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On the subsystems of topological Markov chains

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Abstract. Let S_A be an irreducible and aperiodic topological Markov chain. If $S_{\bar{A}}$ is an irreducible and aperiodic topological Markov chain, whose topological entropy is less than that of S_A, then there exists an irreducible and aperiodic topological Markov chain, whose topological entropy equals the topological entropy at $S_{\bar{A}}$, and that is a subsystem of S_A. If $\bar{S}$ is an expansive homeomorphism of the Cantor discontinuum, whose topological entropy is less than that of S_A, and such that for every $j \in \mathbb{N}$ the number of periodic points of least period j of $\bar{S}$ is less than or equal to the number of periodic points of least period j of S_A, then $\bar{S}$ is topological conjugate to a subsystem of S_A.

Consider an irreducible and aperiodic topological Markov chain. Represent this chain as the shift S_A on a shift space X_A given by a transition matrix A over a finite state space Σ,

$$A(\sigma, \sigma') \in \{0, 1\}, \sigma, \sigma' \in \Sigma,$$
$$X_A = \{(x_i)_{i \in \mathbb{Z}} \in \Sigma^{\mathbb{Z}} : A(x_i, x_{i+1}) = 1, i \in \mathbb{Z}\},$$
$$S_A x = (x_{i+1})_{i \in \mathbb{Z}} \quad (x = (x_i)_{i \in \mathbb{Z}} \in X_A).$$

We use the notation

$$Z(a) = \{x \in \Sigma^{\mathbb{Z}} : x_i = a_i, 0 \leq i < I\}, \quad a \in \Sigma^{[0,I)}, I \in \mathbb{N},$$

and

$$\mathscr{A}[S_A, I] = \{a \in \Sigma^{[0,I)} : A(a_i, a_{i+1}) = 1, 0 \leq i < I\}, \quad I \in \mathbb{N}$$

and, more generally, given a subshift (X, S) of $\Sigma^{\mathbb{Z}}$, we denote

$$\mathscr{A}[S, I] = \{a \in \Sigma^{[0,I)} : X \cap Z(a) \neq \emptyset\}, \qquad I \in \mathbb{N}.$$

Given another irreducible and aperiodic topological Markov chain $S_{\bar{A}}$ such that the entropy condition

$$h(S_A) > h(S_{\bar{A}})$$

is satisfied, the question arises if $S_{\bar{A}}$ is topologically conjugate to the restriction of S_A to a closed S_A-invariant subset of X_A. The answer is here not always affirmative, since the presence in $S_{\bar{A}}$ of periodic points of low period can be an obstruction to isomorphically imbedding $S_{\bar{A}}$ into S_A. In fact, as we shall see in theorem 3, where we characterize the subsystems of S_A, this is the only obstruction. However, it is always possible, as we shall see in theorems 1 and 2, to find an irreducible and aperiodic topological Markov chain that is a subsystem of S_A and has topological entropy equal to $h(S_{\bar{A}})$. The existence of such a chain is significant in connection with certain problems of noiseless coding theory, that deal with a situation where digital data to be recorded or transmitted is to be encoded so as to conform to the design of the device used. (Problems that arise in such a context are e.g. described in [**3**, §§ IV, V].) From the existence of such a chain and a theorem of Adler & Marcus [**1**] it follows that $S_{\bar{A}}$ can be almost homeomorphically imbedded into S_A.

The proofs require a few preparations. First we choose $\alpha, \beta, \omega \in \Sigma$, $\alpha \neq \beta$, $\alpha \neq \omega$,

$$A(\beta, \alpha) = A(\beta, \omega) = 1,$$

together with a block

$$a = (a_i)_{0 \le i < N} \in \mathscr{A}[S_A, N],$$

where

$$a_0 = \alpha, \qquad a_{N-1} = \beta,$$

and where the symbol β appears only once in a. Further, let $\gamma \in \Sigma$, $A(\omega, \gamma) = 1$. Let $S_{A,M}$ be the subshift of finite type that arises by excluding from the system (X_A, S_A) the block $a^{(*)M}$ ($*$ stands for concatenation). Denote

$$\mathscr{C}_M[I] = \{c \in \mathscr{A}[S_{A,M}, I]: c_0 = \gamma, c_{I-1} = \beta\}.$$

