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ON RINGS OF FRACTIONS

BY
T. M. K. DAVISON[1]

1. **Introduction and summary.** Let R be a commutative Noetherian ring with identity, and let M be a fixed ideal of R. Then, trivially, ring multiplication is continuous in the M-adic topology. Let S be a multiplicative system in R, and let $j=j_S\colon R \to S^{-1}R$, be the natural map. One can then ask whether (cf. Warner [3, p. 165]) $S^{-1}R$ is a topological ring in the $j(M)$-adic topology. In Proposition 1, I prove this is the case if and only if $M \subset p(S)$, where

$$p(S) = \bigcap \{P \mid P \text{ prime ideal of } R \text{ such that } P \supset \ker j, P \cap S \neq \phi\}.$$

Hence $S^{-1}R$ is a topological ring for all S if and only if $M \subset p^*(R)$, where

$$p^*(R) = \bigcap \{p(S) \mid S \text{ a multiplicative system}\}.$$

The ideal $p(S)$ occurs in another context: in Proposition 2, I prove that $S^{-1}R$ is an R-algebra of finite type (that is a ring finite extension of R) if and only if $S \cap p(S) \neq \phi$. To globalize this result I prove in Theorem 1 that the all quotient rings of R are of finite type if and only if R is semilocal of dimension at most 1. (This generalizes an old result of Artin–Tate [1, Theorem 4].)

The remainder of the paper is taken up in evaluating $p^*(R)$, (in particular when R is a domain $p^*(R)$ is the pseudo-radical introduced by Gilmer [2]), and discussing the interrelationships with Rad (R), the Jacobson radical of R, and rad (R), the prime radical of R.

2. **Notation and terminology.** In general the notation and terminology is that of Zariski–Samuel [4]. T denotes the set of all nonzero divisors of R, a commutative Noetherian ring with identity. For a given multiplicative system we put

$$S^{\perp} = \{x \in R \mid \exists s \in S \text{ with } xs = 0\}.$$

Thus $S^{\perp} = \ker j_s$, as defined above. $\mathscr{S}$ denotes the set of all multiplicative systems which do not contain 0, hence

$$\mathscr{S} = \{S \text{ mult. system} \mid S \cap S^{\perp} = \phi\}.$$

We say a prime ideal P of R is *S-prime* if and only if $P \supset S^{\perp}$ and $P \cap S \neq \phi$. Thus

$$p(S) = \bigcap \{P \mid P \text{ is an } S\text{-prime ideal}\}.$$

R is *not* considered to be a prime ideal.

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3. Elementary results.

LEMMA 1. *Let $A_1, \ldots, A_n$ be a set of pairwise incommensurable prime ideals in R. For each $i \in [1, n]$ there is an $x_i \in R$ such that $x_i \notin A_i$, but $x_i \in A_j$ all $j \neq i$.*

Proof. Immediate.

LEMMA 2. *Let R_1 be a commutative ring, and G a subgroup of R_1 satisfying $G \supset G^2 \supset G^3 \supset \cdots$. Then the powers of G can be used for a base for a topology on R_1 which gives it the structure of a topological group.*

Then R_1 is a topological ring if and only if given $x_1 \in R_1$ there is a $k \geq 1$ such that $x_1 G^k \subset G$.

Proof. Straightforward (or Bourbaki ?).

Finally we have the following property of minimal prime ideals of R:

LEMMA 3. *Let P be a minimal prime ideal of R (i.e. isolated prime ideal of* (0)). *For all $S \in \mathscr{S}$, P is not an S-prime ideal.*

Proof. Let $P_1, \ldots, P_m$ be the associated prime ideals of (0), and write $(0) = \bigcap Q_i$ in an obvious notation. Put $S_i = R \backslash P_i$.

Assume P is S-prime, then $P \supset S^{\perp}$. Hence [4, Vol. 1, Ch. IV, Theorem 18 (3)]

$$P \supset \bigcap \{Q_i \mid i \text{ such that } Q_i \cap S = \phi\}.$$

Thus $P \supset P_i$ for some i such that $P_i \cap S = \phi$. As P is minimal $P = P_i$, and so $P \cap S = \phi$; which contradicts our assumption that P is S-prime.

