BULL. AUSTRAL. MATH. SOC. 16L60, 16G10, 18A40.
VOL. 67 (2003) [219–224]

RELATIVE APPROXIMATIONS AND MASCHKE FUNCTORS

BIN ZHU

The notion of approximations relative to a functor is introduced and several characterisations of relative (dual) Maschke functors are given by using them. As an application, the injective objects in the category of comodules over a coring are described.

The notions of approximations and of contravariantly finite subcategories were introduced and studied by Auslander and Smalø[**3**] in the connection with the study of the existence of almost split sequences in a subcategory. It turns out that these notions are important in the study of representation theory of Artin algebras. For example, Auslander and Reiten [**1, 2**] proved that certain contravariantly finite subcategories of a module category are in one-to-one correspondence with tilting modules.

From Auslander and Reiten [**1, 2**], for any adjoint pair (F, G) from categories $\mathcal{C}$ to $\mathcal{D}$, the image of $\mathcal{C}$ under F, denoted by $\mathrm{Im}(F)$, is contravariantly finite in $\mathcal{D}$, that is, any object A in $\mathcal{D}$ has a right $\mathrm{Im}(F)$-approximation (for more general results and applications, we refer to [**6, 7**]).

The aim of this note is to introduce the notion of approximations relative to a functor, and, by using it, to give some characterisations of relative Maschke functors which were recently introduced in [**4, 5**]. We shall first give the definitions of F-relative approximations and F-contravariantly finiteness of a subcategory, and then, give some new characterisations of F-Maschke functors. Finally, an application to the description of injective objects in the category of comodules over a coring will be given.

Let $\mathcal{C}$, $\mathcal{D}$ be categories and $F : \mathcal{C} \to \mathcal{D}$ a covariant functor.

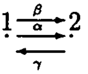
DEFINITION 1. Let $\mathcal{T}$ be a full subcategory of $\mathcal{C}$ and $M \in \mathcal{C}$. A map $f : M_1 \to M$ is called an F-relative right $\mathcal{T}$-approximation of M if M_1 is an object of $\mathcal{T}$ and for any map $g : X \to M$ with $X \in \mathcal{T}$, there is a map $h : FX \to FM_1$ in $\mathcal{D}$ such that $F(g) = h \cdot F(f)$. Dually one can define the notion of F-relative left $\mathcal{T}$-approximation of M.

Received 11th June, 2002
The author was supported partially by NSF 10001017 and by Scientific Foundation for Returned Overseas Chinese Scholars, Ministry of Education. This work was completed when the author was visiting the Academy of Mathematics and System Sciences of China. The author would like to thank Professor Nanhua Xi for his help.

REMARK 1. If the functor F is the identity functor, then we come back to the usual notion of right (or left) $\mathcal{T}$-approximations introduced by Auslander and Smalø in [**2**, **3**].

LEMMA 1. *If $f : M_1 \to M$ is a right $\mathcal{T}$-approximation of M, then it is an F-relative right $\mathcal{T}$-approximation of M. The converse is not true in general.*

PROOF: The proof of the first part is obvious. We present an example to show the last part. Before doing this we first prove the following: Let A be a finite dimensional algebra over a field k, $\mathcal{C} = A$-mod, the category of finite dimensional left modules over A and $\mathcal{T}$ a full subcategory of $\mathcal{C}$. Let $F : A - \text{mod} \to k - \text{mod}$ be the forgetful functor. If $\mathcal{T}$ contains projective A-module A, then every A-module M has an F-relative right $\mathcal{T}$-approximation. To prove this, let $f : M_1 \to M$ be a surjective with $M_1 \in \mathcal{T}$ (such surjective map exists for ${}_AA \in \mathcal{T}$). We claim that f is an F-relative right $\mathcal{T}$-approximation of M: Given a map $g : X \to M$ with $X \in \mathcal{T}$, if we denote by $\phi : F(M) \to F(M_1)$ the right inverse of f in k-mod, then g factors through f by $\phi \cdot g$ in k-mod. Therefore f is an F-relative $\mathcal{T}$-approximation of M. In the rest of the proof, let A be the finite dimensional algebra given by the quiver



with relations $\gamma\alpha = 0 = \alpha\gamma = \beta\gamma$. The subcategory $P^\infty(A)$ consisting of A-modules with finite projective dimension is not contravariantly finite in A-mod (compare [**2**, Section 4]), that is, there is at least one module M without right $P^\infty(A)$-approximations. But it has F-relative $P^\infty(A)$-approximations by the claim above. This finishes the proof. ∎

DEFINITION 2. A full subcategory $\mathcal{T}$ of $\mathcal{C}$ is said to be

(i) *$\mathcal{F}$-relative contravariantly finite* in $\mathcal{C}$ if for each object X in $\mathcal{C}$, there is an F-relative right $\mathcal{T}$-approximation.

