ANALYSIS OF A SIMPLE MECHANISM TO DEPLETE THE KIRKWOOD GAPS

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§1. INTRODUCTION

In most applications resonance problems are implicitely or explicitely modelized by the *Fundamental Model of Resonance* i.e. the pendulum, characterized by its Hamiltonian function :

$$H = \frac{a}{2} I^2 - b \cos \psi \qquad (1)$$

This reduction is performed in two steps : first, the action-angles variables are introduced to get a "one degree of freedom" Hamiltonian system, given by (2) :

$$K = K_o(S) + \varepsilon K_1(S, s) \qquad (2)$$

where K_1 is 2π – periodic in s (the resonant angle).
Secondly, K_o and K_1 are expanded in Taylor's series with respect to S and then truncated.
This method is easy to apply when $K_1(0, s)$ is a non-constant function of s (its simplest form is $K_1(0, s) = \cos s$, which leads to the pendulum by a simple translation). However in many instances $K_1(S, s)$ possesses the "d'Alembert Characteristic" in $(\sqrt{2S}, s)$. This is the case in many orbit-orbit resonances where $\sqrt{2S}$ is proportional to the eccentricity and in some spin-orbit resonances where $\sqrt{2S}$ is there proportional to the obliquity.
In such cases, the simplest form for the truncated Hamiltonian is :

$$K' = \alpha S + \beta S^2 + \varepsilon \sqrt{2S} \cos s \qquad (3)$$

where the reduction to the pendulum (1) is no longer that simple and is only valid on some parts of the phase space and for some values of the three parameters α, β and ε .

The functions (1) and (3) are both *one degree of freedom* hamiltonians; the only advantage of (1) over (3) is that most of the computations can be carried out by means of elliptic functions which are well tabulated.

V. V. Markellos and Y. Kozai (eds.), Dynamical Trapping and Evolution in the Solar System, 189–201.

https://doi.org/10.1017/S0252921100097086 Published online by Cambridge University Press

We think that this advantage is not important enough to justify the approximation and the intricacies involved in the last step.

So we would like to introduce (3) as a *Second Fundamental Model of Resonance* as simple and as well documented as the pendulum (1) but closer to the main resonance problems.

A complete analysis of this model is described elsewhere (Henrard-Lemaître 1983). We plan to summarize this analysis here and to apply it to the problem of depletion of some of the *Kirkwood gaps* in the asteroïd belt.

§2. SCALING THE MODEL

The Hamiltonian (3) depends on three parameters α, β and ε which are not independent. By scaling the time and the actions, we can consider that our model has only one truly independent parameter called δ . Let us define :

$$r = \text{sign}\,\beta \,.\, s \qquad \text{if} \quad \beta\varepsilon < 0$$
$$r = \text{sign}\,\beta \,.\, s + \pi \qquad \text{if} \quad \beta\varepsilon > 0 \tag{4}$$

and the scaled time and momentum :

$$T = \left[\frac{|\beta|\ \varepsilon^2}{4}\right]^{1/3} .\ t \tag{5}$$

$$R = \left[\frac{2\ \beta}{\varepsilon}\right]^{2/3} .\ S \tag{6}$$

The Hamiltonian (3) is replaced by :

$$H(r, R) = -3(\delta + 1)\ R + R^2 - 2\sqrt{2R}\ \cos r \tag{7}$$

and δ is given by :

$$\delta = -\left[\frac{4}{27}\ \frac{\alpha^3}{\beta\varepsilon^2}\right]^{1/3} - 1 \tag{8}$$

The Hamiltonian (7) is not differentiable at $R = 0$; this is why we introduce canonical cartesian variables x and y :

$$x = \sqrt{2\ R}\ \cos r \qquad \text{and} \qquad y = \sqrt{2\ R}\ \sin r \tag{9}$$

and the Hamiltonian (7) becomes :

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$$H_2(y, x) = -\frac{3}{2}(\delta + 1)(x^2 + y^2) + \frac{1}{4}(x^2 + y^2)^2 - 2x \qquad (10)$$

§3. LIBRATION AND CIRCULATION

The equilibria of the Hamiltonian dynamical system are given by the solutions of :

$$\frac{\delta H}{\delta x} = \frac{\delta H}{\delta y} = 0 \qquad . \qquad (11)$$

They are located at points $(x, y) = (x^\star, 0)$, where $x^\star$ is a root of :

$$x^3 - 3(\delta + 1)\, x - 2 = 0 \qquad . \qquad (12)$$

The location of the roots versus δ is given in figure 1. For $\delta > 0$ one of the roots (the leftmost one) is unstable and the other two are stable. Two homoclinic orbits come out of the unstable equilibrium. They enclose an area that we call the *topological libration zone* (see figure 2).