4. Local and global results for continuity of ring multiplication.

Let R, M be as in §1.

LEMMA 4. *Let $S \in \mathscr{S}$. Then $S^{-1}R$ is a topological ring in the $j_S(M)$-adic topology if and only if given $s \in S$ there is a $k \geq 1$ such that*

$$M^k \subset Ms + S^{\perp}.$$

Proof. Let $(x, s) \in S^{-1}R$ be given. By Lemma 2 we need only show there is a $k \geq 1$ such that $(x, s)j(M)^k \subset j(M)$. Since $(x, 1)j(M) \subset j(M)$ for all $x \in R$ it is necessary and sufficient to prove that given $s \in S$ there is a $k \geq 1$ such that $j(M)^k \subset j(M)(s, 1)$. The result now follows.

COROLLARY. *$(S^{-1}R, j(M))$ is a topological ring if and only if $M \subset \bigcap_{s \in S} \sqrt{Rs + S^{\perp}}$.*

Proof. If multiplication is continuous, then given s, $M \subset \sqrt{Ms + S^{\perp}} \subset \sqrt{Rs + S^{\perp}}$. But this is true for all s, whence the result.

Conversely assume the result. Let $s \in S$ be given, then $M \subset \sqrt{Rs + S^{\perp}}$. As R is Noetherian,

$$Rs + S^{\perp} \supset (\sqrt{Rs + S^{\perp}})^k \supset M^k \quad \text{for some } k \geq 1,$$

and so $Ms + S^{\perp} \supset M^{k+1}$.

We now prove the intersection of the above corollary is identical with $p(S)$.

LEMMA 5. $\bigcap_{s\in S} \sqrt{Rs+S^{\perp}}=p(S)$.

Proof. Assume x belongs to the left-hand side. Let P be an S-prime ideal, and choose $s\in P\cap S$. Then as $x\in\sqrt{Rs+s^{\perp}}$ there is a $k\geq 1$, $y\in R$, $z\in S^{\perp}$ such that $x^k=ys+z$. But this implies that $x\in P$ (as $s\in P$, and $z\in P$). Hence x belongs to the right-hand side.

Conversely assume $x\in p(S)$. Let $s\in S$ be given. Every prime ideal containing $\sqrt{Rs+S^{\perp}}$ is S-prime, and so $x\in\sqrt{Rs+S^{\perp}}$. Thus $x\in\bigcap_{s\in S}\sqrt{Rs+S^{\perp}}$.

We combine this result, with the preceding corollary to obtain:

PROPOSITION 1. *A necessary and sufficient condition that $(S^{-1}R, j(M))$ is a topological ring is*

$$M\subset p(S).$$

Hence we deduce.

COROLLARY. *$(S^{-1}R, j_S(M))$ is a topological ring for all $S\in\mathcal{S}$ if and only if*

$$M\subset p^*(R).$$

5. **Local results for ring-finiteness.** In the results that follow we will make use of the fact that the prime ideals of a Noetherian ring satisfy the d.c.c., and so, if there are S-prime ideals then there are minimal such.

PROPOSITION 2. (cf. [2, Lemma 3]). *Let $S\in\mathcal{S}$. The following conditions are equivalent*

(i) *$S^{-1}R$ is an R-algebra of finite type.*

(ii) $S\cap p(S)\neq\phi$.

(iii) *There are only finitely many minimal S-prime ideals.*

Proof. (i)⇒(ii). Assume $\phi\colon R_1[x_1,\ldots,x_n]\to S^{-1}R$, with $R_1=R/A$ for some ideal A of R, is an isomorphism. Choose $(y_i, s_i)\in S^{-1}R$ such that $\phi(x_i)=(y_i, s_i)$. Put $s=s_1\ldots s_n$. Clearly $s\in S$. I claim $s\in p(S)$.

