ON RANDOM VARIABLES WHICH HAVE THE SAME DISTRIBUTION AS THEIR RECIPROCALS

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The motivation for this paper lies in the following remarkable property of certain probability distributions. The distribution law of the r.v. (random variable) X is exactly the same as that of $1/X$, and in the case of a r.v. with p.d.f. (probability density function) $f(x;a,b)$ where a,b are parameters, the p.d.f. of $1/X$ is $f(x;b,a)$. In the latter case the p.d.f. of the reciprocal is obtained from the p.d.f. of X by merely switching the parameters. The existence of random variables with this property is perhaps familiar to statisticians, as is evidenced by the use of the classical 'F' distribution. The Cauchy law is yet another example which illustrates this property. It seems, therefore, reasonable to characterize this class of random variables by means of this rather interesting property. In this paper we make an analytic study of such random variables and discuss their character, confining our attention to non-negative random variables with absolutely continuous distributions.

1. <u>The Functional Equation.</u> Consider the r.v. X with p.d.f. $f(x)$ where $f(x) = 0$ when $x \leq 0$. If we require that X and $1/X$ have the same p.d.f., then $f(x)$ must satisfy the following functional equation.

$$f(x) = (1/x^2)\, f(1/x) . \tag{1}$$

The problem now is to determine $f(x)$. Let

$$Y = \log_e X . \tag{2}$$

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Then

$$(3) \qquad e^{y} f(e^{y}) = e^{-y} f(e^{-y}) .$$

From (3) it is clear that the p.d.f. $g(y)$ of the r.v. Y is symmetric with respect to the origin since $g(y) = e^{y} f(y)$. Conversely, whenever $g(y) = g(-y)$, the r.v. X determined by (2) will have the same distribution as its reciprocal and $f(x) = (1/x\, g(\log_e x))$ is then the solution of (1). We thus have the following theorem.

THEOREM. A necessary and sufficient condition for a non-negative r.v. X, with an absolutely continuous distribution, to have the same p.d.f. as its reciprocal is that the p.d.f. of Y given by the transformation $Y = \log_e X$ be symmetric with respect to the origin.

2. <u>A Structural Property and Decomposability.</u> In this section we consider the construction of random variables with the dual property mentioned in the last section.

Suppose that two independent samples, X, Y, are drawn from a distribution, and the ratio is formed of these samples, say $U = X/Y$. Assuming that the p.d.f. of the samples is $f(\cdot)$ for $x > 0$, $y > 0$ the p.d.f. of U can be shown to be

$$(4) \qquad h(u) = \int_0^\infty y\, f(y)\, f(uy)dy .$$

Writing $V = Y/X$, the p.d.f. of V is

$$(5) \qquad k(v) = \int_0^\infty x\, f(x)\, f(vx)dx .$$

Noting that $V = 1/U$, $X = UY$, $VX = Y$, we obtain from (5)

$$(6) \qquad k(1/u) = \int_0^\infty uy\, f(uy)\, f(y)u\, dy .$$

Therefore, from (4), (5) and (6) we have

$$u^2 h(u) = k(1/u)$$

$$= h(1/u) .$$

Thus we find that the p. d. f. of the ratio of two independent samples from the same distribution satisfies (1). Conversely, if $X = (X_1/X_2)$ is a random variable whose p. d. f. satisfies (1), then X can be decomposed as a ratio if X_1 and X_2 are not required to be independent.

Suppose that the r. v. $Y (Y > 0)$ is independent of the r. v. X. Writing $X_1 = Y\sqrt{X}$ and $X_2 = Y/\sqrt{X}$, it follows that $X = X_1/X_2$, and since $X = X_1/X_2$ for every Y which is independent of X, this decomposition is not unique. However, if X_1 and X_2 are to be independent, the ratio decomposition will not be available. As an example we consider the following problem.

Let $Y_i = \log_e X_i$ $(i = 1, 2)$, and define $Y = \log_e X = \log_e X_1 - \log_e X_2$. Thus $Y = Y_1 - Y_2$, and if $X = X_1/X_2$, Y_1 and Y_2 are independent of each other and have the same distribution. If we now take the p. d. f. $g(y)$ of Y as $g(y) = y^2 e^{-y^2/2}/\sqrt{2\pi}$, it can be shown that although $g(y)$ is symmetric about the origin it cannot be expressed as the sum of the p. d. f. ' s of independent non-degenerate random variables.

A few examples of p. d. f. ' s satisfying (1) are given below.

<u>Examples:</u>

(a) The zero-truncated Cauchy distribution

$$f(x) = (2/\pi)/(1+x^2) \qquad x \geq 0 .$$

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Indeed the p.d.f. $f(x) = (1/\pi)/(1+x^2)$, $-\infty < x < \infty$, also satisfies (1).

