BOUNDARIES FOR THE EQUIPOTENTIAL CURVES IN THE ELLIPTIC RESTRICTED THREE BODY PROBLEM

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ABSTRACT

In the elliptic restricted three body problem an invariant relation between the velocity square of the third body and its potential is studied for long time intervals as well as for different values of the eccentricity. This relation, corresponding to the Jacobian integral in the circular problem, contains an integral expression which can be estimated if one assumes that the potential of the third body remains finite. Then upper and lower boundaries for the equipotential curves can be derived. For large eccentricities or long time intervals the upper boundary increases, while the lower decreases, which can be interpreted as shrinking respectively growing zero velocity curves around the primaries.

1. INTRODUCTION

In a paper by Szebehely and Giacaglia (1964) the equations of motion for the infinitesimal third body in the elliptic restricted three body problem are presented in the same analytical form as for the circular case. From these equations, they derived an invariant relation for the velocity square, which corresponds to the Jacobian integral in the circular problem. In our paper, using this relation, we search for boundaries for the equipotential curves, which limit the motion of the third body to well defined parts of the plane.

The problem is described in a synodic, pulsating, barycentric coordinate system (ξ,η); the primaries are situated on the ξ axis and their masses are taken to be $(1-\mu)$ und μ (Fig.1). Starting from the usual synodic barycentric coordinate system (ξ^*,η^*), the pulsating system (ξ,η) is obtained, when the variable distance r between the primaries:

$$r = \frac{a(1-e^2)}{1+e\cos f}$$

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(a, e and f being the semimajor axis, the eccentricity and the true anomaly of the elliptic orbit) is chosen to be the unit of length.

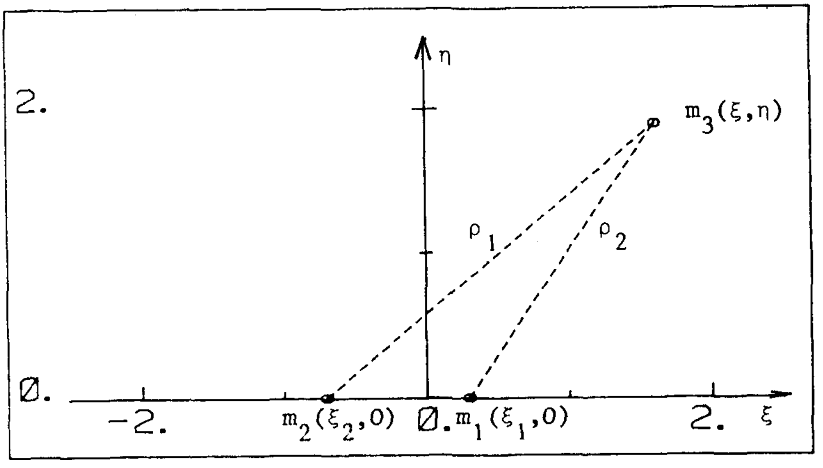

Fig. 1.: Synodic, pulsating barycentric coordinate system (ξ,η).

This results in fixed abscissae for these bodies:

$$\xi_1 = \xi_1^*/r = \mu$$
$$\xi_2 = \xi_2^*/r = -(1-\mu) \tag{1.1}$$

A transformation to the true anomaly as the independent variable leads to the desired form for the equations of motion for the third body (primes denote derivatives with respect to the true anomaly f):

$$\xi'' - 2\eta' = \frac{\partial\omega}{\partial\xi}$$
$$\eta'' + 2\xi' = \frac{\partial\omega}{\partial\eta} \tag{1.2}$$

The function ω depends on the potential function Ω:

$$\Omega\ (\xi(f),\eta(f)) = \frac{1}{2}\left[(1-\mu)\ \rho_1{}^2 + \mu\rho_2{}^2\right] + \frac{1-\mu}{\rho_1} + \frac{\mu}{\rho_2}$$

$$\omega(\xi,\eta,f) = \frac{\Omega(\xi(f),\eta(f))}{1+e\ \cos\ f} \tag{1.3}$$

where ρ_1,ρ_2 are the normed distances from the third body to the primaries:

$$\rho_1{}^2 = (\xi-\xi_1)^2 + \eta^2 \tag{1.4}$$

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$$\rho_2^{\;2} = (\xi-\xi_2)^2 + \eta^2$$

The invariant relation for the velocity square in the pulsating system is given by the formula (Szebehely and Giacaglia, 1964):

$$\xi'^2 + \eta'^2 = 2\omega(\xi,\eta,f) - 2\int\frac{\partial\omega}{\partial f}\,df - C \qquad (1.5)$$

and corresponds to the Jacobian integral for e = 0.

It is shown by Erdi (1982) that it is possible to evaluate the integral in Equation (1.5) for the case of Trojan asteroids using an asymptotic solution for their motion. In a paper by Delva and Dvorak (1979) a series expansion was used to study an equivalent of equation (1.5).

