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ON THE TRANSCENDENCE OF SOME INFINITE SERIES

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Abstract. The paper deals with a criterion for the sum of a special series to be a transcendental number. The result does not make use of divisibility properties or any kind of equation and depends only on the random oscillation of convergence.

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1. Introduction. Erdös [**2**] proved that if $\{a_n\}_{n=1}^{\infty}$ is an increasing sequence of positive integers such that $\lim_{n\to\infty} a_n^{\frac{1}{2^n}} = \infty$ then the number $S = \sum_{n=1}^{\infty} \frac{1}{a_n}$ is irrational. In [**4**] it is shown that if $\lim_{n\to\infty} \log_3 \log_2 a_n > 1$ and $a_n \in \mathbb{N}$ for all $n \in \mathbb{N}$ then S is a transcendental number. Many other criteria for S to be transcendental can be found in [**1**], [**5**], [**6**] or [**8**] but divisibility properties or fullfilling special equations are necessary. It seems to be the case that in general it is not easy to decide when S is a transcendental number if $\limsup_{n\to\infty} \log_3 \log_2 a_n < 1$ holds and divisibility properties or fullfilling certain equations are not required. In this paper we give conditions on sequences $\{a_n\}_{n=1}^{\infty}$ with $\limsup_{n\to\infty} \log_3 \log_2 a_n < 1$ such that S is transcendental. We prove the following.

THEOREM 1.1. *Let $\{a_n\}_{n=1}^{\infty}$ be an eventually non-decreasing sequence of positive integers such that $a_n > 2^n$ for every sufficiently large n. Suppose that $a_n < 2^{3^{\frac{1}{4}n}}$ and that $a_{2n} > 2^{3^{\frac{3}{4}(2n)}}$ for infinitely many n. Then the number $\sum_{n=1}^{\infty} \frac{1}{a_n}$ is transcendental.*

EXAMPLE 1.1. Let $a_1 = a_2 = 1$. For every $s = 0, 1, 2, 3, \ldots$ set

$$a_n = \begin{cases} 2^{[3^{\frac{1}{4}(5n-42^{3^s})}]} + 3 & \text{if } 2^{3^s} < n \le 2.2^{3^s} \\ 2^{[3^{\frac{6}{4}2^{3^s}}]} + 2^{[3^{\frac{1}{4}n}]} + 3 & \text{if } 2.2^{3^s} < n \le 2^{3^{s+1}}. \end{cases}$$

Then the number $\sum_{n=1}^{\infty} \frac{1}{a_n}$ is transcendental.

It is unclear to the authors if there exists a sequence $\{a_n\}_{n=1}^{\infty}$ of positive integers such that $\sum_{n=1}^{\infty} \frac{1}{a_n}$ is an algebraic number and $a_n > 2^{(\frac{5}{2})^n}$ for all $n \in \mathbb{N}$.

2. Main results. In the sequel, for a real number x we use $[x]$ to denote the greatest integer less than or equal x. Theorem 1.1 is an immediate consequence of the following theorem.

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THEOREM 2.1. *Let* α, β, γ *and* ν *be real numbers with* $0 < \beta < \alpha < \log_2 3$, $0 \leq \nu < 1$ *and* $\gamma > 0$. *Assume that* $\{a_n\}_{n=1}^{\infty}$ *and* $\{b_n\}_{n=1}^{\infty}$ *are sequences of positive integers with* $\{a_n\}_{n=1}^{\infty}$ *eventually non-decreasing such that for every sufficiently large n*

$$b_n < a_n^{\nu} \log_2^{\gamma} a_n \tag{1}$$

and

$$a_n > 2^n . \tag{2}$$

Suppose that there exists a positive real number k with

$$k < \frac{(\alpha - \beta)}{\log_2(\frac{2}{1-\nu} + 1) - \alpha} \tag{3}$$

such that for infinitely many n

$$a_n < 2^{2^{\beta n}} \tag{4}$$

and

$$a_{n+[k.n]} > 2^{2^{\alpha(n+[k.n])}} . \tag{5}$$

Then the number $\sum_{n=1}^{\infty} \frac{b_n}{a_n}$ *is transcendental.*

As an immediate consequence of Theorem 2.1 we obtain the following corollary.