Besides the libration zone we distinguish two circulation zones (an internal one, inside the smallest homoclinic orbit and an external one, outside the largest homoclinic orbit).

To describe the capture into resonance we need a new concept. We shall call the *area index* of a trajectory the area enclosed by it (plus the area enclosed by the smallest homoclinic orbit in the case of libration trajectories). In this way there is a one-to-one correspondence between trajectories and area indexes (see figure 3).

§4. SOLUTIONS FOR MODERATE δ

In figures 4 and 5 some of the trajectories of the problem are plotted for some values of δ . In (Henrard - Lemaître 1983) we give more information about the orbits (periods, area indexes, energy constants).

For large values of δ asymptotic formulae can be calculated, obtained by expansions with respect to Δ , where $\Delta = -3(\delta + 1)$. (13)

§5. EVOLUTION THROUGH RESONANCE

Let us assume that the parameter δ slowly varies with the time, i.e.

$$|\dot{\delta}| \leqslant \eta \qquad\qquad |\ddot{\delta}| \leqslant \eta^2 \qquad (14)$$

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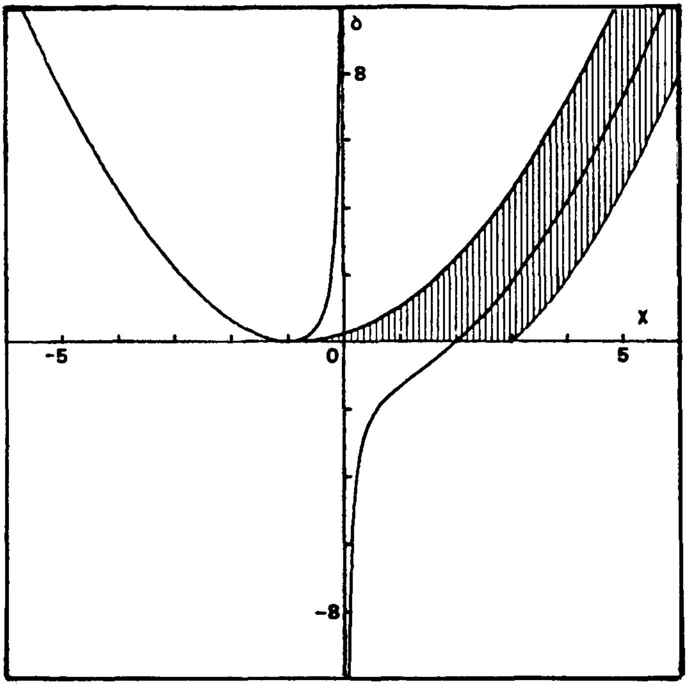

figure 1 : location of the equilibria .

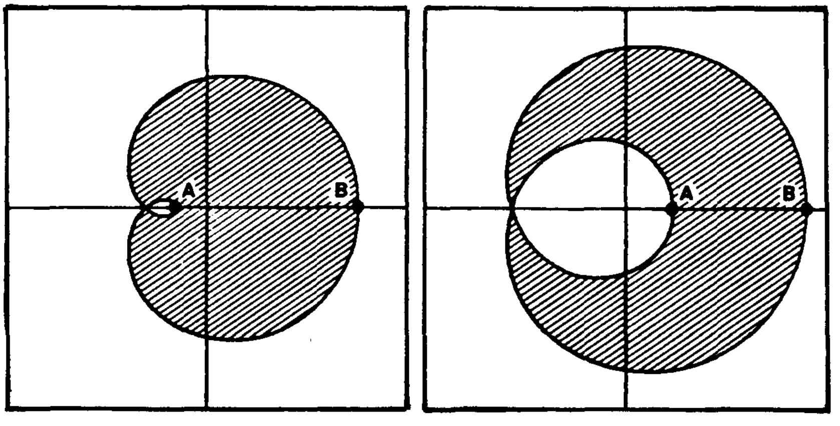

figure 2 : topological libration zone .