It is straightforward (but tedious) to prove using ϕ^{-1} that given $(y, t)\in S^{-1}R$ there is a $k\geq 0$, and $x\in R$ such that $(s^k y, t)\sim(x, 1)$ (in the case of domains this is trivial). Let P be an S-prime ideal, and choose $t\in P\cap S$. By the preceding remark we see that $s^m-xt\in S^{\perp}$ for some $m\geq 0$, $x\in R$. Thus $s^m\in xt+S^{\perp}\subset P+S^{\perp}\subset P$ and so $s\in P$. Whence $s\in p(S)$.

(ii)⇒(iii). Assume $s\in S\cap p(S)$, and put

$$\sqrt{Rs+S^{\perp}}=P^{(1)}\cap P^{(2)}\cap\cdots\cap P^{(k)},$$

where all $P^{(j)}$ are prime, and in fact S-prime. Let P be an S-prime ideal. Clearly $P\supset Rs+S^{\perp}$, and so $P\supset P^{(j)}$ for some j. Hence the minimal S-prime ideals are among the $P^{(j)}$, for $j=1, 2,\ldots, k$.

(iii)$\Rightarrow$(i). If there are no S-prime ideals then in particular a maximal ideal is either disjoint from S or does not contain $S^{\perp}$. Thus the natural injection of $R/S^{\perp}$ into $S^{-1}R$ is onto, and so an isomorphism.

Now assume $P^{(1)}, \ldots, P^{(k)}$, $k \geq 1$, are the minimal S-prime ideals of R. Take $s_i \in P^{(i)} \cap S$, and put $t = \prod_{i=1}^{k} s_i$. In the ring $\bar{R} = R/S^{\perp}$ we denote images by $\bar{r}$, etc., so $\{1, \bar{t}, \bar{t}^2, \ldots\}$ is a multiplicative system in $\bar{R}$. We denote the ring of fractions by $\bar{R}[1/t]$. I claim $S^{-1}R \simeq \bar{R}[1/t]$. The following three statements make this clear, using the universal property of $S^{-1}R$. Consider the natural map f obtained from composing $R \to \bar{R} \to \bar{R}[1/t]$ namely $r \mapsto \bar{r}/1$:

(a) $s \in S \Rightarrow f(s)$ is a unit in $\bar{R}[1/t]$. For since $\sqrt{Rs + S^{\perp}}$ is an intersection of S-prime ideals it must contain t, which is in each S-prime ideal. Thus $\exists x \in R$, $m \geq 0$ such that $xs - t^m \in S^{\perp}$; and so $\bar{x}\bar{s} = \bar{t}^m$, whence the result, as $\bar{t}$ is a unit.

(b) $x \in R, f(x) = 0 \Rightarrow x \in S^{\perp}$. This is clear as

$$\begin{aligned} \frac{\bar{x}}{1} = \frac{\bar{0}}{1} &\Leftrightarrow \exists m \geq 0 \qquad \bar{t}^m \bar{x} = \bar{0} \\ &\Leftrightarrow \exists m \geq 0 \qquad x t^m \in S^{\perp} \\ &\Leftrightarrow x \in S^{\perp}, \qquad \text{as } t^m \in S. \end{aligned}$$

(c) Each element of $\bar{R}[1/t]$ is of the form $f(x)f(s)^{-1}$ for some $x \in R$, $s \in S$. This is clear also, as in fact each element of $\bar{R}[1/t]$ is of the form $f(x)f(t^m)^{-1}$ for some $x \in R$, $m \geq 0$; and $t^m \in S$.

6. **Global results for ring-finiteness.** The result of this section (Theorem 1) is a generalization of a theorem of Artin–Tate [1, Theorem 4], and also contains part of Gilmer's theorem 1 (see [2]).