COROLLARY 2.1. *Let* α, β *and* γ *be real numbers with* $0 < \beta < \alpha < 1$ *and* $\gamma > 0$. *Assume that* $\{a_n\}_{n=1}^{\infty}$ *and* $\{b_n\}_{n=1}^{\infty}$ *are sequences of positive integers with* $\{a_n\}_{n=1}^{\infty}$ *eventually non-decreasing such that for every sufficiently large n*

$$b_n < \log_2^{\gamma} a_n$$

and

$$a_n > 2^n .$$

Suppose that there exists positive real number k with

$$k < \frac{(\alpha - \beta)}{1 - \alpha}$$

such that for infinitely many n

$$a_n < 2^{3^{\beta n}}$$

and

$$a_{n+[k.n]} > 2^{3^{\alpha(n+[k.n])}} .$$

Then the number $\sum_{n=1}^{\infty} \frac{b_n}{a_n}$ *is transcendental.*

REMARK 2.1. Let the sequences $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ satisfy all conditions (1)–(5). Then Theorem 2.1 implies that the number $\sum_{n=1}^{\infty} \frac{b_n}{a_n}$ is transcendental. If in addition

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there exists a fixed $\epsilon > 0$ such that

$$a_n < 2^{(2-\epsilon)^n}$$

holds for all sufficiently large n then there exists a sequence $\{c_n\}_{n=1}^{\infty}$ of positive integers such that $\sum_{n=1}^{\infty} \frac{b_n}{c_n a_n}$ is a rational number. For more information see [**3**].

3. Proof. *Proof.* (of Theorem 2.1) Let N be a sufficiently large positive integer satisfying (4) and (5). Assume that δ is a sufficiently small positive real number.

Let us define the finite sequence $\{c_t\}_{t=N}^{N+[k.N]}$ by

$$c_t = \begin{cases} a_t^t, & \text{if } t = N \\ a_t^{\frac{1}{\left(\frac{2}{1-\nu}+1+\delta\right)^{t-N}}}, & \text{if } t = N+1, N+2, \ldots, N+[k.N]. \end{cases}$$

Set

$$c_T = \max_{t=N,N+1,\ldots,N+[k.N]} c_t \,. \tag{6}$$

If $c_T = c_N$ then from (4) and (5) we obtain

$$2^{N2^{\beta N}} > a_N^N = c_N \geq c_{N+[k.N]} = a_{N+[k.N]}^{\frac{1}{\left(\frac{2}{1-\nu}+1+\delta\right)^{[k.N]}}} > 2^{\frac{2^{\alpha(N+[k.N])}}{\left(\frac{2}{1-\nu}+1+\delta\right)^{[k.N]}}} = 2^{2^{\alpha(N+[k.N])-[k.N]\log_2\left(\frac{2}{1-\nu}+1+\delta\right)}}.$$

Applying $\log_2$ twice to the above inequality we get

$$\log_2 N + \beta N > \alpha(N+[k.N]) - [k.N]\log_2\left(\frac{2}{1-\nu}+1+\delta\right)$$

and this is a contradiction with (3).

Therefore $c_T \neq c_N$ and thus

$$c_T \geq \max_{j=N,N+1,\ldots,T-1} c_j \,.$$

From this and from the fact that the sequence $\{a_n\}_{n=1}^{\infty}$ is eventually non-decreasing we obtain that

$$a_T \geq \left(\max_{j=N,N+1,\ldots,T-1} c_j\right)^{\left(\frac{2}{1-\nu}+1+\delta\right)^{T-N}} > \prod_{i=N}^{T-1}\left(\max_{j=N,N+1,\ldots,T-1} c_j\right)^{\left(\frac{2}{1-\nu}+\delta\right)\cdot\left(\frac{2}{1-\nu}+1+\delta\right)^{i-N}}. \tag{7}$$

Here the second inequality comes from the fact that

$$\frac{\left(\frac{2}{1-\nu}+1+\delta\right)^{T-N}}{\left(\frac{2}{1-\nu}+1+\delta\right)-1} > \frac{\left(\frac{2}{1-\nu}+1+\delta\right)^{T-N}-1}{\left(\frac{2}{1-\nu}+1+\delta\right)-1} = \left(\frac{2}{1-\nu}+1+\delta\right)^{T-N-1} + \left(\frac{2}{1-\nu}+1+\delta\right)^{T-N-2} + \cdots + 1\,.$$

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The fact that $\{a_n\}_{n=1}^{\infty}$ is the eventually non-decreasing sequence and inequality (7) yield

$$a_T > \left(\prod_{i=N}^{T-1}\Bigl(\max_{j=N,N+1,\dots,T-1} c_j\Bigr)^{\left(\frac{2}{1-\nu}+1+\delta\right)^{i-N}}\right)^{\frac{2}{1-\nu}+\delta} \geq \left(\prod_{i=N}^{T-1} c_i^{\left(\frac{2}{1-\nu}+1+\delta\right)^{i-N}}\right)^{\frac{2}{1-\nu}+\delta}$$
$$= \left(a_N^N \prod_{i=N+1}^{T-1} a_i\right)^{\frac{2}{1-\nu}+\delta} \geq \left(\prod_{i=1}^{T-1} a_i\right)^{\frac{2}{1-\nu}+\delta}.$$