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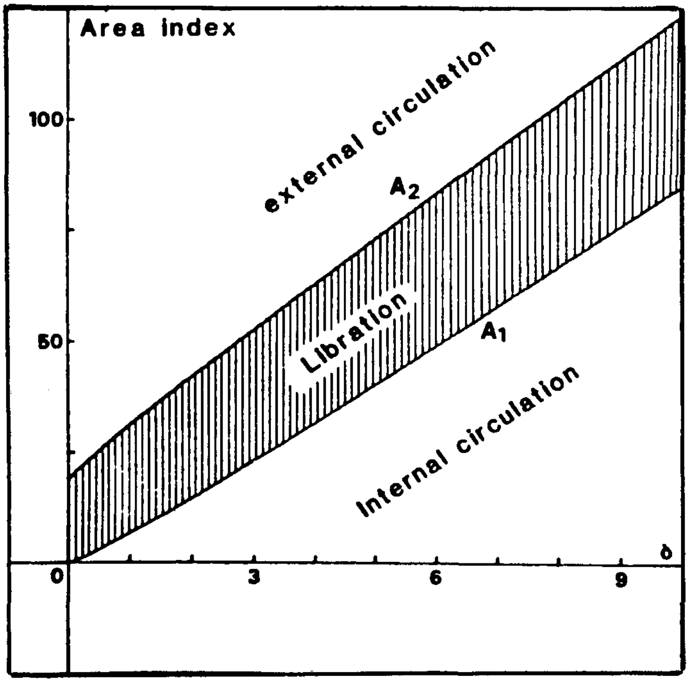

figure 3 : the critical areas and the three zones in the area index diagram .

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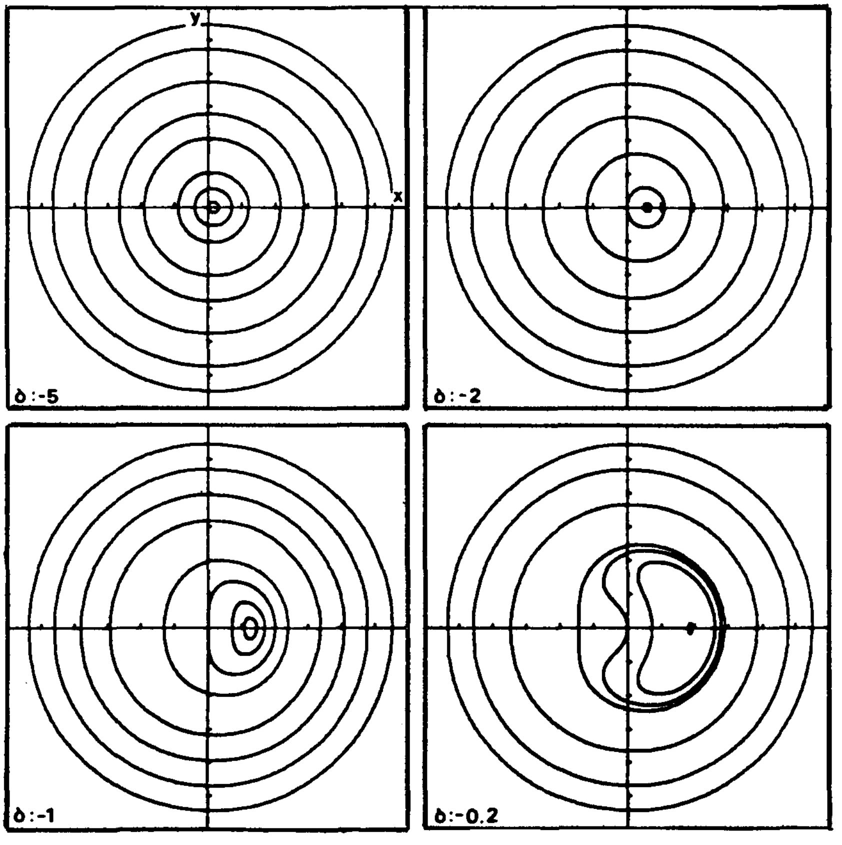

figure 4 : trajectories in the plane (x, y) for negative δ .