(ii) *$\mathcal{F}$-relative covariantly finite* in $\mathcal{C}$ if for each object Y in $\mathcal{C}$, there is an F-relative left $\mathcal{T}$-approximation.

(iii) *$\mathcal{F}$-relative functorially finite* in $\mathcal{C}$ if $\mathcal{T}$ is both $\mathcal{F}$-relative contravariantly and $\mathcal{F}$-relative covariantly finite in $\mathcal{C}$.

REMARK 2. If the functor F is the identity functor, then we arrive back to the usual notion of contravariantly (or covariantly or functorially) finite subcategories introduced by Auslander and Smalø in [**2**, **3**].

LEMMA 2. *If $\mathcal{T}$ is a contravariantly finite (or covariantly finite) subcategory in $\mathcal{C}$, then it is a F-relative contravariantly finite (respectively, F-relative covariantly finite) subcategory in $\mathcal{C}$. The converse is not true in general.*

PROOF: By Lemma 1. the proof for the first part is obvious. For the proof of last part, let A be the algebra in the proof of Lemma 1, $\mathcal{T}$ the subcategory $P^\infty(A)$ and $F : A - \text{mod} \to k - \text{mod}$ the forgetful functor. Then $P^\infty(A)$ is F-relative contravariantly finite but not contravariantly finite in A-mod (compare [**2**, Section 4]). ∎

https://doi.org/10.1017/S0004972700033682 Published online by Cambridge University Press

Now we recall a result due to Auslander and Reiten (compare [**1**, Section 1], a more general version can be found in [**7**]). This result is the starting point of this note.

LEMMA 3. *Let (F, G) be an adjoint pair from category $\mathcal{C}$ to $\mathcal{D}$. Then* $\mathrm{Im}(F)$ *is contravariantly finite in $\mathcal{D}$ and for any $X \in \mathcal{D}$, the counit map $\varepsilon_X : FG(X) \to X$ is a right* $\mathrm{Im}(F)$*-approximation of X. Dually* $\mathrm{Im}(G)$ *is covariantly finite in $\mathcal{C}$ and for any Y in $\mathcal{C}$, the unit map $\eta_Y : Y \to GF(Y)$ is a left* $\mathrm{Im}(G)$*-approximation of Y.*

We now recall the notions of relative injective and of Maschke functors from [**5**, Section 3] or [**4**, Chapter 3].

DEFINITION 3. Let $F : \mathcal{C} \to \mathcal{D}$ and $H : \mathcal{C} \to \mathcal{E}$ be covariant functors. An object $M \in \mathcal{C}$ is called F-relative H-injective if the following condition is satisfied: for any $i : C \to C'$ in $\mathcal{C}$ with $F(i) : F(C) \to F(C')$ a split monomorphism in $\mathcal{D}$, and for every $f : C \to M$ in $\mathcal{C}$, there exists $g : H(C') \to H(M)$ in $\mathcal{E}$ such that $H(f) = g \cdot H(i)$.

F is called an H-Maschke functor if any object of $\mathcal{C}$ is F-relative H-injective.

An F-relative $1_{\mathcal{C}}$-injective is also called an F-relative injective object. An $1_{\mathcal{C}}$-Maschke functor is also called a Maschke functor.

$P \in \mathcal{C}$ is called F-relative H-projective if P is F^{op}-relative H^{op}-injective, where $F^{op} : \mathcal{C}^{op} \to \mathcal{D}^{op}$ is the functor opposite to F.

F is called a dual H-Maschke functor if any object of $\mathcal{C}$ is F-relative H-projective.

Our next result gives some characterisations of (dual) H-Maschke functors.

