ON FELLER'S KERNEL AND THE DRICHLET NORM

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§ 0. Introduction

Recently J. L. Doob [2] evaluated the Dirichlet integral of the BLD harmonic funtion on a Green space in terms of its fine boundary values and θ-kernel of L. Naïm.

On the other hand, the general theory of additive functionals of Markov processes enables us to define the concept of the Dirichlet norm of functions with respect to Markov processes.

In § 1 we shall prove that Naïm's kernel is equal to the kernel U, a generalization of the kernel introduced by W. Feller [3], on the Martin exit boundary of a Green space.

In § 2 we shall treat the case of a multidimensional diffusion process corresponding to a self-adjoint elliptic differential operator. In this case we shall define the Dirichlet norm of the harmonic function and represent it in terms of Feller's kernel U, under certain regularity conditions.

The kernel U plays an essential role in the investigation of the boundary problems concerning Markov processes ([3], [8]).

The author wishes to express his thanks to Prof. N. Ikeda and Prof. T. Watanabe for their kind suggestions and encouragement.

§ 1. Feller's kernel and Naïm's kernel

Let $p(t, x, y)$ be the Brownian transition density on a Green space R. For $x, y \in \mathrm{R}$, put $G_\alpha(x, y) = \int_0^{+\infty} e^{-\alpha t} p(t, x, y) dt$ $(\alpha \geqq 0)$, and $G(x, y) = G_0(x, y)$.

Let M be the totality of the minimal points of the Martin boundary of R, and put $\hat{\mathrm{R}} = \mathrm{R} \cup \mathrm{M}$. To each point $y \in \hat{\mathrm{R}}$, there corresponds the Martin K-function $K(x, y)$, $x \in \mathrm{R}$, which will sometimes be written as $K_y(x)$. When $y \in \mathrm{R}$, $K(x, y) = \dfrac{G(x, y)}{G(x_0, y)}$, where x_0 is a fixed reference point of R.

Received December 2, 1963.

https://doi.org/10.1017/S0027763000011399 Published online by Cambridge University Press

For $y\in\hat{\mathrm{R}}-\{x_0\}$, we denote by $\mathfrak{B}^y=\{\Omega, \mathrm{B}, P_x^y, x\in\mathrm{R}', X_t, t\geqq 0\}$ the conditional Brownian motion on R' whose transition density is $p^y(t, x, z)=\dfrac{K_y(z)}{K_y(x)}p(t, x, z)$, $(t>0, x, z\in\mathrm{R}')$, where $\mathrm{R}'=\mathrm{R}-\{y\}$.

We adopt the notation $X_{\sigma_\infty -}(w)=\lim_{t\uparrow\sigma_\infty} X_t(w)$ where σ_∞ is the life time of the path w. It holds that $X_{\sigma_\infty -}(w)=y$ with P_x^y-measure 1. ([1]). The following definition is due to Kunita and Watanabe [6].

Definition 1. *We call $y\in\mathrm{M}$ an exit boundary point if and only if there exists at least one point x of* R *such that $P_x^y(\sigma_\infty<+\infty)>0$. If $P_x^y(\sigma_\infty<\infty)=0$ for every point x of* R, *then we call $y\in\mathrm{M}$ a passive boundary point.*

The totality of the exit boundary points will be denoted by $(\mathrm{M})_{ex}$. Put $\hat{\mathrm{R}}_1=\mathrm{R}\cap(\mathrm{M})_{ex}$. Then we can check that $P_x^y(\sigma_\infty<+\infty)=1$ holds for $y\in\hat{\mathrm{R}}_1-\{x_0\}$, $x\in\mathrm{R}'$.

Now put

$K_\alpha(x, y)=K(x, y)E^y_x(e^{-\alpha\sigma_\infty})$, for $y\in\hat{\mathrm{R}}-\{x_0\}$, $x\in\mathrm{R}'$, and $\alpha>0$.