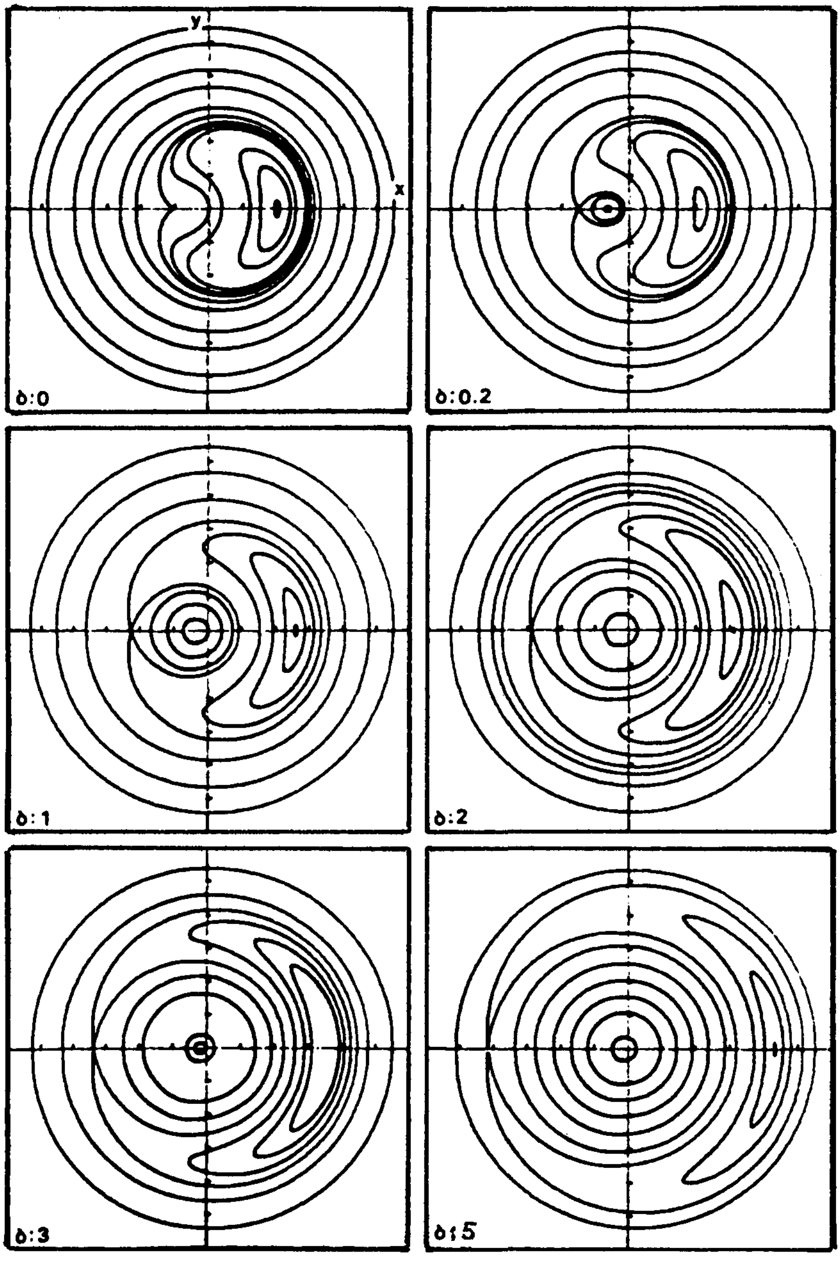

figure 5 ; trajectories in the plane (x, y) for positive δ .

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where η is some small quantity.

The solutions of system (7) are not anymore the closed curves described above but they are close to them.
They stay for a moderate period of time on a solution of system (7) which we call the *guiding trajectory*.

The guiding trajectory evolves and to identify it, we use the *adiabatic invariant* theory (Henrard 1982a) which states that the area enclosed by the guiding trajectory does not change by more than η for times smaller than η^{-1}

A problem appears when the guiding trajectory becomes an homoclinic orbit at some point in the evolution. In that case the classical theory of the adiabatic invariant breaks down. Under some assumptions (verified for the system (7)) the evolution of the system can be described in the following terms : (Henrard - 1982 b).