LEMMA 1. *For every $\delta > 0$ there are $M, I \in \mathbb{N}$ such that*

$$|\mathscr{C}_M[I']| > e^{(h(S_A)-\delta)I'}, \qquad I' > I.$$

Proof. Let L be such that

$$\log A^{L-2}(\sigma, \sigma') > \left(h(S_A) - \frac{\delta}{2}\right) L, \qquad \sigma, \sigma' \in \Sigma$$

and let M be such that

$$NM > L.$$

One has $\mathscr{C}_M[I'] \supset \mathscr{B}[I']$, where

$$\mathscr{B}[I'] = \{(b_i)_{0 \le i < I'} \in \mathscr{A}[S_A, I']: b_0 = \gamma, b_{I'-1} = \beta, b_{kL-1} = \beta, b_{kL} = \omega, 0 \leqq k < I'L^{-1}\}.$$

If

$$I' > 2Lh(S_A)\delta^{-1}$$

then

$$|\mathscr{B}[I']| > e^{(h(S_A)-\delta)I'}. \qquad \square$$

We first turn our attention to an irreducible and aperiodic topological Markov chain $S_{\bar{A}}$ where $h(S_A) > h(S_{\bar{A}})$. Let $\bar{A}$ be over a state space $\bar{\Sigma}$, and choose (going to a two-block system if necessary) a $\bar{\rho} \in \bar{\Sigma}$ such that $\bar{A}(\bar{\rho}, \bar{\rho}) = 0$. For $I \in \mathbb{N}$, where $\bar{A}^I$ has all entries positive, we are going to describe an irreducible and aperiodic subshift of finite type $T_{\bar{A},I}$ as the restriction of the shift to a shift-invariant closed subset $Y_{\bar{A},I}$ of the shift space

$$((\bar{\Sigma} - \{\bar{\rho}\}) \times \{1\}) \cup (\bar{\Sigma} \times \{0\})^{\mathbb{Z}}.$$

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We stipulate that only those two-blocks $((\bar{\sigma}, \tau), (\bar{\sigma}', \tau'))$ appear as subblocks of sequences in $Y_{\bar{A},I}$ where $\bar{A}(\bar{\sigma}, \bar{\sigma}') = 1$, and we stipulate that all $(I+2)$-blocks $(c_i)_{0 \le i \le I+1}$ that appear as subblocks of the sequences in $Y_{\bar{A},I}$ conform to one of the following eight specifications. These are designed to ensure the irreducibility and aperiodicity of $T_{\bar{A},I}$ and also to ensure that the set

$$\bar{F} = \{y \in Y_{\bar{A},I} : y_0 \in (\bar{\Sigma} - \{\bar{\rho}\}) \times \{1\}\}$$

satisfies

$$\bar{F} \cap T^i_{\bar{A},I} \bar{F} = \emptyset, \qquad 0 < i < I,$$

and

$$Y_{\bar{A},I} = \bigcup_{0 \le i < I} T^i_{\bar{A},I} \bar{F}.$$

(1) $c_0 \in (\bar{\Sigma} - \{\bar{\rho}\}) \times \{1\}$,
$c_i \in \bar{\Sigma} \times \{0\}, \quad 0 < i < I$,
$c_I \in (\bar{\Sigma} - \{\bar{\rho}\}) \times \{1\}$,
$c_{I+1} \in \bar{\Sigma} \times \{0\}$.

(2) $c_0 \in (\bar{\Sigma} - \{\bar{\rho}\}) \times \{1\}$,
$c_i \in \bar{\Sigma} \times \{0\}, \quad 0 < i < I$,
$c_I = (\bar{\rho}, 0)$,
$c_{I+1} \in (\bar{\Sigma} - \{\bar{\rho}\}) \times \{1\}$.

