A NOTE ON THE THERMOELASTIC PROBLEM FOR A PENNY-SHAPED CRACK

by JOHN TWEED†

(Received 12 November, 1968)

1. The problem of determining the state of stress in the vicinity of a penny-shaped crack which is opened by thermal means has been considered by Olesiak and Sneddon [**1**]. In that paper no simple closed expressions were given either for the stress-intensity factor at the tip of the crack or for the normal component of the surface displacement. The purpose of this note is to show how such expressions may be derived.

In terms of the Hankel operator $\mathscr{H}_\nu$ defined by the equation

$$\mathscr{H}_\nu[g(\xi);\rho] = \int_0^\infty \xi g(\xi) J_\nu(\xi\rho)\,d\xi,$$

we know [**2**] that (in the usual notation) the solution of the equations of thermoelasticity in cylindrical coordinates (ρ, ϕ, z), which tends to zero as $(\rho^2+z^2)^{\frac{1}{2}} \to \infty$ and which satisfies the condition $\sigma_{\rho z}(\rho, 0) = 0$, is such that

$$u_z(\rho,0) = b\mathscr{H}_0[\xi^{-1}\psi(\xi);\rho], \tag{1}$$

$$\theta(\rho,0) = \mathscr{H}_0[f(\xi);\rho], \tag{2}$$

$$\frac{\partial\theta(\rho,0)}{\partial z} = -\mathscr{H}_0[\xi f(\xi);\rho], \tag{3}$$

$$\sigma_{zz}(\rho,0) = -\frac{bE}{2(1+\eta)\beta^2}\mathscr{H}_0[f(\xi)+2(\beta^2-1)\psi(\xi);\rho], \tag{4}$$

where $\beta^2 = 2(1-\eta)/(1-2\eta)$ and the functions $f(\xi)$ and $\psi(\xi)$ are to be determined from the boundary conditions on the boundary plane $z = 0$.

2. We now consider the problem of determining the distribution of stress in the vicinity of the penny-shaped crack $0 \leqq \rho \leqq 1$, $z = 0$ when the temperature on the crack surface is a prescribed function $q(\rho)$ of ρ. If the crack is free from stress, we have the boundary conditions

$$\theta(\rho,0) = q(\rho) \qquad (0 \leqq \rho \leqq 1), \tag{5}$$

$$\frac{\partial\theta(\rho,0)}{\partial z} = 0 \qquad (\rho > 1), \tag{6}$$

$$\sigma_{zz}(\rho,0) = 0 \qquad (0 \leqq \rho \leqq 1), \tag{7}$$

$$u_z(\rho,0) = 0 \qquad (\rho > 1), \tag{8}$$

† This paper was prepared as a part of the work of the Applied Mathematics Research Group at North Carolina State University through the Grant AF-AFOSR-444-66 and is under the joint sponsorship of AFOSR, ARO, and ONR through the Joint Services Advisory Group.

by means of which we may determine the temperature and deformation fields in the half-space $z \geqq 0$ and hence, by symmetry, in the whole space.

The pair of equations (5) and (6) is equivalent to the pair of dual integral equations

$$\mathscr{H}_0[f(\xi);\rho] = q(\rho) \quad (0 \leqq \rho \leqq 1), \qquad \mathscr{H}_0[\xi f(\xi);\rho] = 0 \quad (\rho > 1),$$

whose solution is known [3] to be given by the equation

$$f(\xi) = \xi^{-1}\int_0^1 \phi'(t)\cos(\xi t)\,dt, \tag{9}$$

where the function $\phi(t)$ is defined by the formula

$$\phi(t) = \frac{2}{\pi}\int_0^t \frac{\rho q(\rho)\,d\rho}{(t^2-\rho^2)^{\frac{1}{2}}}. \tag{10}$$

Similarly, the pair of equations (7) and (8) is equivalent to the pair of dual integral equations

$$2(\beta^2-1)\mathscr{H}_0[\psi(\xi);\rho] = -q(\rho) \quad (0 \leqq \rho \leqq 1), \qquad \mathscr{H}_0[\xi^{-1}\psi(\xi);\rho] = 0 \quad (\rho > 1),$$

whose solution is given in [3] in the form

$$\psi(\xi) = -\frac{1}{2(\beta^2-1)}\int_0^1 \phi(t)\sin(\xi t)\,dt, \tag{11}$$

where $\phi(t)$ is again given by equation (10).