Let $F : \mathcal{C} \to \mathcal{D}$, $G : \mathcal{D} \to \mathcal{C}$, $H : \mathcal{C} \to \mathcal{E}$ and $H' : \mathcal{D} \to \mathcal{E}'$ be covariant functors. We denote by $HDS(G)$ the subcategory of $\mathcal{E}$ consisting of objects $H(X)$, where X is an object of $\mathcal{C}$ such that there is a morphism $g : X \to G(Y)$ with $H(g)$ a split monomorphism from $H(X)$ to $H(G(Y))$.

Similarly, $H'DS(F)$ denotes the subcategory of $\mathcal{E}'$ consisting of objects $H(X')$, where X' is an object of $\mathcal{D}$ such that there is a morphism $g' : F(Y') \to X'$ with $H'(g')$ a split epimorphism from $H'(F(Y'))$ to $H'(X')$.

If $H = 1_{\mathcal{C}}$, then $HDS(G)$ (denoted by $DS(G)$ in this case) is the subcategory of $\mathcal{C}$ consisting of direct summands of $G(Y), Y \in \mathcal{D}$. Similar remark applies to $H'DS(F)$.

THEOREM 4. *Assume that the functor $F : \mathcal{C} \to \mathcal{D}$ has a right adjoint $G : \mathcal{D} \to \mathcal{C}$ and $H : \mathcal{C} \to \mathcal{E}$ is a covariant functor. Then the following statements are equivalent:*

(1) *$M \in \mathcal{C}$ is F-relative H-injective;*
(2) *$H(\eta_M) : H(M) \to HGF(M)$ has a left inverse in $\mathcal{E}$;*
(3) *There is a map $f : M \to G(X)$ in $\mathcal{C}$, such that $H(f) : H(M) \to HG(X)$ has a left inverse in $\mathcal{E}$;*
(4) *$H(M) \in HDS(G)$.*

In particular, F is an H-Maschke functor if and only if every object X of $H(\mathcal{C})$ is in $HDS(G)$, that is, $H(\mathcal{C}) = HDS(G)$.

PROOF: The equivalence between (1) and (2) is [5, Theorem 3.4]. The directions (2)⇒(3) and (3)⇒(4) are obvious. We prove the direction (4)⇒(1): Since (F, G) is an adjoint pair from $\mathcal{C}$ to $\mathcal{D}$, by Lemmas 3 and 1, we have that $\eta_M : M \to GF(M)$ is an F-relative left $\mathrm{Im}(G)$-approximation of M, where M is any object of $\mathcal{C}$. By (4), we have a map $f : M \to G(Y)$ in $\mathcal{C}$, such that $H(f) : H(M) \to HG(Y)$ is a split monomorphism, where $Y \in \mathcal{D}$. Then there is a map $g : H(M) \to HG(Y)$ in $\mathcal{E}$, such that $H(f) = g \cdot H(\eta_M)$. Therefore the splitness of $H(\eta_M)$ follows from the splitness of $H(f)$. By [5], we have (1). For the proof of last statement, we note that F is an H-Maschke functor if and only if every object M of $\mathcal{C}$ is F-relative H-injective if and only if for any object M of $\mathcal{C}$ we have $H(M) \in HDS(G)$ if and only if $H(\mathcal{C}) = HDS(G)$. □

Let us remark here that the equivalence between (1) and (2) in Theorem 4 is known as Theorem 3.4. in [5]. The conditions (3) and (4) are new even in the case that H is the identity functor on $\mathcal{C}$.

Let H be the identity functor, we get a new characterisation of Maschke functors as follows.

COROLLARY 5. *Assume that the functor $F : \mathcal{C} \to \mathcal{D}$ has a right adjoint $G : \mathcal{D} \to \mathcal{C}$. Then $M \in \mathcal{C}$ is F-injective if and only if $M \in DS(G)$. Moreover, F is a Maschke functor if and only if every object $M \in \mathcal{C}$ is in $DS(G)$, that is, $\mathcal{C} = DS(G)$.*

Dually, we have the following

THEOREM 6. *Assume that the functor $F : \mathcal{C} \to \mathcal{D}$ has a right adjoint $G : \mathcal{D} \to \mathcal{C}$ and $H' : \mathcal{D} \to \mathcal{E}'$ is a covariant functor. Then the following statements are equivalent:*

(1) *$P \in \mathcal{D}$ is G-relative H'-projective;*

(2) *$H'(\varepsilon_P) : H'FG(P) \to H'(P)$ has a right inverse in $\mathcal{E}'$;*

(3) *There is a map $g : F(X') \to P$ in $\mathcal{D}$ such that $H'(f) : H'F(X') \to H'(P)$ has a right inverse in $\mathcal{E}'$;*

(4) *$H'(P) \in H'DS(F)$.*

In particular, G is a dual H-Maschke functor if and only if every object M' of $H'(\mathcal{D})$ is in $H'DS(F)$, that is, $H'(\mathcal{D}) = H'DS(F)$.