(3) $c_0 = (\bar{\rho}, 0)$,
$c_1 = (\bar{\Sigma} - \{\bar{\rho}\}) \times \{1\}$,
$c_i \in \bar{\Sigma} \times \{0\}, \quad 1 < i \le I$,
$c_{I+1} \in (\bar{\Sigma} - \{\bar{\rho}\}) \times \{1\}$.

(4) $c_0 = (\bar{\rho}, 0)$,
$c_1 = (\bar{\Sigma} - \{\bar{\rho}\}) \times \{1\}$,
$c_i \in \bar{\Sigma} \times \{0\}, \quad 1 < i \le I$,
$c_{I+1} = (\bar{\rho}, 0)$.

(5) There is a j, $1 < j < I$, such that
$c_i \in \bar{\Sigma} \times \{0\}, \quad 0 \le i < j-1$,
$c_{j-1} = (\bar{\rho}, 0)$,
$c_j \in (\bar{\Sigma} - \{\bar{\rho}\}) \times \{1\}$,
$c_i \in \bar{\Sigma} \times \{0\}, \quad j < i \le I+1$.

(6) There is a $\bar{\sigma} \in \bar{\Sigma} - \{\bar{\rho}\}$ such that
$c_0 = (\bar{\sigma}, 0), \sum_{\bar{\alpha} \ne \bar{\rho}} \bar{A}^{I-1}(\bar{\alpha}, \bar{\sigma}) > 0$,
$c_1 \in (\bar{\Sigma} - \{\bar{\rho}\}) \times \{1\}$,
$c_i \in \bar{\Sigma} \times \{0\}, \quad 1 < i \le I$,
$c_{I+1} \in (\bar{\Sigma} - \{\bar{\rho}\}) \times \{1\}$.

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(7) There is a $\bar{\sigma} \in \bar{\Sigma} - \{\bar{\rho}\}$ such that

$c_0 = (\bar{\sigma}, 0), \sum_{\bar{\alpha} \neq \bar{\rho}} \bar{A}^{I-1}(\bar{\alpha}, \bar{\sigma}) > 0,$

$c_1 \in (\bar{\Sigma} - \{\bar{\rho}\}) \times \{1\},$

$c_i \in \bar{\Sigma} \times \{0\}, \quad 1 < i \leq I,$

$c_{I+1} = (\bar{\rho}, 0).$

(8) There is a j, $1 \leq j \leq I$, and a $\bar{\sigma} \in \bar{\Sigma}$ such that

$$c_0 = (\bar{\sigma}, 0), \sum_{\bar{\alpha} \neq \bar{\rho}} A^{I-j+1}(\bar{\alpha}, \bar{\sigma}) > 0$$

$c_i \in \bar{\Sigma} \times \{0\}, \quad 1 < i < j-1,$

$c_{j-1} \in (\bar{\Sigma} - \{\bar{\rho}\}) \times \{0\},$

$c_j \in (\bar{\Sigma} - \{\bar{\rho}\}) \times \{1\},$

$c_i \in \bar{\Sigma} \times \{0\}, \quad j < i \leq I+1.$

THEOREM 1. *The topological entropy of $T_{\bar{A},I}$ equals the topological entropy of $S_{\bar{A}}$.*

Proof. Let $I' \geq I+2$. An inspection of the above eight rules shows that for every

$$(\bar{\sigma}_i)_{0 \leq i < I'} \in \mathscr{A}[S_{\bar{A}}, I']$$

there is at least one

$$(\bar{\sigma}_i, \tau_i)_{0 \leq i < I'} \in \mathscr{A}[T_{\bar{A},I}, I']$$

and also, that there are at most $I+1$ such elements of $\mathscr{A}[T_{\bar{A},I}, I']$. □

Remark that the proof of this theorem can also be formulated by saying that the mapping that drops the second components is an at most $I+1$ to 1 continuous mapping of $Y_{\bar{A},I}$ onto $X_{\bar{A}}$.

Our next theorem can be obtained as a corollary of theorems 1 and 3. In presenting the following proof, we hope to illustrate the method that is again to be used in the proof of theorem 3.