Using the properties of Bessel functions of the first kind of order zero, we can easily show that

$$2(\beta^2-1)\mathscr{H}_0[\psi(\xi);\rho] = \frac{\phi(1)H(\rho-1)}{(\rho^2-1)^{\frac{1}{2}}} - \int_0^1 \frac{\phi'(t)H(\rho-t)\,dt}{(\rho^2-t^2)^{\frac{1}{2}}},$$

from which it follows that the stress intensity factor K defined by the equation

$$K = \lim_{\rho\to 1+} (\rho-1)^{\frac{1}{2}}\sigma_{zz}(\rho,0) \tag{12}$$

may be easily calculated from the formula

$$K = -\frac{Eb}{\beta^2(1+\eta)}\cdot\frac{\phi(1)}{2\sqrt{2}}. \tag{13}$$

Similarly from equations (1), (11) we deduce that

$$u_z(\rho,0) = -\frac{b}{2(\beta^2-1)}\int_\rho^1 \frac{\phi(t)\,dt}{(t^2-\rho^2)^{\frac{1}{2}}} \qquad (0 \leqq \rho \leqq 1). \tag{14}$$

3. We have similar results when the flux of heat across the crack surfaces is prescribed. Equations (5) and (6) are then replaced by

$$\frac{\partial\theta(\rho,z)}{\partial z} = Q(\rho)H(1-\rho). \tag{15}$$

From equation (3) and the Hankel inversion theorem it follows at once that in this case

$$f(\xi) = -\xi^{-1}\int_0^1 \rho Q(\rho)J_0(\xi\rho)\,d\rho. \tag{16}$$

It is now easily seen that

$$\phi(t) = -\int_0^1 \rho Q(\rho)\,d\rho \int_0^\infty \frac{\sin\xi t}{\xi} J_0(\xi\rho)\,d\xi \tag{17}$$

and in particular that

$$\phi(1) = -\tfrac{1}{2}\pi\int_0^1 \rho Q(\rho)\,d\rho,$$

so that the stress intensity factor K is given by the formula

$$K = \frac{\pi Eb}{4\sqrt{2}\,(1+\eta)\beta^2}\int_0^1 \rho Q(\rho)\,d\rho. \tag{18}$$

The normal component of the surface displacement is given by equations (14) and (17); it does not seem possible to reduce these to a single simple formula (cf. [**1**]).

4. We consider two special cases.

(i) If the faces of the crack are kept at a temperature θ_0 below the reference temperature, so that $q(\rho) = -\theta_0$, it follows that $\phi(t) = -(2\theta_0 t)/\pi$ and hence that

$$K = \frac{Eb\theta_0}{\sqrt{2}\,\pi\beta^2(1+\eta)}$$

and

$$u_z(\rho,0) = \frac{b\theta_0}{\pi(\beta^2-1)}(1-\rho^2)^{\frac{1}{2}}.$$

(ii) If there is a constant flux Q_0 across the faces of the crack,

$$K = \frac{\pi EbQ_0}{8\sqrt{2}\,(1+\eta)\beta^2}.$$

https://doi.org/10.1017/S0017089500000744 Published online by Cambridge University Press

REFERENCES

1. Z. Olesiak and I. N. Sneddon, The distribution of thermal stress in an infinite solid containing a penny-shaped crack, *Archiv. Rat. Mech. Anal.* **4** (1960), 238–254.

2. I. N. Sneddon, Boundary value problems in thermoelasticity, *Proc. Symp. Wisconsin*, 1959 (University of Wisconsin Press), 231–241.

3. I. N. Sneddon, The elementary solution of dual integral equations, *Proc. Glasgow Math. Assoc.* **4** (1960), 108–113.

4. I. N. Sneddon, A note on the problem of the penny-shaped crack, *Proc. Cambridge Philos. Soc.* **61** (1965), 609–611.

UNIVERSITY OF GLASGOW