Lemma 1.

i) *For any $y\in\hat{\mathrm{R}}-\{x_0\}$, $\alpha>0$, the following equation holds.*

$$K(x, y)=\alpha\int_{\mathrm{R}}G_\alpha(x, z)K(z, y)dz+K_\alpha(x, y) \qquad \text{for } x\in\mathrm{R}'. \tag{1}$$

ii) *For any $y\in\hat{\mathrm{R}}_1-\{x_0\}$, $\alpha>0$, it holds that*

$$K(x, y)=\alpha\int_{\mathrm{R}}G(x, z)K_\alpha(z, y)dz+K_\alpha(x, y) \qquad \text{for } x\in\mathrm{R}'. \tag{2}$$

$$\lim_{\alpha\to+\infty}K_\alpha(x, y)=0 \qquad \text{for } x\in\mathrm{R}'. \tag{3}$$

iii) *If y in* M *is a passive boundary point,*

$$K_\alpha(x, y)=0 \qquad \textit{for any } \alpha>0,\ x\in\mathrm{R}.$$

Proof. We only sketch the proof of ii). If $y\in\mathrm{R}_1-\{x_0\}$

$$\frac{\alpha}{K(x, y)}\int_{\mathrm{R}}G(x, z)K_\alpha(z, y)dz=\alpha\int_{\mathrm{R}}\frac{K_y(z)}{K_y(x)}G(x, z)\frac{K_\alpha(z, y)}{K(z, y)}dz$$
$$=\alpha E_x^y\Big(\int_0^{\sigma_\infty}E_{x_t}^y(e^{-\alpha\sigma_\infty})dt\Big).$$

where E_x^y denotes the expectation with respect to P_x^y-measure. According to the Markov property of $\mathfrak{B}^y$ and the fact that σ_∞ is a Markov time, we find that the last term is equal to

$$\alpha E_x^y\left(\int_0^{\sigma_\infty} e^{-\alpha(\sigma_\infty - t)}dt\right) = P_x^y(\sigma_\infty < +\infty) - E_x^y(e^{-\alpha\sigma_\infty}) = 1 - \frac{K_\alpha(x,y)}{K(x,y)}.$$

Thus the equation (2) holds. Further we have

$$\lim_{\alpha\to+\infty}\frac{K_\alpha(x,y)}{K(x,y)} = \lim_{\alpha\to+\infty} E_x^y(e^{-\alpha\sigma_\infty}) = P_x^y(\sigma_\infty = 0) = 0,$$

for any x in R′.

From (1) and the resolvent equation, it follows that $K_\alpha(x,y) = \dfrac{G_\alpha(x,y)}{G(x_0,y)}$ for $y \in \mathrm{R} - \{x_0\}$, $x \in \mathrm{R}'$.

Definition 2. *For $x, y \in \hat{\mathrm{R}} - \{x_0\}$ and $\alpha > 0$, $U_\alpha(x,y)$ is defined by*

$$U_\alpha(x,y) = \alpha\int_{\mathrm{R}} K(z,x)K_\alpha(z,y)\,dz. \tag{4}$$

By the next lemma, we find that U_α is non-decreasing in α. We put $U(x,y) = \lim_{\alpha\to+\infty} U_\alpha(x,y)$, and we shall call it *Feller's kernel* with respect to the Brownian motion on R.

$U_\alpha(x,y)$ has the meaning even if x is a non-minimal boundary point. Therefore we can regard it as a K-potential as a function of x in the sense of Naïm.

Lemma 2.

i) *$U_\alpha(x,y)$ is not identically infinity, and*

$$\text{if } \alpha < \beta, \text{ then } U_\alpha(x,y) \leqq U_\beta(x,y) \quad \text{for } x, y \in \hat{\mathrm{R}} - \{x_0\}. \tag{5}$$

ii) *If $x, y \in \hat{\mathrm{R}}_1 - \{x_0\}$,*

$$U_\alpha(x,y) = \alpha\int_{\mathrm{R}} K_\alpha(z,x)K(z,y)dz = U_\alpha(y,x). \tag{6}$$

Proof.

i) When y is the passive boundary point, U_α is zero. If $y \in \hat{\mathrm{R}}_1 - \{x_0\}$, by (2), we can see that for $x \in \mathrm{R}' - \{x_0\}$

$$U_\alpha(x,y) = \alpha\int_{\mathrm{R}} \frac{G(z,x)}{G(x_0,x)} K_\alpha(z,y)dz = \frac{K(x,y) - K_\alpha(x,y)}{G(x,x_0)} < +\infty. \tag{7}$$

Further this equation shows that the inequality (5) holds for $x \in \mathrm{R}' - \{x_0\}$, therefore according to one of the principal properties of the K-potential ([7] p. 200), (5) is true for any $x \in \hat{\mathrm{R}} - \{x_0\}$.