Let us assume that the guiding trajectory is first a circulation trajectory. After going through an homoclinic orbit, it may become either a circulation trajectory of the other type or a libration trajectory. This choice depends on the phase at the time of transition. It is expressed in terms of probability, assuming that all phases are equally probable. In both cases the area enclosed by the guiding trajectory undergoes a discontinuity but then stays constant again. A discontinuity in the area index appears, which we represent in terms of the *free vibration amplitude* defined by :

$$F = \left[\frac{\text{Area}}{\pi} \right]^{1/2} \tag{15}$$

This concept is based on the following idea : when δ is large in absolute value (a long time before or after transition) the trajectory is almost a circle, with F as radius and the origin as center. F can be interpreted as normalised eccentricity.

We must distinguish two types of behaviour : δ increasing or δ decreasing with the time. In our paper (Henrard - Lemaître 1983) we give a description of both cases and a detailed example for the first one. This is why we develop here only the second case, with an application of it in section 6 .
First we assume that our guiding trajectory is an internal circulation trajectory. When δ decreases to zero the area of the largest possible internal circulation goes to zero. Hence the guiding trajectory is forced to undergo a transition and to become an external circulation orbit.
No capture into libration is possible, because the area of the libration zone is also decreasing with δ .
Secondly, we assume that our guiding trajectory is an external circulation trajectory. When δ decreases to zero, it must stay an external

circulation orbit, because the area enclosed by the largest homoclinic orbit decreases.
Thirdly, we assume that our guiding trajectory is a libration one. When δ decreases with the time, it may become an external circulation orbit by undergoing a transition or if no transition has occured when $\delta = 0$, it becomes automatically a circulation.

If we note the free vibration amplitude after transition F_f and before transition F_i, we can plot the jump $(F_f - F_i)$ with respect to the initial value F_i (figure 6). This jump is always positive and can be interpreted as a jump in the normalized eccentricity.

§6. APPLICATION

The evolution in resonance we have just described could be of some interest to explain the well-known *Kirkwood Gaps* in the asteroïd belt.

We consider a circular restricted three-body problem Sun-Jupiter-Asteroïd, to describe the motion of a minor planet in a first-order resonance zone with Jupiter's mean motion n_J $\left(\frac{nA}{nJ} \cong \frac{p+1}{p}\right)$.

In terms of Schubart's variables N, S, U and s (see Schubart - 1966), the Hamiltonian function is given by :

$$K = - n_J (p + 1) (N - S) - \frac{\mu^2}{2(N - S)^2 p^2} + \mu a_J \frac{m_J}{M_A} P \left(\frac{a}{a_J}, e, s\right) \qquad (16)$$

where $\mu = G.M_A$, M_A being the *mass acting on the asteroïd*, m_J is Jupiter's mass, a is the semi-major axis, e is the eccentricity and s the resonant angle.

After expansion with respect to S and truncation, we obtain the Hamiltonian (3) :

$$K' = \alpha S + \beta S^2 + \varepsilon \sqrt{2S} \cos s$$

with

$$\alpha = n_J . (p + 1) - \frac{\mu^2}{p^2 N^3}$$

$$\beta = - \frac{3}{2} \frac{\mu^2}{p^2 N^4} \qquad (17)$$

$$\varepsilon = \frac{\mu}{a_J} \frac{m_J}{M_A} \frac{b}{\sqrt{N}}$$

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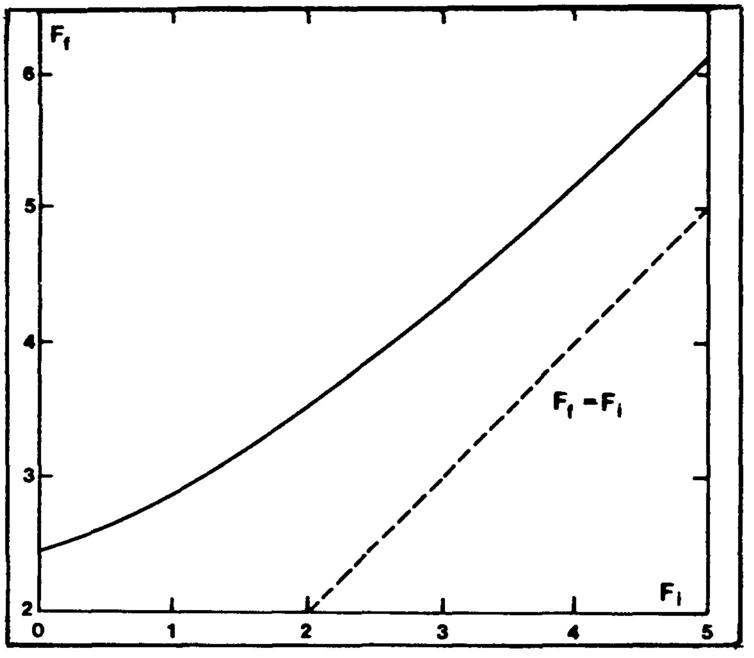