This implies that

$$a_T^{1-\nu} = \left(a_T^{\frac{1+\frac{\delta}{2}(1-\nu)}{1+\frac{\delta}{2}(1-\nu)}}\right)^{1-\nu} = a_T^{\frac{1-\nu}{1+\frac{\delta}{2}(1-\nu)}} \cdot a_T^{\frac{\frac{\delta}{2}(1-\nu)^2}{1+\frac{\delta}{2}(1-\nu)}} > a_T^{\frac{\frac{\delta}{2}(1-\nu)^2}{1+\frac{\delta}{2}(1-\nu)}} \cdot \left(\prod_{i=1}^{T-1} a_i\right)^2. \tag{8}$$

Now we will prove that for every sufficiently large N

$$\sum_{n=N+1}^{\infty} \frac{b_n}{a_n} < \frac{2\log_2^2 a_{N+1}}{a_{N+1}^{1-\nu}}. \tag{9}$$

From (1), (2) and the fact that $\{a_n\}_{n=1}^{\infty}$ is an eventually non-decreasing sequence of positive integers we obtain that

$$\sum_{n=N+1}^{\infty} \frac{b_n}{a_n} < \sum_{n=N+1}^{\infty} \frac{\log_2^{\gamma} a_n}{a_n^{1-\nu}} = \sum_{N<n\leq \log_2 a_{N+1}} \frac{\log_2^{\gamma} a_n}{a_n^{1-\nu}} + \sum_{\log_2 a_{N+1}<n} \frac{\log_2^{\gamma} a_n}{a_n^{1-\nu}} < \frac{\log_2^{1+\gamma} a_{N+1}}{a_{N+1}^{1-\nu}}$$
$$+ \sum_{\log_2 a_{N+1}<n} \frac{\log_2^{\gamma} a_n}{a_n^{1-\nu}} < \frac{\log_2^{1+\gamma} a_{N+1}}{a_{N+1}^{1-\nu}} + \sum_{\log_2 a_{N+1}<n} \frac{n}{2^{n(1-\nu)}} \leq \frac{2\log_2^{1+\gamma} a_{N+1}}{a_{N+1}^{1-\nu}}.$$

Let T satisfies (6). Inequalities (8) and (9) imply that for every sufficiently large T

$$\left|\sum_{n=1}^{\infty} \frac{b_n}{a_n} - \sum_{n=1}^{T-1} \frac{b_n}{a_n}\right| = \left|\sum_{n=1}^{\infty} \frac{b_n}{a_n} - \frac{\prod_{n=1}^{T-1} a_n \sum_{n=1}^{T-1} \frac{b_n}{a_n}}{\prod_{n=1}^{T-1} a_n}\right| = \left|\sum_{n=T}^{\infty} \frac{b_n}{a_n}\right| \leq \frac{2\log_2^{1+\gamma} a_T}{a_T^{1-\nu}}$$
$$< \frac{2\log_2^{1+\gamma} a_T}{a_T^{\frac{\frac{\delta}{2}(1-\nu)^2}{1+\frac{\delta}{2}(1-\nu)}} \cdot \left(\prod_{i=1}^{T-1} a_i\right)^2} = \frac{2\log_2^{1+\gamma} a_T}{\left(a_T^{1-\nu}\right)^{\frac{\frac{\delta}{2}(1-\nu)}{1+\frac{\delta}{2}(1-\nu)}} \cdot \left(\prod_{i=1}^{T-1} a_i\right)^2}$$
$$< \frac{2\log_2^{1+\gamma} a_T}{a_T^{\frac{\frac{\delta^2}{4}(1-\nu)^3}{\left(1+\frac{\delta}{2}(1-\nu)\right)^2}} \cdot \left(\prod_{i=1}^{T-1} a_i\right)^{2+\frac{\delta(1-\nu)}{1+\frac{\delta}{2}(1-\nu)}}}.$$

Set $q_T = \prod_{n=1}^{T-1} a_n$, $p_T = \prod_{n=1}^{T-1} a_n \sum_{n=1}^{T-1} \frac{b_n}{a_n}$ and $\epsilon = \frac{\delta(1-\nu)}{1+\frac{\delta}{2}(1-\nu)}$. Because

$$\frac{2\log_2^2 a_T}{a_T^{\frac{\frac{\delta^2}{4}(1-\nu)^3}{\left(1+\frac{\delta}{2}(1-\nu)\right)^2}}} \text{ tends to zero when } T \text{ tends to infinity}$$

we obtain the inequality

$$\left|\sum_{n=1}^{\infty} \frac{b_n}{a_n} - \frac{p_T}{q_T}\right| < \frac{1}{q_T^{2+\epsilon}} \tag{10}$$

which holds true for all sufficiently large T.

The fact that we can find infinitely many pairs (p_T, q_T) satisfying (10) and the Roth's Theorem [7] imply that the number $\sum_{n=1}^{\infty} \frac{b_n}{a_n}$ is transcendental.

□

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