ii) For $x \in \hat{R}_1 - \{x_0\}$ and the compact subset D of R, let $K_x^{R-D}(z)$ be the *extrémale* of $K_x(z)$ relative to $R-D$ ([7] p. 192). $K_x^{R-D}(z)$ can be written in the potential form,

$$\int_R G(z, z')\mu_x^{R-D}(dz'),$$

where μ_x^{R-D} is a non-negative measure on R. We note that

$$K_x^{R-D}(z) \uparrow K_x(z) \text{ when } D \uparrow R. \quad \text{Putting } K_{\alpha,x}^{R-D}(z) = \int_R G_\alpha(z, z')\mu_x^{R-D}(dz'),$$

we find, by the resolvent equation and (1), that

$$\lim_{D\uparrow R} K_{\alpha,x}^{R-D}(z) = K_\alpha(z, x).$$

On account of (1) and (2),

$$\alpha\int_R K_x^{R-D}(z) K_\alpha(z, y)\, dz = \alpha\int_R K_{\alpha,x}^{R-D}(z) K(z, y) dz$$

for any $y \in \hat{R}_1 - \{x_0\}$.
Applying Fatou's lemma,

$$U_\alpha(x, y) = \alpha\int_R K(z, x) K_\alpha(z, y) dz \geqq \alpha\int_R K_\alpha(z, x) K(z, y) dz = U_\alpha(y, x).$$

Since x and y are arbitrary in $\hat{R}_1 - \{x_0\}$, we have

$$U_\alpha(y, x) = U_\alpha(x, y),$$

which was to be proved.

Now, for $x, y \in \hat{R} - \{x_0\}$, let $\theta(x, y)$ be Naïm's kernel. By definition, it is equal to $\dfrac{2\,K(x, y)}{qG(x_0, x)}$ when $x \in R - \{x_0\}$, $y \in \hat{R} - \{x_0\}$, and equal to

$$\lim_{D\to R}\int_R \theta(z, y) G(x_0, z)\mu_x^{R-D}(dz) \text{ when } x, y \in \hat{R} - \{x_0\}.[1)]$$

Obviously $\theta(x, y)$ is symmetric in x, y. We shall prove the following theorem.

THEOREM 1. *For any* $x, y \in \hat{R}_1 - \{x_0\}$,

(8) $$U(x, y) = \frac{q}{2}\theta(x, y).$$

[1)] We denote by q either 2π (if $N=2$) or the product of $N-2$ and the unit ball boundary area (if $N>2$), where N is the dimension of R.

https://doi.org/10.1017/S0027763000011399 Published online by Cambridge University Press

Proof. Let y be fixed arbitrarily in $\hat{R}_1 - \{x_0\}$. If $x \in R - \{x_0\}$ on account of (3) and (7),

$$U(x, y) = \lim_{\alpha \to +\infty} U_\alpha(x, y) = \frac{K(x, y)}{G(x_0, x)} = \frac{q}{2}\theta(x, y). \tag{9}$$

When $x \in (M)_{ex}$, according to the principal properties of K-potintial and θ-kernel ([7] p. 214), we find that

$$U_\alpha(x, y) = \lim_{\substack{x' \to x \\ x' \in R}} \frac{K(x', y) - K_\alpha(x', y)}{G(x_0, x)} \leqq \lim_{\substack{x' \to x \\ x' \in R}} \frac{K(x', y)}{G(x_0, x')} = \frac{q}{2}\theta(x, y)$$

Thus we have

$$U(x, y) \leqq \frac{q}{2}\theta(x, y) \qquad \text{for } x \in (M)_{ex} \text{ and } y \in \hat{R}_1 - \{x_0\}.$$

On the other hand,

$$\int_R U_\alpha(x, z) G(x_0, z) \mu_y^{R-D}(dz) = \alpha \int_R K_\alpha(z, x) K_y^{R-D}(z) dz$$

$$\leqq \alpha \int_R K_\alpha(z, x) K(z, y) dz = U_\alpha(x, y).$$

Letting α tend to infinity, we find that

$$\int_R U(x, z) G(x_0, z) \mu_y^{R-D}(dz) \leqq U(x, y).$$

But since $U(x, z)$ in the integrand of the left member is equal to $\frac{q}{2}\theta(z, x)$ by (6) and (9), letting the compact subset D tend to R, we obtain the following inequality.

$$\frac{q}{2}\theta(x, y) \leqq U(x, y). \qquad \text{for } x \in (M)_{ex} \text{ and } y \in \hat{R}_1 - \{x_0\}.$$

Remark. Assume that $M = (M)_{ex}$ and that there is no non-minimal boundary point. According to Doob [2], the above theorem implies that, for any BLD harmonic function u on R, the Dirichlet integral $D(u)$ of u can be represented in the following form.

$$D(u) = \int_M \int_M (u'(x) - u'(y))^2 U(x, y) \mu(dx) \mu(dy),$$

where u' is the find boundary value of u and $\mu(E)$ is the harmonic measure of a subset E of M relative to the reference point x_0.

https://doi.org/10.1017/S0027763000011399 Published online by Cambridge University Press

§ 2. The Dirichlet norm in the case of A-diffusion

Let D be a bounded domain of the N-dimensional Euclidean space, whose boundary ∂D is a $(N-1)$-dimensional hypersurface of class $\mathbf{C}^3$.