Let H' be the identity functor, we get a new characterisation of dual Maschke functor as follows.

COROLLARY 7. *Assume that the functor $F : \mathcal{C} \to \mathcal{D}$ has a right adjoint $G : \mathcal{D} \to \mathcal{C}$. Then Then $N \in \mathcal{D}$ is F-projective if and only if $N \in DS(F)$. Moreover G is a dual Maschke functor if and only if every object $M \in \mathcal{D}$ is in $DS(F)$, that is, $\mathcal{D} = DS(F)$.*

In the following we shall give an applications of Theorems 4 and 6.

Let A be a ring and C an A-coring with comultiplication Δ_C and counit ε_C. A right C-comodule is a right A-module M together with a right A-module map $\rho^r : M \to M \otimes_A C$

such that

$$(\rho^r \otimes_A I_c) \circ \rho^r = (I_M \otimes_A \Delta_C) \circ \rho^r$$
$$(I_M \otimes_A \varepsilon_C) \circ \rho^r = I_M.$$

Let $\mathcal{M}^C$ denote the category of all right C-comodules and $\mathcal{M}_A$ the category of all right A-modules. We look at the forgetful functor $F : \mathcal{M}^C \to \mathcal{M}_A$. The functor F has a right adjoint $G = - \otimes_A C$. For details, we refer to [**4**].

PROPOSITION 8. *Let A be a semisimple ring and C an A-coring. Then $M \in \mathcal{M}^C$ is injective if and only if there is a right A-module Q such that M is a direct summand of $Q \otimes_A C$.*

PROOF: Since A is semisimple, for any injective homomorphism f in $\mathcal{M}^C$, $F(f)$ is a split monomorphism in $\mathcal{M}_A$. Then $M \in \mathcal{M}^C$ is injective if and only if M is F-relative injective. By Corollary 5, M is F-relative injective if and only if there is a right A-module Q such that M is a direct summand of $Q \otimes_A C$. This finishes the proof. ∎

REMARK 2. The proposition generalises in [**5**, Corollary 4.9].

We call an A-coring is semisimple if each right C-comdule is injective. As a consequence of Proposition 8, we have the following.

COROLLARY 9. *Let A be a semisimple ring and C an A-coring. Then the following statements are equivalent*

(1) *C is semisimple;*
(2) *$\mathcal{M}^C = DS(G)$;*
(3) *F is a Maschke functor.*

REFERENCES

[1] M. Auslander and I. Reiten, 'Homological finite subcategories', in *Representation Theory of Algebras and Related Topics(Kyoto, 1990)*, London Mathematical Society Lecture Note Series, **168** (Cambridge University Press, Cambridge, 1992), pp. 1–42.

[2] M. Auslander and I. Reiten, 'Applications of contravariantly finite subcategories', *Adv. Math.* **86** (1991), 111–152.

[3] M. Auslander and S.O. Smalø, 'Almost split seqences in subcategories', *J. Algebra.* **69** (1981), 426–454.

[4] S. Caenepeel, S. Militaru and S. Zhu, *Frobenius and separable functors for generalized module categories and nonlinear equations*, Lecture Note in Mathematics **1787** (Springer-Verlag, Berlin, 2002).

[5] S. Caenepeel and G. Militaru, 'Maschke functors, semisimple functors and separble functors of the second kind. Applications.', *J. Pure Appl. Algebra* (to appear).

[6] S. Caenepeel and B. Zhu, 'Separable bimodules and approximations', preprint.

[7] B. Zhu, 'Contravariantly finite subcategories and adjunctions', *Algebra Colloq.* **8** (2001), 307–314.

Department of Mathematical Sciences
Tsinghua University
100084 Beijing
China
e-mail: bzhu@math.tsinghua.edu.cn