THEOREM 2. *For I sufficiently large, $T_{\bar{A},I}$ is topologically conjugate to a subsystem of S_A.*

Proof. By lemma 1 we can find $M, I \in \mathbb{N}$ such that

$$|\mathscr{C}_M[I' - NM - 1]| > |\mathscr{A}[S_{\bar{A}}, I']|, \qquad I' \geq I.$$

We can then have one-to-one into mappings

$$\phi_I : \mathscr{A}[S_{\bar{A}}, I] \to \mathscr{C}_M[I - NM - 1],$$

and

$$\phi_{I+1} : \mathscr{A}[S_{\bar{A}}, I+1] \to \mathscr{C}_M[I - NM].$$

We form the block

$$r = a^{(*)M} * (\omega).$$

Observe that r cannot overlap itself properly, and define a one-to-one continuous and shift-invariant mapping Φ of $Y_{\bar{A},I}$ into X_A where Φ carries a $\bar{y} \in Y_{\bar{A},I}$ into the

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$x \in X_A$ that is determined by requiring

$$(x_i)_{0 \le i < I} = r * \phi_I((\bar{\sigma}_i)_{0 \le i < I}),$$
$$\bar{y}_i = (\bar{\sigma}_i, \tau_i), \qquad 0 \le i < I$$
$$\bar{y}_0, \bar{y}_I \in (\bar{\Sigma} - \{\bar{\rho}\}) \times \{1\},$$

and

$$(x_i)_{0 \le i \le I} = r * \phi_{I+1}((\bar{\sigma}_i)_{0 \le i \le I}),$$
$$\bar{y}_i = (\bar{\sigma}_i, \tau_i), \qquad 0 \le i \le I,$$
$$\bar{y}_0, \bar{y}_{I+1} \in (\bar{\Sigma} - \{\bar{\rho}\}) \times \{1\}.$$ □

We now turn our attention to an expansive homeomorphism S of the Cantor discontinuum. We assume S given as a subshift (X, S) of some full shift space, and we denote by $\mathcal{N}_S[J, K]$, $K \ge J$, the set of all $a \in \mathcal{A}[S, K]$ such that there is for every j, $0 < j < J$, at least one k, $0 \le k < K - j$ such that

$$a_k \ne a_{k+j}.$$

We have the following lemma of Kakutani–Rohlin type.

LEMMA 2. *For all $J, K \in \mathbb{N}$, $K \ge J$, there is a closed open set $F \subset X$ such that*

$$F \cap S^j F = \emptyset, \quad 0 < j < J$$

and

$$\bigcup_{-J<j<J} S^j F \supset \bigcup_{a \in \mathcal{N}_S[J,K]} X \cap Z(a).$$

Proof. Enumerate

$$\{Z(a): a \in \mathcal{N}_S[J, K]\} = \{C_l: 1 \leqq l \leqq L\},$$

and then obtain inductively an increasing sequence F_l, $1 \le l \le L$, of closed open sets by

$$F_1 = C_1$$

$$F_{l+1} = F_l \cup \left(C_{l+1} \cap \left(X - \bigcup_{-J<j<J} S^j F_l\right)\right) \qquad 1 < l < L.$$

Set $F = F_L$. □

In the sequel Π_j will indicate the set of periodic points of a homeomorphism that has least period j.

THEOREM 3. *Let S be an irreducible and aperiodic topological Markov chain, and let $\bar{S}$ be an expansive homeomorphism of the Cantor discontinuum such that*

$$h(S) > h(\bar{S})$$

and

$$|\Pi_j(S)| \ge |\Pi_j(\bar{S})|, \qquad j \in \mathbb{N}, \tag{1}$$

Then $\bar{S}$ is topologically conjugate to a subsystem of S.