2. ESTIMATION OF THE INTEGRAL $\int\frac{\partial\omega}{\partial f}\,df$

In equation (1.5) for the velocity square, the integral term on the right hand side causes variations of the zero velocity curves, which define the regions of motion. We now estimate boundaries for its value for long intervals of the time. For this purpose it is necessary to restrict the motion of the third body in such a way, that the value of the potential function $\Omega(\xi,\eta)$ on the orbit always remains bounded by an upper limit M(M>0) for all times:

$$\Omega(\xi,\eta) \le M \qquad (2.1)$$

or we have to exclude collisions with the primaries and very large distances from them. A lower limit for $\Omega(\xi,\eta)$ is the value in the Lagrangian points L_4 and L_5; the inequality

$$m = \Omega(\xi_{L_4},\eta_{L_4}) = \frac{3}{2} \le \Omega(\xi,\eta) \le M \qquad (2.2)$$

is valid on the orbit for all times. To be able to estimate the integral

$$I = \int\frac{\partial\omega}{\partial f}\,df = \int\Omega(\xi(f),\eta(f))\,\frac{e \sin f}{(1+e\cos f)^2}\,df \qquad (2.3)$$

we will use the inequality (2.2) and a theorem on integration (Smirnow, (1973):

if two functions $\Omega(\xi(f),\eta(f))$ and g(f) are integrable (and thus bounded) on an interval $[f_a,f_b]$ and if g(f) does not change its sign on the interval, the following inequalities hold for the integral of $(\Omega.g)$:

for $g(f) \ge 0$:

$$m\int_{f_a}^{f_b} g\,df \le \int_{f_a}^{f_b} \Omega\, g\,df \le M\int_{f_a}^{f_b} g\,df \qquad (2.4)$$

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for $g(f) \leq 0$:

$$M \int_{f_a}^{f_b} g\, df \leq \int_{f_a}^{f_b} \Omega\, g\, df \leq m \int_{f_a}^{f_b} g\, df$$

Since we are interested in long time intervals or long intervals of the true anomaly, we choose the integration interval to be an integer multiple of 2π:

$$f_a = f_o = 0 \qquad (2.5a)$$

$$f_b = f_{2n} = 2n\pi \quad , \quad n \in N$$

and a decomposition of the interval:

$$f_k = k\pi \quad , \qquad k = 0,\ldots,2n$$

$$f_o = 0 \leq f_1 \leq \ldots \leq f_k \leq \ldots \leq f_{2n} \qquad (2.5b)$$

Defining the function g(f) by the equation

$$g(f) = \frac{e \sin f}{(1+e \cos f)^2} \qquad (2.6)$$

it is positive or zero on all intervals $[f_{2j},f_{2j+1}]$ and negative or zero on $[f_{2j+1},f_{2j+2}]$, $j=0,\ldots,n-1$. Calculating its integral on these intervals and using (2.4), we find the inequalities:

$$m \frac{2e}{1-e^2} \leq \int_{f_{2j}}^{f_{2j+1}} \Omega\, g\, df \leq M \frac{2e}{1-e^2} \qquad (2.7a)$$

$$-M \frac{2e}{1-e^2} \leq \int_{f_{2j+1}}^{f_{2j+2}} \Omega\, g\, df \leq -m \frac{2e}{1-e^2} \qquad (2.7b)$$

Summation then gives for the total integral I (2.3):

$$-n(M-m) \frac{2e}{1-e^2} \leq I \leq n(M-m) \frac{2e}{1-e^2} \qquad (2.8)$$

3. BOUNDARIES FOR THE EQUIPOTENTIAL CURVES

The result (2.8) is used to study the possible changes of the zero velocity curves with changing eccentricity and length of the integration interval. The curves are defined by the equation

$$F(\xi,\eta,f) = 2\omega - 2I - C = 0 \qquad (3.1)$$

or by the condition that

$$G(\xi,\eta,f) = 2\ \{\frac{1}{2}\ [(1-\mu)\rho_1{}^2 + \mu\rho_2{}^2] + \frac{1-\mu}{\rho_1} + \frac{\mu}{\rho_2}\} = (2I+C)\ (1+e\cos f) \tag{3.2}$$

The relation holds and is meaningful for any fixed value of f. From the inequality (2.8) and estimating the factor (1+e cos f), the curves $G(\xi,\eta,f)$ can vary within the following limits:

$$C - eC - \frac{4en(M-m)}{1+e} \le G(\xi,\eta,f) \le C + e\ C + \frac{4en(M-m)}{1-e} \tag{3.3}$$

For values of the lower boundary smaller than $2\Omega\ (\xi_{L_4},\eta_{L_4}) = 3$ no solutions for (3.2) are found, no equipotential curve will limit the motion of the third body. For any value of both boundaries greater than $2\Omega(\xi_{L_4},\eta_{L_4})$, zero velocity curves exist and define forbidden regions for the motion (Fig.2)