figure 6 : the free vibration amplitude after transition (F_f) versus its value (F_i) before transition for a passage through resonance with decreasing δ (for internal circulation orbits).

(b is the coefficient of e cos s in the expansion of the perturbation P) and by scaling, we get our "Second Fundamental Model of Resonance" (7), where δ is calculated as :

$$\delta = \frac{2}{3}\left(\frac{n_J}{n_A}\,\frac{p+1}{p} - 1\right)\left(\frac{1}{3}\,\frac{a_J^2}{a^2}\,\frac{M_A^2}{m_J^2}\,\frac{1}{b^2}\right)^{1/3} - 1 \quad . \tag{18}$$

If δ slowly decreases with the time (a few units is enough) we can reproduce exactly the same behaviour as in section 5 , which can be interpreted as a depletion of the libration zone. We can note an increase in the eccentricity of the asteroïds which undergo a transition from internal to external circulation.

However, a question still remains : how to explain that variation in δ ?

An idea would be to situate this phenomena during the solar nebulae dispersion. In a first approximation at least the effect of the nebula upon Jupiter and upon the asteroïd can be modelized by a change in the mass of the Central body. This change depends upon the amount of material included in the orbit of Jupiter or of the asteroïd and we have to distinguish between M_J : the mass of the pseudo-Sun as seen by Jupiter and M_A : the mass of the pseudo-Sun as seen by the Asteroïd.

Following Torbett and Smoluchowski (1980) we adopt Cameron's model for the description of the early solar system and more precisely we use the numerical data of example C of Torbett and Smoluchowski's paper.

Concurrently with the formation of the protosolar object, an accretion disk of the Lynden-Bell and Pringle(1974) variety forms and ultimately reaches a maximum mass M_D of $\sim 1.7\ M_S$ with a radius R_D of $\sim$ 600 AU.
At this point it reverses its growth and begins to disperse with a fraction of its mass going into the protosolar object and the rest being ejected out of the system.

Before dispersion the masses of the protosolar object (M_C) and of the disk (M_D) are :

$$M_C = 0{,}7\ M_S \qquad M_D = 1.7\ M_S \tag{19}$$

where M_S is the present solar mass.

During dispersion, the variation of M_J and M_A can be simply modelized by (Torbett-Smoluchowski 1980) :

$$M_J = M_C + \frac{\pi}{2} M_D \frac{a_J}{R_D}$$
$$M_A = M_C + \frac{\pi}{2} M_D \frac{a}{R_D} \qquad (20)$$

which gives an initial value to the ratio $\frac{M_J}{M_A}$:

$$\frac{M_J}{M_A} = \frac{0.1044}{0.1 + 0.0044 \frac{a_A}{A_J}} \quad \text{at time} \quad t = 0 \qquad (21)$$

(Today after disparition of the disk the ratio equals 1).

So for an asteroïd initially situated in the 2/1 resonance zone ($p = 1$, $\frac{a_A}{a_J} = 0.629$) this ratio varies from 1.0158 to 1 , which gives a variation in δ of 1.808 .
(We can notice that by angular momentum conservation $\frac{n_A}{n_J}$ is proportional to $\frac{M_A^2}{M_J^2}$).

Doing the same for an asteroïd in the 3/2 resonance zone ($p = 2$, $\frac{a_A}{a_J} = 0.763$) the ratio $\frac{M_J}{M_A}$ varies from 1.010 to 1 and δ decreases of 0.898 .

In conclusion, the mechanism of depletion of the Kirkwood Gaps we have just outlined seems promising.
It apparently predicts the right size for the gaps, the increase in eccentricity and a more marked depletion for the resonance 2/1 than for 3/2 . Of course it has to be investigated much more carefully both in its dynamical and cosmogonical aspects.

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