Proof. We let $\bar{S}$ be given as a subshift of some shift space $\bar{X}$, and we continue to let the topological Markov chain to be given as S_A. In view of lemma 1 we have an $M \in \mathbb{N}$ such that

$$h(S_{A,M}) > h(\bar{S}). \tag{2}$$

There is then an $L \in \mathbb{N}$ such that

$$|\Pi_j(S_{A,M})| \geq |\Pi_j(\bar{S})|, \quad j > L. \tag{3}$$

Let now $P, Q \in \mathbb{N}$ be such that all entries of A^{NP} are positive, and such that

$$Q > P + L + M \tag{4}$$

and form the blocks

$$s = a^{(*)P} * (\omega),$$
$$t = a^{(*)Q} * (\omega).$$

By lemma 1, and by (2), there is a $J \in \mathbb{N}$, $J > (2P+Q)N+3$ such that

$$|\mathscr{C}_M[J' - (2P+Q)N - 3]| > |\mathscr{A}[\bar{S}, J']|, \qquad J' \geq J. \tag{5}$$

Denote

$$\mathscr{P}_j = \{b \in \mathscr{A}[S_A, j]: Z(b) \cap \Pi_j(S_A) \neq \varnothing\}, \qquad 0 < j < L,$$
$$\mathscr{P}_j = \{b \in \mathscr{A}[S_{A,M}, j]: Z(b) \cap \Pi_j(S_{A,M}) \neq \varnothing\}, \quad L \leq j < J,$$
$$\bar{\mathscr{P}}_j = \{\bar{b} \in \mathscr{A}[\bar{S}, j]: Z(\bar{b}) \cap \Pi_j(\bar{S}) \neq \varnothing\}, \qquad 0 < j < J.$$

By (1) and (3) one can assign in a one-to-one manner to every $\bar{b} \in \bar{\mathscr{P}}_j$ a $b \in \mathscr{P}_j$, $0 < j < J$, such that, if

$$(\bar{b}_0, \bar{b}_1, \ldots, \bar{b}_{j-1}) \to (b_0, b_1, \ldots, b_{j-1}),$$

then

$$(\bar{b}_{j-1}, \bar{b}_0, \ldots, \bar{b}_{j-2}) \to (b_{j-1}, b_0, \ldots, b_{j-2}), \quad 0 < j < J.$$

By (5) one can also have one-to-one into mappings

$$\psi_j: \mathscr{A}[\bar{S}, j] \to \mathscr{C}_M[j - (2P+Q)N - 3], \qquad J \leq j < 2J.$$

Finally we select for all $\bar{b} \in \bar{P}_j$, $0 < j < J$, blocks

$$s_-(b), s_+(b) \in \mathscr{A}[S_A, NP+1]$$

such that

$$(s_-(b))_{NP} = \beta, \qquad (s_+(b))_0 = \gamma,$$
$$A(\bar{b}_{j-1}, s_-(b)_0) = A((s_+(b))_{NP}, \bar{b}_0) = 1.$$

There is a $K \in \mathbb{N}$, $K > J^2$, such that

$$|\mathscr{A}[\bar{S}, K] - \mathscr{N}_{\bar{S}}[J, K]| = \sum_{0 < j \leq J} |\Pi_j(\bar{S})|,$$

for if no such K existed, then it would follow by a compactness argument that for some j, $0 < j \leq J$, $\bar{S}$ had more than $|\Pi_j(\bar{S})|$ periodic points of least period j. We apply now lemma 3 to obtain a closed open set $F \subset \bar{X}$ such that

$$F \cap S^i F = \varnothing, \qquad 0 < i < J,$$

$$\bigcup_{-J < i < J} S^i F \supset \bigcup_{\bar{d} \in \mathscr{N}_{\bar{S}}[J,K]} \bar{X} \cap Z(\bar{d}).$$

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Define then an $\tilde{F}$ by setting

$$F_{-}=\Big(\bigcap_{-2J<i<0}(\bar{X}-\bar{S}^{i}F)\Big)\cap F,$$

$$F_{+}=F\cap\Big(\bigcap_{0<i<2J}(\bar{X}-S^{-i}F)\Big)$$

$$\tilde{F}=\bar{S}^{J}F_{-}\cup F\cup S^{-J}F,$$

and have

$$\tilde{F}\cap\bar{S}^{i}\tilde{F}=\varnothing,\qquad 0<i<J,$$

$$\bigcap_{-J<i<J}(\bar{X}-S^{i}\tilde{F})\subset\bigcup_{0<j<J}\bigcup_{\bar{d}\in\bar{\mathcal{P}}_j}\bigcap_{-J<i<J}\bar{X}\cap S^{i}Z(\bar{d}).\tag{6}$$

