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Corrigendum to "On $\mathbb{Z}$-modules of Algebraic Integers"

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Abstract. We fix a mistake in the proof of Theorem 1.6 in the paper *On $\mathbb{Z}$-modules of algebraic integers.* Canad. J. Math. **61**(2009), no. 2, 264–281.

1 Introduction

An algebraic integer q is called a *Pisot* number if it is a real number greater than one with the property that all of its conjugates (other than itself) lie inside the unit circle. In [1], we began a study of the rings that arise from adjoining a Pisot number q to $\mathbb{Z}$. In particular, we claimed to show that if q is a Pisot number, then under general conditions (see [1, Theorems 1.3, 1.5, 1.6]) there are only finitely many Pisot numbers r with the property that $\mathbb{Z}[q] = \mathbb{Z}[r]$. Yann Bugeaud[1] has pointed out that our proof of Theorem 1.6 relied on a misstatement of the Schmidt Subspace Theorem in [1]. We restate our Theorem 1.6 here.

Theorem 1.6 *Let r be a Pisot number with the property that all of its conjugates lie in the extension $\mathbb{Q}(r)$ of $\mathbb{Q}$. Then there are only finitely many Pisot numbers q with the property that $\mathbb{Z}[q] = \mathbb{Z}[r]$.*

The purpose of this note is to give a correct proof of Theorem 1.6.

2 Correction

We begin with a simple lemma that will allow us to eventually apply the Schmidt Subspace Theorem.

Lemma 2.1 *Let K be a number field with $[K : \mathbb{Q}] = n$ and let $c_1, \dots, c_n \in K$, not all zero. Then there are only finitely many Pisot numbers $q \in K$ with $\mathbb{Q}(q) = K$ and with conjugates $q = q_1, q_2, \dots, q_n \in K$ and such that $\sum_{i=1}^n c_i q_i = 0$.*

Proof Suppose that c_i is nonzero. Then since the Galois group of the splitting field of q acts transitively on the conjugates of q, there is some σ such that $\sigma(q_i) = q$. It follows that

$$q = -\sigma(c_i)^{-1} \sum_{j \neq i} \sigma(c_j)\sigma(q_j),$$

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and so

$$|q| < \sum_{j \neq i} |\sigma(c_j c_i^{-1})|,$$

as all conjugates of q other than q are less than one in modulus. Since the Pisot numbers in a number field are discrete, we see that there are only finitely many solutions. ■

We state the Schmidt Subspace Theorem.

Theorem 2.2 (Schmidt Subspace Theorem [2, Chapter VI]) *Let $C, \varepsilon > 0$. If $L_1, \dots, L_n$ are n linearly independent linear homogeneous functions of $\mathbf{x} = (x_1, \dots, x_n)$ with algebraic integer coefficients, then the set of points $\mathbf{x} \in \mathbb{Z}^n$ such that*

$$|L_1(\mathbf{x}) \cdots L_n(\mathbf{x})| < C\|x\|^{-\varepsilon}$$

lies on a finite union of proper subspaces of $\mathbb{Q}^n$.

We note that in the original proof of Theorem 1.6 (see [1]), the flaw in our argument comes from the incorrect assertion that Schmidt's subspace theorem gives that the set of points $\mathbf{x} \in \mathbb{Z}^n$ such that $|L_1(\mathbf{x}) \cdots L_n(\mathbf{x})| < C\|x\|^{-\varepsilon}$ is finite.

As it turns out, Lemma 2.1 is all we need to deduce finiteness in the statement of Theorem 1.6 once we invoke the Schmidt Subspace Theorem properly.

Proof of Theorem 1.6 Let $d = [\mathbb{Q}(r) : \mathbb{Q}]$. We let $r = r_1, \dots, r_d$ denote the conjugates of r. For each Pisot number q with the property that $\mathbb{Z}[q] = \mathbb{Z}[r]$, we can write $q = c_0 + c_1 r + \cdots + c_{d-1} r^{d-1}$ for some unique vector $(c_0, \dots, c_{d-1}) \in \mathbb{Z}^d$.

We note that the original argument in the proof of Theorem 1.6 (see [1, pp. 279–280]), combined with the correct statement of the Subspace Theorem, shows that, assuming there are infinitely many Pisot numbers q for which $\mathbb{Z}[q] = \mathbb{Z}[r]$, then there is an infinite set of such q lying in some proper $\mathbb{Q}$-vector subspace W of $\mathbb{Q}(r)$. Thus there exists some $m < d$ and linearly independent elements $t^{(1)}, \dots, t^{(m)} \in \mathbb{Q}(r)$ such that $\{t^{(1)}, \dots, t^{(m)}\}$ forms a basis for W as a $\mathbb{Q}$-vector space. By assumption, there are infinitely many Pisot numbers q of the form $q = b_1 t^{(1)} + \cdots + b_m t^{(m)}$ with $b_1, \dots, b_m \in \mathbb{Q}$. For each $i \in \{1, \dots, m\}$, we let $t^{(i)} = t_1^{(i)}, \dots, t_d^{(i)}$ denote the conjugates of $t^{(i)}$.

Observe that if $q = b_1 t^{(1)} + \cdots + b_m t^{(m)}$ is a Pisot number, then the d conjugates $q = q_1, \dots, q_d$ of q have a representation

$$q_j = \sum_{i=1}^{m} b_i t_j^{(i)}.$$

Consider the $m \times d$ matrix A whose (i, j) entry is $t_j^{(i)}$. As $m < d$, the columns of A are linearly dependent. Hence there is a nonzero complex $d \times 1$ vector $\mathbf{v}$ such that $A\mathbf{v} = \mathbf{0}$. By assumption there are infinitely many rational row vectors $\mathbf{b} = [b_1, \dots, b_m]$ for which $\mathbf{b}A = [q_1, \dots, q_d]$, where q_1 is Pisot and $q_1, \dots, q_d$ are the conjugates of q_1. Thus $[q_1, \dots, q_d]\mathbf{v} = 0$ for infinitely many Pisot numbers $q = q_1 \in \mathbb{Q}(r)$ with conjugates $q = q_1, \dots, q_d$. This contradicts Lemma 2.1. The result follows. ■

3 Additional Corrections

The following typos occur in [1]:

(1) on line 6 of page 279, $1 \le i, j \le d$ should be $0 \le i, j \le d - 1$;
(2) on line 12 of page 279, q_2 and q_d should be q_1 and q_{d-1} respectively;
(3) on line 14 of page 279, q_2 should be q_1.

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References

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