# ON THE COMPACITY OF THE ORTHOGONAL GROUPS

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It is a well known fact on Lorenz groups that a quadratic form $f$ is definite if and only if the corresponding orthogonal group $O_n(R_\infty, f)$, where $R_\infty$ is the real number field, is compact. In this note, we shall show that the analogue of this holds for the case of the $p$-adic orthogonal group $O_n(R_p, f)$, where $R_p$ is the rational $p$-adic number field, as a special result of the more general statement on the completely valued fields.

Let $K$ be a field with non-trivial valuation $|\ |$, and of characteristic $\neq 2$. Let $V$ be an $n$-dimensional vector space over $K$ and let $u_i$ $(i=1, \ldots, n)$ be some fixed basis of $V$ over $K$. If we define norm of $x = \sum_{i=1}^{n} x_i u_i \in V$ by $\|x\| = \max_{i=1,\ldots,n} |x_i|$, then the space $V$ is topologized as usual.[1] Now, let $E$ be the algebra of endomorphisms of $V$ over $K$. Using the above basis, we also define norm of transformation $X = (x_{ij})$ by $\|X\| = \max_{i,j=1,\ldots,n} |x_{ij}|$. It is easy to see that $\|X \cdot Y\| \leqq n\|X\| \cdot \|Y\|$. Thus, $E$ becomes a normed algebra over $K$. A subset $S$ of a normed space is called bounded if for some number $b > 0$ we have $\|x\| < b$ for all $x \in S$. For our normed space $V$, boundedness is independent of the choice of basis $u_i$. The same is true for the normed space $E$. If $K$ is locally compact, then a bounded and closed subset of a normed space over $K$ is the same thing as a compact subset. Now, let $f$ be a non-degenerate symmetric bilinear form on $V$. The orthogonal group $O_n(K, f)$ is obviously a closed subset of $E$. If $f$ and $g$ are congruent, it is easy to see that their groups are homeomorphically isomorphic and if one of them is bounded in $E$ so is the other. We say that a from $f$ is of index $\nu$ if $\nu$ is the maximum dimension of $U \subset V$ such that $U$ is a totally isotropic subspace of $V$.[2] $\nu = 0$ means that $f(x, x) = 0$ implies $x = 0$.

We prove the following

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1) See [1] p. 18.

2) See [2] p. 17.

THEOREM 1. *Let $K$ be a completely (non-trivially) valued field with characteristic $\neq 2$ and let $f$ be a non-degenerate symmetric bilinear form over $K$. Then the index $\nu$ of $f$ is zero if and only if the orthogonal group $O_n(K,f)$ is bounded in $E$.*

*Proof.* If $n=1$, since then $\nu=0$ always and the group is of order 2, the statement is trivial. So we assume that $n \geqq 2$. Suppose that $\nu \geqq 1$. Then $f$ is congruent to the form $g$ whose matrix is of type

$$G=\begin{pmatrix} 0 & 1 & \\ 1 & 0 & \\ & & * \end{pmatrix}^{3)}$$

Since ${}^t\begin{pmatrix} x & 0 \\ 0 & x^{-1} \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} x & 0 \\ 0 & x^{-1} \end{pmatrix}=\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ for all $x(\neq 0) \in K$, it follows that

$$X=\begin{pmatrix} x & 0 & & & \\ 0 & x^{-1} & & & \\ & & 1 & & \\ & & & \ddots & \\ & & & & 1 \end{pmatrix}$$

belongs to $O_n(K,g)$ for each $x(\neq 0) \in K$. Thus, $O_n(K,g)$ is not bounded in $E$. Hence, $O_n(K,f)$ is also not bounded. This proves the sufficiency. It is to be noted that we do not use the completeness of $K$.

Next, we shall prove the necessity.[4] Here the completeness of $K$ is used essentially. Assume that $O_n(K,f)$ is not bounded. Without loss of generality, we may suppose that the matrix of $f$ is of type

$$F=\begin{pmatrix} a_1 & & \\ & \ddots & \\ & & a_n \end{pmatrix} \quad \text{where } |a_i| \leqq 1,\ i=1,\dots,n$$

By our assumption, for any $N>0$ there exists an $X \in O_n(K,f)$ such that $\|X\|>N$. Suppose that $\|X\|=|x_{pq}|$. Comparing the $(q,q)$-components of both sides in ${}^tXFX=F$, we get $\sum_{i=1}^{n} a_i x_{iq}^2=a_q$. Multiplying $x_{pq}^{-2}$ on both sides, we see that

[3] See [3] Satz 5.

[4] The following proof is inspired by Theorem 2, Dieudonné [4].