Thus one can define a continuous shift invariant mapping Ψ of $\bar{X}$ into X_A where Ψ carries an $\bar{x}\in\bar{X}$ into the $x\in X_A$ that is determined by the following rules:

(1) If

$$\bar{x}\in Z(\bar{b})\cap\bigcap_{-J<i<J}(\bar{X}-\bar{S}^{i}\tilde{F}),\qquad \bar{b}\in\bar{\mathcal{P}}_j,\quad 0<j<J$$

then

$$(x_i)_{0\le i<j}=b.$$

(2) If

$$\bar{x}\in\Big(\bigcap_{-2J<i<0}(\bar{X}-\bar{S}^{i}\tilde{F})\Big)\cap S^{j}Z(\bar{b})\cap\tilde{F}\cap\Big(\bigcap_{0<i<I_+}(\bar{X}-\bar{S}^{-i}\tilde{F})\Big)\cap\bar{S}^{I_+}\tilde{F},$$

$$J<I_+<2J,\quad \bar{b}\in\bar{\mathcal{P}}_j,\quad 0<j\le J,$$

then

$$(x_i)_{-[(2J-1)j^{-1}]j\le i<I_+}=b^{(*)[2J-1]j^{-1}}*s_-(b)*t*s*\psi((\bar{x}_i)_{0\le i<I_+}).$$

(3) If

$$\bar{x}\in S^{I_-}\tilde{F}\Big(\bigcap_{-I<i<0}(\bar{X}-\bar{S}^{i}\tilde{F})\Big)\cap\tilde{F}\cap Z(b)\cap\Big(\bigcap_{0<i<2J}(\bar{X}-\bar{S}^{-i}\tilde{F})\Big),$$

$$J<I_-<2J,\quad \bar{b}\in\bar{\mathcal{P}}_j,\quad 0<j\le J,$$

then

$$(x_i)_{-I_-<i\le[(2J-1)j^{-1}]j}=\psi((\bar{x}_i)_{-I_-<i\le 0})*s*t*s_+(b)*b^{(*)[2J-1]j^{-1}}.$$

(4) If

$$\bar{x}\in S^{I_-}\tilde{F}\cap\Big(\bigcap_{-I_-<i<0}(X-S^{i}\tilde{F})\Big)\cap\tilde{F}\cap\Big(\bigcap_{0<i<I_+}(\bar{X}\cap\bar{S}^{-i}\tilde{F})\Big)\cap\bar{S}^{I_+}\tilde{F},$$

$$J<I_-,\quad I_+<2J,$$

then

$$(x_i)_{-I_-+L'\le i\le I_+}=\psi((\bar{x}_i)_{-I_-\le i<0})*s*t*s*\psi((\bar{x}_i)_{0\le i<I_+}).$$

The structure of the blocks s and t and (4) show that $x\in F$ if and only if

$$\Psi x\in S_A^{(2P+Q)N+3}Z(t).$$

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Since we have (6) and since the assignments that entered into the construction of Ψ are one-to-one we conclude that $\Psi\bar{x}$ uniquely determines $\bar{x}$. □

There is a version of the finite generator theorem for ergodic measure preserving transformations of finite entropy, that realizes such a transformation by means of an invariant probability measure of any irreducible and aperiodic topological Markov chain, whose topological entropy exceeds the entropy of the transformation ([**4**], [**2** § 28]). One can say that a corollary of theorem 3 achieves for minimal expansive homeomorphisms of the Cantor discontinuum what the finite generator theorem does for measure preserving transformations.

Corollary. Let S be an irreducible and aperiodic topological Markov chain and let $\bar{S}$ be a minimal expansive homeomorphism of the Cantor discontinuum such that

$$h(S) > h(\bar{S}).$$

Then $\bar{S}$ is topologically conjugate to a subsystem of S_A.

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