THEOREM 1. *Let R be a commutative Noetherian ring. Then the following statements are equivalent.*

(a) *R is semilocal, and* $\dim R \leq 1$.
(b) *R has only finitely many prime ideals.*
(c) *$S^{-1}R/R$ is finite for all $S \in \mathcal{S}$.*

Proof. (a)$\Rightarrow$(b) is trivial.

(b)$\Rightarrow$(c) is true by §5, Proposition 2(iii).

(c)$\Rightarrow$(a). Let $\{P_1, \ldots, P_n\}$ be all the prime ideals of R of height 0; since R is Noetherian these are finite in number. Put $T = R \backslash \bigcap_{i=1}^{n} P_i$. Clearly $T^{\perp} \subset \sqrt{(0)} = \bigcap_{i=1}^{n} P_i$, so $P \supset T^{\perp}$ for all prime ideals P. Furthermore $P \cap T = \phi$ if and only if $P = P_j$ for some $j \in [1, n]$. Hence, by Lemma 3, P is T-prime if and only if $\operatorname{ht}(P) \geq 1$. Let $P'_1, \ldots, P'_k$ be the minimal T-prime ideals (these are finite in number since $T^{-1}R$ is an R-algebra of finite type). Let $P'_{k+1}, \ldots, P'_m$ denote the prime ideals of height 0 which are not contained in any P^1_j, for $j \leq k$.

I claim $P'_1, \ldots, P'_m$ are maximal ideals. Let $\{a_1, \ldots, a_m\}$ be chosen (by Lemma 1)

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so that $a_i \notin P'_i$, $a_i \in \bigcap_{j \neq i} P'_j$. Assume $a \notin P'_j$. Consider $b = a + \Sigma' a_q$, where Σ' denotes the sum over all indices q such that $a \in P'_q$. Clearly $b \notin \bigcup_{i=1}^m P'_i$. If b is not a unit, let P' be an isolated prime ideal of Rb. By the principal ideal theorem [4, Vol. 1, p. 238], ht $P' \leq 1$. If ht $P' = 1$ then P' is a T-prime ideal, and so $P' \supset P'_q$ for some $q \leq k$. If $P' > P'_q$ then ht $(P'_q) = 0$, and so P'_q is not T-prime, which is absurd. Thus $P' = P'_q$, and so $b \in P'_q$ which is contrary to the construction of b. So we must have ht $P' = 0$. But then $P' \subset \bigcup_{q=1}^m P'_q$, by our choice of the set of ideals $\{P'_1, \ldots, P'_m\}$ which implies $b \in \bigcup_{q=1}^m P'_q$; which is also absurd. Thus b is a unit, and so P'_j is maximal.

There are no more maximal ideals, for if M is maximal either it is T-prime, and so belongs to $\{P'_1, \ldots, P'_k\}$ or it is not T-prime and so belongs to $\{P_1, \ldots, P_n\}$.

Finally each prime ideal has height ≤ 1. For if P is a prime ideal with ht $P \geq 2$ then P is T-prime, and so $P \supset P'_j$ for some $j \leq k$. Since P'_j is maximal $P = P'_j$. Let $P > P' > P''$ with P', P'' prime. Then P' is a T-prime ideal, and $P'_j > P'$ contradicts the fact that P'_j is a minimal T-prime ideal.

Compare the proof of (c)$\Rightarrow$(b) with Theorem 8 in [3].

7. **Properties of** $p^*(R)$. Let $P_1, \ldots, P_n$ be the isolated prime ideals of (0). We find a more convenient description of $p^*(R)$ in the next result.

PROPOSITION 3. $p^*(R) = \bigcap_{i=1}^n p(S_i)$.

Proof. It is clear that $p^*(R) \subset \bigcap_{i=1}^n p(S_i)$ directly from the definition. If $x \notin p^*(R)$ there is a prime P, and an $S \in \mathscr{S}$ such that P is S-prime and $x \notin P$. Now $P \supset P_i$ for some $i \leq n$. If $P > P_i$ then P is S_i-prime, and so $x \notin p(S_i)$. Hence $x \notin \bigcap_{i=1} p(S_i)$. If $P = P_i$ we contradict Lemma 3. Hence $x \notin p^*(R) \Rightarrow x \notin \bigcap_{i=1}^n p(S_i)$.