the inequality $\left|\sum_{i=1}^{n} a_i x_i^2\right| < |a_q| N^{-2}$ has a solution $x_i$ such that $|x_i| \leqq 1$, $|x_p| = 1$. Now, if $K$ is locally compact then the unit cube, i.e. the set of $x$ with $\|x\| \leqq 1$ in $V$ is compact. Thus, for increasing $N$ we may select a sequence of vectors $x_N$ in the unit cube satisfying an inequality as above one of whose component, say $p_N$-th, is of value 1. Taking a subsequence, if necessary, we may assume that $p_N$ are all equal. It is obvious that $x = \lim_{N\to\infty} x_N$ gives a non-trivial solution of $f(x, x) = 0$. Thus, the necessity is proved for our special case, i.e. the case when $K$ is archimedean (that is, when $K$ is real or complex field) or $K$ is a finite extension of the Hensel $p$-adic number field $R_p$ with some prime $p$ or a field of power series of one variable over a finite field of characteristic $\neq 2$. Therefore, there remains to be considered a case of a non-archimedean field $K$. We shall construct a non-trivial solution of $f(x, x) = 0$ by successive approximation. We fix an element $c \in K$ such that $|c| < 1$, and put $d = 2a_1 \ldots a_n \cdot c$. Then, from the above argument, the inequality $\sum_{i=1}^{n} a_i x_i^2 < |d|^3$ has a solution $x_i$ such that $|x_i| \leqq 1$, $|x_p| = 1$. Then, we shall show by induction on $\mu$ that the inequality $\left|\sum_{i=1}^{n} a_i x_{i,\mu}^2\right| < |d|^{\mu+2}$ has a solution $x_{i,\mu}$ such that $|x_{i,\mu}| \leqq 1$, $|x_{p,\mu}| = 1$. For $\mu = 1$, it suffices to take $x_{i,1} = x_i$. Next, we assume that we have a solution for some $\mu$. Put $\sum_{i=1}^{n} a_i x_{i,\mu}^2 = d^\mu e$, $e = d^2 f$. We have $|f| < 1$. And set $y = -e(2a_p x_{p,\mu})^{-1}$. Then, we get $|y| = |a_1 \ldots \overset{p}{\vee} \ldots a_n| \, |c| \, |d| \, |f| \, |x_{p,\mu}|^{-1} < |d| < 1$. Using this $y$, we put $x_{i,\mu+1} = x_{i,\mu}$ $(i \neq p)$, $x_{p,\mu+1} = x_{p,\mu} + d^\mu y$. Since the valuation is non-archimedean, we have $|x_{i,\mu+1}| = |x_{i,\mu}|$ $i = 1, \ldots, n$. From the definition of $y$, we have $\sum_{i=1}^{n} a_i x_{i,\mu+1}^2 = \sum_{i=1}^{n} a_i x_{i,\mu}^2 + 2a_p x_{p,\mu} d^\mu y + a_p d^{2\mu} y^2 = d^\mu(e + 2a_p x_{p,\mu} y) + a_p d^{2\mu} y^2 = a_p d^{2\mu} y^2$. Therefore, it follows that $\left|\sum_{i=1}^{n} a_i x_{i,\mu+1}^2\right| \leqq |d|^{2\mu} |y|^2 < |d|^{2\mu+2} \leqq |d|^{\mu+3}$. Thus, we get $n$ Cauchy sequences $\{x_{i,\mu}\}$ in $K$. Since $K$ is complete, there exist $x_i = \lim_{\mu\to\infty} x_{i,\mu}$. It is obvious that $x = \sum_{i=1}^{n} x_i u_i$ is a non-trivial solution of the equation $f(x, x) = 0$. This proves the necessity assertion.

As an immediate consequence of Theorem 1 we get the following

THEOREM 2. *Let $K$ be a locally compactly valued field with characteristic $\neq 2$. Then, the index $\nu$ of $f$ is zero if and only if the group $O_n(K, f)$ is compact.*[5]

[5] Mr. A. Hattori has communicated to the writer an elegant alternative proof. Here we sketch his proof. Let $P$ be the projective space corresponding to $V$. If we define the open set in $P$ as the totality of lines in $V$ each of which intersects with some given open set in $V$, then $P$ becomes a compact space. If $\nu = 0$, then there is a homeomorphism between $P$ and the set $S$ of all symmetries with respect to the hyperplanes in $V$. Here, the topology in $S$ is the one induced from $E$. Thus $S$ is a compact set. Therefore, $O_n(K, f) = S^n$ (Cartan-Dieudonné) is also compact.

Now we shall apply the above results to the orthogonal group over a field $K$ of algebraic numbers or algebraic functions of one variable over a finite field of characteristic $\neq 2$. Let $K_{\mathfrak{p}}$ be a $\mathfrak{p}$-adic completion of $K$ with respect to a place $\mathfrak{p}$ in $K$. Suppose that a form $f$ is given in $K$. Naturally $f$ may be considered as a form over $K_{\mathfrak{p}}$ and $O_n(K,f)$ is contained in $O_n(K_{\mathfrak{p}}, f)$.[6] Let $\nu$ and $\nu_{\mathfrak{p}}$ be the global and local indices of $f$ respectively. According to *Hasse's principle*, we have the relation $\nu = \min_{\mathfrak{p}} \nu_{\mathfrak{p}}$ between these indices.[7] If $\nu \geqq 1$, since we do not use the completeness of valuation in the proof of sufficiency in Theorem 1, if $\mathfrak{p}$ is any place of $K$, then $O_n(K,f)$ is unbounded with respect to the $\mathfrak{p}$-adic topology. Conversely, if $\nu = 0$, then by the above principle we get $\nu_{\mathfrak{p}} = 0$ for some $\mathfrak{p}$. Therefore $O_n(K_{\mathfrak{p}}, f)$ is compact for such $\mathfrak{p}$ (Theorem 2) and we see that $O_n(K,f)$ is bounded in the $\mathfrak{p}$-adic topology.

Thus we get

THEOREM 3. *Let $K$ be a field of algebraic numbers or algebraic functions of one variable over a finite field of characteristic $\neq 2$. Then a form $f$ is a zero-form*[8] *if and only if the orthogonal group $O_n(K,f)$ is unbounded for all $\mathfrak{p}$-adic topologies in $K$.*

## REFERENCES


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[4] J. Dieudonné, Sur les groupes orthogonaux rationnels à trois et quatre variables, Compt. rend., t. **233** (1951).



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[6] By Cayley's parametrization we can see that $O_n(K,f)$ is dense in $O_n(K_{\mathfrak{p}}, f)$. But this fact is unnecessary to prove our Theorem 3.

[7] See [3] Satz 19. Though only the number field case is treated in [3], we know that the principle is also valid for the function field case.

[8] This means that $f$ represents zero non-trivially.

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