(b) The log-normal distribution.

$$f(x) = (\sigma/x\sqrt{2\pi}) \exp\{-[(\sigma \log x)^2/2]\}, 0 < x < \infty .$$

(c) $f(x) = 6x/(1+x)^4 \qquad x > 0 .$

(d) $f(x) = 6x^2/(1+x^6) \qquad x > 0 .$

(e) Let (X_i, Y_i) have a bivariate normal distribution with the same mean and variances σ_1^2, σ_2^2 for $i = 1, 2, \ldots, n$. Defining s_1^2, s_2^2 by the quantities $\sum_{i=1}^{n} (X_i - \overline{X})^2/n$ and $\sum_{i=1}^{n} (Y_i - Y)^2/n$ respectively, the distribution of $V = (s_1/\sigma_1)/(s_2/\sigma_2)$ is ([1], [2]),

$$f(v) = \frac{2(1-\rho^2)^{(n-1)/2} v^{n-2}}{B[\frac{(n-1)}{2}, \frac{(n-1)}{2}] (1+v^2)^{n-1}} \left\{1 - \frac{4\rho^2 v^2}{(1+v^2)^2}\right\}^{-n/2}$$

(f) Consider the ratio V of two positive and independent normal variables. If the means are the same, m, and the variances are σ_1^2, σ_2^2 respectively, the p.d.f. of V is

$$f(v; \sigma_1^2, \sigma_2^2) = \frac{1}{\sqrt{2\pi}} \frac{m(\sigma_1^2 + \sigma_2^2 v)}{(\sigma_1^2 + \sigma_2^2 v^2)^{3/2}} \exp\left\{-\frac{m^2(1-v)^2}{2(\sigma_1^2 + \sigma_2^2 v^2)}\right\} .$$

(g) The Beta-distribution

$$f(x; a, b) = [1/B(a, b)] x^{a-1}/(1+x)^{a+b} \qquad x > 0, \quad a, b > 0 .$$

(h) $f(x; a, b) = (2\sqrt{ab}/\pi)/(a+bx^2) \qquad x \geq 0 .$

(i) Let X_m, X_n be the largest members of two independent samples of sizes m and n respectively from a rectangular distribution. The p.d.f. of $U = (X_m / X_n)$ is [3].

$$f(u;m,n) = \begin{cases} \frac{mn}{m+n} u^{m-1} & 0 < u \leq 1 \\ \frac{mn}{m+n} u^{-n-1} & 1 \leq u \leq \infty . \end{cases}$$

3. Remarks. From the functional equation (1) it becomes clear the point $x = 1$ is an invariant point, and that with a knowledge of the probabilities from $(0, 1)$, the probabilities from $(1, \infty)$ can be determined. The point $x = 1$ plays the same role as the point $x = 0$ plays in the functional equation $f(x) = f(-x)$. The p.d.f.'s satisfying (1) have median at $x = 1$. For we have $F(1/x) = 1 - F(x)$, and for x to be the median $F(x) = 1 - F(x)$. So $F(x) = F(1/x)$ and we obtain $x = (1/x) = 1$, since the range of interest is the positive half line.

Suppose that the first k moments, about the origin, of X exist. The (k-2) moments (about the origin) of $1/x$ also exist and satisfy the following relation (except for $k = 2$): $\mu'_{k, X} = \mu'_{(k-2), 1/X}$.

Finally let $y = r(x)$ be a differentiable function of x with $r'(x) \neq 0$. The p.d.f. of $r(x)$ is $r'(x) f(r(x))$, where $f(x)$ is the p.d.f. of X. If $1/r(x)$ exists, the p.d.f. of $1/r(x)$ is $r'(x)f(r(x))/(r(x))^2$. But since $f(r(x)) = [1/(r(x))^2] f(1/r(x))$, we have

$$r'(x)f(r(x)) = [1/(r(x))^2] r'(x) f(1/r(x)) .$$

Thus if $P(y)$ denotes the p.d.f. of Y, we note that $P(y)$ satisfies the functional equation $P(y) = (1/y^2) P(1/y)$. Illustration (e) shows this property.

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REFERENCES

1. S. S. Bose, (1935), On the distribution of the ratio of variances of two samples drawn from a given normal bivariate correlated population, Sankhya 2, p. 65.

2. D. J. Finney, (1938), The distribution of the ratio of estimates of the two variances in a sample from a normal bivariate population, Biometrika, 30, p. 190.

3. V. N. Murty, (1950), Distribution of the quotient of maximum values in samples from a rectangular distribution, Journal of the American Statistical Association, 50, p. 1136.

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