COROLLARY. *For any pair of rings R', R'' we have*

$$p^*(R' \times R'') \simeq p^*(R') \times p^*(R'').$$

Proof. The minimal prime ideals of $R' \times R''$ are easy to describe, given the minimal primes of R' and R''. Now apply the proposition.

The next result in conjunction with the one following shows that p^* has the effect of indicating dimension.

PROPOSITION 4. $p^*(R) = R$ *if and only if* dim $R = (0)$. *That is if and only if R is Artinian.*

Proof. If $p^*(R) = R$ then $p(S_i) = R$, each i, and so there are no prime ideals properly containing P_i, for any i. Hence each P_i is maximal and so the result follows. Conversely if R is Artinian then R is a finite product of primary rings. Now if R' is primary $p^*(R') = R$, and so the result follows by applying the Corollary to Proposition 3.

As $p(S_i) \supset P_i$ for each i it is clear that $p^*(R) > \text{rad}\,(R)$. With respect to Rad (R) the situation is a little more complex. First we need the following result.

LEMMA 7. *Let R' be an arbitrary ring. There exist rings R'' and R''' such that* (i) *the minimal prime ideals of R'' have depth ≥ 1* (*equivalently the maximal prime ideals have height ≥ 1*). (ii) *R''' is Artinian, with the property that $R' \simeq R'' \times R'''$.*

Note if R' has property (i) *or* (ii) *itself this representation is trivial, i.e. $R' \simeq R''$ or $R' \simeq R'''$.*

Proof. Let $P'_1, \ldots, P'_n$ be the minimal prime ideals of (0) in R'. Assume P'_k, $P'_{k+1}, \ldots, P'_n$ are maximal ideals, and put $B = \bigcap_{j=k}^{n} Q'_j$ where $(0) = \bigcap_{i=1}^{m} Q'_i$ in an obvious notation. Then there is A such that $A \cap B = (0)$, $A + B = R$, namely $\bigcap_{i=1}^{k-1} Q'_i \cap \bigcap_{i=n+1}^{m} Q'_i$. It is now standard [4, Vol. 1, Theorem 32, p. 178] that $R' \simeq R'/A \times R'/B$. Take $R'' = R'/A$, $R''' = R'/B$. These rings clearly have the desired property.

PROPOSITION 5. *$p^*(R) \subset \text{Rad}(R)$ if and only if there is no minimal prime ideal of R which is also maximal.*

Proof. Assume no minimal prime ideal of R is maximal. Let M be a maximal prime, then there is a minimal prime P_i properly contained in M, so $p(S_i) \subset M$. Thus $p^*(R) \subset \bigcap M = \text{rad}(R)$.

Assume conversely that $p^*(R) \subset \text{Rad}(R)$. Write $R = R' \times R''$ using Lemma 7, with R'' Artinian. Then $p^*(R) = p^*(R') \times p^*(R'')$, whereas $\text{Rad}(R) = \text{Rad}(R') \times \text{Rad}(R'')$ and so $p^*(R'') \subset \text{Rad}(R'')$—but this is absurd (apply Proposition 4), and so we must have $R'' = (0)$. Thus R has no Artinian part, which implies that no minimal prime of R is maximal.

COROLLARY. *$(R, p^*(R))$ is a Zariski ring if and only if there is no minimal prime ideal of R which is also maximal.*

In fact it is clear that more than Proposition 5 is true: $p^*(R) \subset \bigcap \{P \mid P \text{ prime of height} \geq 1\}$. One more result of the type is worth stating.

PROPOSITION 6. *$p^*(R) = \text{rad}(R)$ if and only if $S_i^{-1}R$ is not ring-finite for any i.*

COROLLARY. *If R is semiprimary then $p^*(R) > \text{rad}(R)$.*

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MCMASTER UNIVERSITY,
HAMILTON, ONTARIO

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