Consider the self-adjoint elliptic differential operator A expressed in terms of the local coodinates as

$$Au(x) = \frac{1}{\sqrt{a(x)}}\frac{\partial}{\partial x^i}\left(a^{ij}(x)\sqrt{a(x)}\frac{\partial u(x)}{\partial x^j}\right)$$

where $a^{ij}(x)$ is the contravariant tensor on $\overline{D}$ which is strictly positive definite on $\overline{D}$, and $a(x) = \det(a^{ij}(x))^{-1}$.[2] We assume that $a^{ij}(x)$ is a function of class $\mathbf{C}^3$ on $\overline{D}$.

Let $p(t, x, y)$ be the fundamental solution of the diffusion equation

$$\frac{\partial u(t, x)}{\partial t} = Au(t, x) \qquad \text{for } x \in D,$$

with the boundary condition, $u(t, x) = 0$ for $x \in \partial D$.

Let $\{\Omega, \mathbf{B}, P_x, x \in D, X_t, t \geqq 0\}$ be a continuous strong Markov process with the transition probability $p(t, x, y)dy$ and it will be called an absorbing barrier A-diffusion [4].

According to S. Ito [5] and Ikeda, Ueno, Tanaka and Sato [4], we can verify the following related facts to this diffusion, We first note that $p(t, x, y)$ is symmetric in x, y. Put $G_\alpha(x, y) = \int_0^{+\infty} e^{-\alpha t} p(t, x, y)\, dt$, $(\alpha \geqq 0)$ and $G(x, y) = G_0(x, y)$, for $x, y \in D$, and put $h_\alpha(x, \xi) = h_\alpha(\xi, x) \left(= \frac{\partial}{\partial n_\xi} G_\alpha(x, \xi)\right)$ $(\alpha \geqq 0)$ and $h(x, \xi) = h_0(x, \xi)$ for $x \in D$, $\xi \in \partial D$. Further we define $U_\alpha(\xi, \eta) = \alpha \int_D h_\alpha(\xi, z) h(z, \eta)\, dz$ for $\xi, \eta \in \partial D$. Then we have

$$h(x, \xi) = \alpha \int_D G(x, z) h_\alpha(z, \xi)\, dz + h_\alpha(x, \xi) \qquad \text{for } x \in D,\ \xi \in \partial D \text{ and } \alpha > 0, \tag{10}$$

$$U_\alpha(\xi, \eta) = \int_0^{+\infty} (1 - e^{-\alpha t}) \frac{\partial^2}{\partial n_\xi \partial n_\eta} p(t, \xi, \eta)\, dt \qquad \text{for } \xi, \eta \in \partial D \text{ and } \alpha > 0. \tag{11}$$

We call

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[2] We denote by dx, $d\xi$ and $\frac{\partial}{\partial n_\xi}$, the volume element in D, surface element in ∂D and normal derivative to the boundary point ξ respectively, each of which is determined by the fundamental tensor $a^{ij}(x)$. Here the summation sign $\sum_{i,j}$ is omitted as usual.

$$U(\xi, \eta) = \lim_{\alpha \uparrow +\infty} U_\alpha(\xi, \eta) = \int_0^{+\infty} \frac{\partial^2}{\partial n_\xi \partial n_\eta} p(t, \xi, \eta) dt$$

Feller's kernel for our diffusion.

For the continuous function u on ∂D, we define Hu by

$$Hu(x) = \int_{\partial D} h(x, \xi) u(\xi) d\xi.$$

We have $\lim_{t \uparrow \sigma_\infty} Hu(X_t) = u(X_{\sigma_\infty -})$ and $Hu(x) = E_x(u(X_{\sigma_\infty -}))$ for $x \in D$, where σ_∞ is the life time of the path and $X_{\sigma_\infty -} = \lim_{t \uparrow \sigma_\infty} X_t \in \partial D$. In our case $P_x(\sigma_\infty < \infty) = 1$ holds for any $x \in D$.

Now let us show that $v(x) = E_x((u(X_{\sigma_\infty -}) - Hu(X_0))^2)$ is a potential.