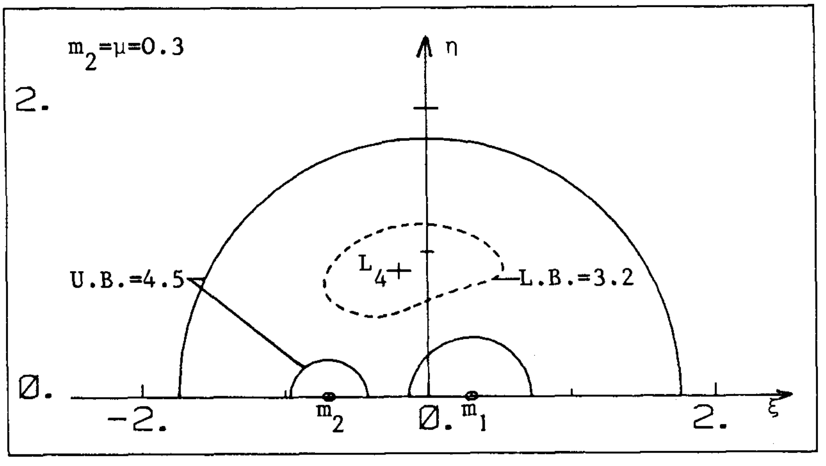

Fig.2: Equipotential curves for the lower and upper boundary of $G(\xi,\eta,f)$, if greater then $2\Omega(\xi_{L_4},\eta_{L_4})$.

The constant C is calculated from the moment with f = o:

$$C = \frac{2\Omega(\xi(0),\eta(0))}{1+e} - [\xi'(0)^2 + \eta'(0)^2] \tag{3.4}$$

3.A Boundaries for various values of the eccentricity e

In the case e=0 the inequality (3.3) reduces to the known form $G(\xi,\eta) = C$

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which defines the regions of motion once for all values of f.

In the case e≠0 and C>0, the lower boundary decreases with increasing values of e, while the upper one increases. For large values of e(e<1) the equipotential curves G(ξ,η,f) can vary within broad limits, defining small or no forbidden regions for the lower and large ones, becoming more and more closed around the primaries, for the upper boundary (Fig.3). It is clear that, if the body trespasses the restriction of inequality (2.1), these regions of motion are not valid and no predictions on their long time behaviour are allowed.

We conclude that from a theoretical point of view and within the restrictions (2.1) for the motion, the larger the eccentricity is, the larger variations in the regions of motion may occur, permitting more possibilities for the orbit of the third body.

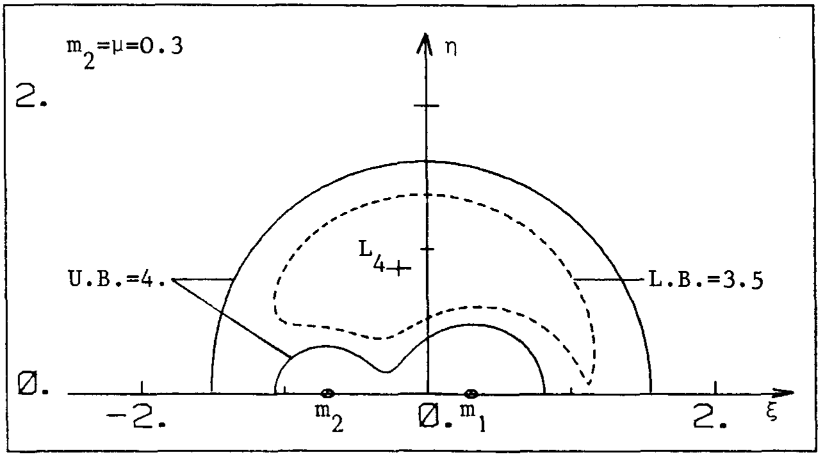

Fig.3a: Zero velocity curves for small e.

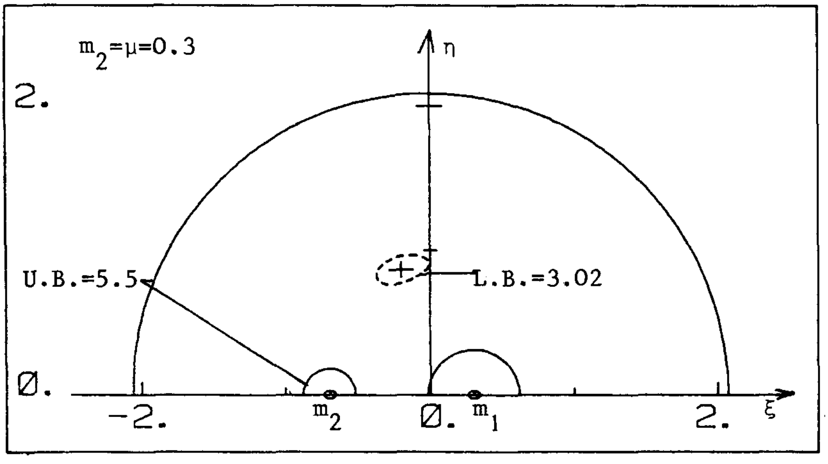

Fig.3b: Zero velocity curves for large e.

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3.B Boundaries for long time intervals

Since we chose the integration interval to have the length $n(2\pi)$, we consider the boundaries (3.3) for large values of the integer n. It is easily seen, that increasing the value of n causes the same effect as increasing the eccentricity e. The situation for long time intervals corresponds therefore to the one in Fig.3b: long time intervals can allow more possibilities of variation for the equipotential curves.

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