3 and 5.4 mean that the spectral analysis holds in the space $\mathcal{E}(L)$ with respect to the elemental family given by the collection of all functions

$$\psi_n^{\pm}, \quad n \in \mathbf{N}; \quad e_{-1/2,x}^{\langle 1 \rangle}, \quad x \in H$$

and the functions

$$ae_{i\lambda - 1/2,x} + be_{-\langle i\lambda + 1/2 \rangle,x}, \quad x \in H, |a|^2 + |b|^2 > 0.$$

Nevertheless, some additional information can be deduced with the aid of Theorems 3.7 and 4.4. Namely, we obtain

PROPOSITION 5.5. *Let $V \subset \mathcal{E}(L)$ be a closed and G-invariant subspace. Assume that for some $n \in \mathbf{Z}$ the space V_n contains some A-analytic function. Then V_n contains a function of the form*

$$\varphi_n(k(\theta)a(t)\rho_0) := te^{in\theta - t/2}$$

or the function

$$ae_n^{\langle i\lambda - 1/2 \rangle} + be_n^{-\langle i\lambda + 1/2 \rangle}, \quad |a|^2 + |b|^2 \neq 0.$$

6. Comments and applications. The space $\mathcal{E}(L)$ is mapped under the dual Radon transform (2.7) continuously into $\mathcal{E}(H)$. Since the mapping B commutes with the group action, the space $\ker B$ is closed and G-invariant. By the very definition a function f belongs to $\mathcal{N}$ if and only if for each $t \in \mathbf{R}$ one has $\tau(t)f \in \ker B$. In particular $\mathcal{N} \subset \ker B$ since $\mathcal{N}$ is τ-invariant. In the sequel we shall see that $\mathcal{N} \neq \ker B$. This implies that $\ker B$ constitutes an example of a G-invariant subspace which is not τ-invariant.

Let

$$\varphi_\lambda(gk) := Be_{i\lambda - 1/2,x_0}(gx_0) = \int_K \langle x_0 | gkp_0 \rangle^{i\lambda - 1/2} dk.$$

The function φ_λ is a zonal spherical function on H. It is known [**5**] that $\varphi_{-\lambda} = \varphi_\lambda$ for each $\lambda \in \mathbf{C}$. In particular it means that

$$e_{i\lambda - 1/2,x_0} - e_{-i\lambda - 1/2,x_0} \in \ker B.$$

By the G-invariance of $\ker B$ and the formula (2.1) the same is true for an arbitrary point $x \in H$. By applying the derivation with respect to λ one obtains

$$e^{\langle 1 \rangle}_{i\lambda - 1/2, x} \in \ker B \quad \text{for all } \lambda \in \mathbf{C} \text{ and } x \in H.$$

All these functions do not belong to $\mathcal{N}$, hence $\mathcal{N} \neq \ker B$. At the same time we observe that in $\ker B$ appear elementary functions of all frequencies $\lambda \in \mathbf{C}$.

On the Pompeiu problem on L. Let $A \subset \mathbf{R}^n$ be a compact set and let $f \in L^\infty(\mathbf{R}^n)$. The Pompeiu problem consists in asking if the condition:

$$\int_{gA} f(x)dx = 0 \quad \text{for all rigid motions } g$$

implies $f = 0$ almost everywhere on $\mathbf{R}^n$. If the answer is positive one says that A has the Pompeiu property. The problem is equivalent to the question if the translations of the characteristic function 1_A span a dense subset in $L^1(\mathbf{R}^n)$. By the Tauberian theorem of Wiener the set A has the Pompeiu property if and only if the Fourier transform $\hat{1}_A$ is everywhere different from zero.

Various generalizations of the Pompeiu problem were considered on $\mathbf{R}$ and on symmetric spaces of rank one ([**1**], [**2**], [**3**], [**6**], [**7**].)

The results of the last section permit us to formulate a theorem of Pompeiu type for the manifold L.

THEOREM 6.1. *Let $T \in \mathcal{E}'(L)$. The system of equations*

$$T(L_g f) = 0, \quad \text{for each } g \in G$$

has in $\mathcal{E}(L)$ only the trivial solution $f \equiv 0$ if and only if the following conditions are satisfied:

a) *For every $\lambda \in \mathbf{C}$ the functions*

$$H \ni x \longrightarrow T(e_{i\lambda - 1/2, x}) \quad \text{and} \quad H \in x \longrightarrow T(e_{-\langle i\lambda + 1/2 \rangle, x})$$

are linearly independent.

b) $T(e^{\langle 1 \rangle}_{-1/2, x}) \neq 0$ *for some $x \in H$.*

c) *For every $n \in \mathbf{N}$ there exists $Q \in \mathbf{Z}$ such that $|q| \geqq n$ and*

$$T(\psi_{q,n,0}) \neq 0.$$

Proof. Let us denote

$$V := \{f \in \mathcal{E}(L) | T(L_g f) = 0, g \in G\}.$$

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The space V is closed and G-invariant. If $V \neq 0$ then according to Proposition 5.2 and Theorem 5.3 one of the following conditions is satisfied:

$$1) \quad ae_{i\lambda-1/2,x} + be_{-\langle i\lambda+1/2\rangle,x} \in V \quad \text{for some } \lambda \in \mathbf{C},$$
$$|a|^2 + |b|^2 \neq 0 \quad \text{and all } x \in H,$$
$$2) \quad e^{\langle 1\rangle}_{-1/2,x}V, \quad \text{for all } x \in H,$$

or

3) $V \cap \mathcal{N} \neq 0$ and consequently F^+_{-n} or F^-_{-n} belongs to V for $n \in \mathbf{N}$.

In the first case the condition a) is not satisfied: the case 2) contradicts b) and the case 3) contradicts c). Then the conditions a) b) c) are sufficient.

Now, if a) is not satisfied then for some $a, b \in \mathbf{C}$ and $\lambda \in \mathbf{C}$ we have $(a, b) \neq (0, 0)$ and

$$T(ae_{i\lambda-1/2,x} + be_{-\langle i\lambda+1/2\rangle,x}) = 0 \quad \text{for all } x \in H.$$

The nontrivial and G-invariant space spanned by the functions in parenthesis belong to V.

If b) is not satisfied then the function

$$e^{\langle 1\rangle}_{-1/2,x}$$

is a solution of the system in question.

If for some $n \in \mathbf{N}$ and all $q \in \mathbf{Z}$ such that $q \geqq n$ (or all q such that $q \leqq -n$) we have

$$T(\psi_{q,n,0}) = 0$$

then the space F^+_{-n} (or F^-_{-n}) belongs to V. This ends the proof.

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