ON KILMISTER'S CONDITIONS FOR THE EXISTENCE OF LINEAR INTEGRALS OF DYNAMICAL SYSTEMS

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Kilmister (1) has considered dynamical systems specified by coordinates $q^\alpha(\alpha = 1, 2, \ldots, n)$ and a Lagrangian

$$L = \tfrac{1}{2}a_{\alpha\beta}\dot{q}^\alpha\dot{q}^\beta + a_\alpha\dot{q}^\alpha + a$$

(with summation convention). He sought to determine generally covariant conditions for the existence of a first integral, $b_\alpha\dot{q}^\alpha = \textit{constant}$, linear in the velocities. He showed that it is not, as is usually stated, necessary that there must exist an ignorable coordinate (equivalently, that b_α must be a Killing field:

$$b_{\alpha;\beta} + b_{\beta;\alpha} = 0,$$

where covariant derivation is with respect to $a_{\alpha\beta}$). On the contrary, a singular integral, in the sense that $b_\alpha\dot{q}^\alpha = 1$ for all time if satisfied initially, need not be accompanied by an ignorable coordinate.

In fact, Kilmister has shown that necessary and sufficient conditions for the existence of a linear first integral are

$$b_{\alpha;\beta} + b_{\beta;\alpha} = 2\theta b_\alpha b_\beta, \quad \text{.....(1)}$$

$$b^\rho(a_{\rho;\beta} - a_{\beta;\rho}) = \phi b_\beta, \quad \text{.....(2)}$$

$$b^\rho a_{,\rho} = \phi - \theta. \quad \text{.....(3)}$$

In this note, a simple consequence of these equations will be exploited. Namely, since the kinetic energy matrix $a_{\alpha\beta}$ must be positive definite, $b_\alpha b^\alpha \neq 0$, so that, from (2), $\phi = 0$. Moreover, from (1)

$$b_{\alpha;\beta}b^\alpha b^\beta = \theta(b_\alpha b^\alpha)^2, \text{ or } \theta = (-1/2b_\alpha b^\alpha)_{,\beta}b^\beta.$$

Finally, (3) says $(a - 1/2b_\alpha b^\alpha)_{,\beta}b^\beta = 0$.

For $n = 2$ these equations can be integrated completely. Choosing a coordinate system (x, y) for which $b^\alpha = (1, 0)$, $b_\alpha = a_{\alpha 1}$, and (1) becomes

$$a_{\alpha\beta,1} = a_{11,1}a_{\alpha 1}a_{\beta 1}/a_{11}^2,$$

which is satisfied identically for $\alpha = \beta = 1$ and for $\alpha = 1, \beta = 2$ gives

$$a_{12} = a_{11}A(y).$$

For $\alpha = \beta = 2$ we find $a_{22} = a_{11}A^2(y) + B^2(y)$. $A(y)$ and $B(y)$ are arbitrary.

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(2) becomes $a_{1,2} = a_{2,1}$ or $a_\alpha = V_{,\alpha}$, say. But since such terms will not contribute to the equations of motion, they may be dropped from the Lagrangian. (3) requires $a = \dfrac{1}{2a_{11}} + C(y)$. Thus

$$L = \tfrac{1}{2}[a_{11}\dot{x}^2 + 2a_{11}A(y)\dot{x}\dot{y} + a_{11}A^2(y)\dot{y}^2 + B^2(y)\dot{y}^2] + \frac{1}{2a_{11}} + C(y).$$

Let $\xi = x + \int A(y)dy$ and $\eta = \int B(y)dy$. Then

$$L = \tfrac{1}{2}[a_{11}\dot{\xi}^2 + \dot{\eta}^2] + \frac{1}{2a_{11}} + D(\eta).$$

a_{11} is an arbitrary positive function of ξ and η. This Lagrangian is mentioned by Kilmister, who implied it to be of less general interest. The integration procedure used here fails when $n > 2$.

The equation (1) has been noticed by Rayner (2) while studying rigid motions in general relativity. There, of course, a_α and a have no analogues. Kilmister's result corresponds to the fact that if a motion b^α satisfies (1) (implying that the motion is rigid) and if u^α is a unit tangent to a geodesic, then $b_\alpha u^\alpha = 0$ along the entire geodesic if it is satisfied at one point.

REFERENCES

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