Put
$$\begin{aligned} S_t &= Hu(X_t) - Hu(X_0) \qquad 0 \leqq t < \sigma_\infty \\ &= u(X_{\sigma_\infty -}) - Hu(X_0) \qquad t \geqq \sigma_\infty \end{aligned}$$

Since S_t is an additive functional and $E_x(S_{+\infty}) = 0$, we obtain the equality,

$$v(x) = E_x(S^2_{+\infty};\ t \geqq \sigma_\infty) + E_x(S^2_t;\ t < \sigma_\infty) + E_x(v(X_t);\ t < \sigma_\infty),$$

which implies that v is an excessive function.

Moreover if D_n, $n = 1, 2, \ldots$, are open subsets of D such that $\overline{D}_n \subset D_{n+1}$, $D_n \uparrow D (n \uparrow +\infty)$ and if σ_n is the first leaving time from D_n of the path, then

$$E_x(v(X_{\sigma_n})) = v(x) - E_x(S^2_{\sigma_n}) \longrightarrow 0 \qquad (n \to +\infty).$$

Thus v is a potential, that is, there exists a non negative measure ν on D such that

$$v(x) = \int_D G(x, y) \nu(dy). \tag{12}$$

Denote by $D(u)$ the total mass of ν which is uniquely determined by u.[3)]

LEMMA 3. *It holds that*

$$D(u) = \lim_{\alpha \to +\infty} \alpha \int_{\partial D} \int_D h_\alpha(\xi, x) v(x) dx d\xi. \tag{13}$$

Proof. By (10) and (12), we can see that

$$\begin{aligned} \alpha \int_{\partial D} \int_D h_\alpha(\xi, x) v(x) dx d\xi &= \int_{\partial D} \int_D (h(\xi, x) - h_\alpha(\xi, x)) \nu(dx) d\xi \\ &\uparrow \int_{\partial D} \int_D h(\xi, x) \nu(dx) d\xi = \nu(D) \qquad (\alpha \uparrow +\infty) \end{aligned}$$

3) $\nu(dx) = a^{ij}(x) \dfrac{\partial Hu(x)}{\partial x^i} \dfrac{\partial Hu(x)}{\partial x^j} dx$ at least when u is the function of class C^3.

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THEOREM 2. *If Hu is the function of class* C^1 on, $\overline{D}$, then

$$D(u)=\int_{\partial D}\int_{\partial D}(u(\xi)-u(\eta))^2 U(\xi,\eta)d\xi d\eta. \tag{14}$$

Proof. We find by simple caluculations,

$$\int_{\partial D}\int_{\partial D}(u(\xi)-u(\eta))^2 U_\alpha(\xi,\eta)d\xi d\eta-\alpha\int_{\partial D}\int_D h_\alpha(\xi,x)v(x)dxd\xi \tag{15}$$
$$=\alpha\int_{\partial D}\int_D (Hu(x)-u(\xi))^2 h_\alpha(\xi,x)dxd\xi,$$

for the continuous function u on ∂D.

According to Lemma 3, (14) is true if and only if the right side of (15) tends to zero as α tends to infinity.

Take a fixed Euclidean coodinate, by which we represent x, ξ as $(x^1, x^2; \cdots x^N)$, $(\xi^1,\xi^2,\ldots,\xi^N)$ respectively. When Hu is in the class C^1 on $\overline{D}$, there exists a positive constant C such that

$$|Hu(x)-u(\xi)|\leqq C\sum_{i=1}^N |x^i-\xi^i|, \qquad \text{for any } x\in D \text{ and any } \xi\in\partial D.$$

Further relying upon the following estimate of the fundamental solution given in [4],

$$\frac{\partial}{\partial n_\xi}p(t,\xi,x)\le (C_1 t^{-1/2}+C_2)t^{-N/2}\exp\left(-\frac{C_3\sum_{i=1}^N|x^i-\xi^i|^2}{4t}\right), \qquad x\in D,\ \xi\in\partial D,$$

(C_1, C_2, C_3 are some positive constants),
we can see for each j, $(1\leqq j\leqq N)$,

$$\alpha\int_D h_\alpha(\xi,x)|x^j-\xi^j|^2 dx^1dx^2\cdots dx^N$$
$$=\alpha\int_0^{+\infty}e^{-\alpha t}\left[\int_D\frac{\partial}{\partial n_\xi}p(t,\xi,x)|x^j-\xi^j|^2dx^1dx^2\cdots dx^N\right]dt\leqq\frac{C'}{\sqrt{\alpha}}, \qquad (\alpha>1),$$

where C' is a positive constant independent of ξ and α.
Therefore the right side of (15) vanishes as α tends to infinity.

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