Ergod. Th. & Dynam. Sys. (1982), **2**, 1–13
Printed in Great Britain

Iterated extensions

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(*Received* 21 *May* 1981 *and revised* 22 *March* 1982)

Abstract. The notion of an iterated extension of a flow is introduced and studied. In particular it is shown how eigenfunctions occur in a natural way. This is then exploited to produce an example of a weakly mixing minimal set with a non-weakly mixing quasi-factor.

Introduction

The flow (Y, T) is an extension of the flow (X, T) if there exists an epimorphism of (Y, T) onto (X, T). One way of producing extensions of (X, T) is by means of cocycles of (X, T) into a compact group K. Thus let σ be a cocycle on (X, T) to K. Then one forms the skew product flow $(K\times_\sigma X, T)$ where the action of T on $K\times X$ is given by the map

$$(k, x, t)\mapsto(k\sigma(x, t), xt): K\times X\times T\to K\times X.$$

When the phase group T is isomorphic to the integers, the set of cocycles on (X, T) to K may be identified with the set of continuous functions on X to K and this may be exploited to iterate the extension.

In order to illustrate the basic definitions and facilitate the reading of the paper we describe this construction informally.

A flow in this paper is a pair (X, t) consisting of a compact Hausdorff space X and a homeomorphism t of X onto itself. (We use the same letter, t, for every flow considered.) K will stand for a compact abelian topological group, e the identity element of K, and $Z(X; K)$ the set of 1-cocycles from $X\times\mathbb{Z}$ into K.

Now start with a minimal pointed flow (X, x_0) and a cocycle $\sigma\in Z(X; K)$. Let (Y, y_0) be the pointed flow ext (X, σ). Thus $Y\subset K\times X$ is the orbit closure of $y_0=(e, x_0)$ in the flow on $K\times X$ given by

$$(k, x)t=(k\sigma(x, t)x, t).$$

We let $F_\sigma: X\to K$ be the function on X defined by $F_\sigma(x)=\sigma(x, t)$. Next we consider the function

$$\delta\sigma(k, x)^{-1}\delta\sigma((k, x)t^n)=\sigma(x, t^n),$$

$\delta\sigma$ is the function on Y which 'co-bounds' σ. Our next step is to define a cocycle on Y by means of the function $\delta\sigma$. Namely let $\sigma_1\in Z(Y, K)$ be given by

$$\sigma_1(y, t)=F_{\sigma_1}(y)=\delta\sigma(y).$$

Let $Y_1 = \text{ext}(Y, \sigma_1) \subset K \times K \times X$; i.e. Y_1 is the orbit closure of $z_0 = (e, e, x_0)$ in the flow on $K \times K \times X$ given by

$$(l, k, x)t = (lF_{\sigma_1}(k, x), kF_\sigma(x), xt).$$

(Writing down the orbit of z_0 in Y_1 we have

$$(e, e, x_0)t^n = (\sigma_1(y_0, t^n), \sigma(x_0, t^n), x_0t^n) = (\delta\sigma_1(z_0t^n), \delta\sigma(y_0t^n), x_0t^n).)$$

The function $\delta\sigma_1: Y_1 \to K$, $\delta\sigma_1(l, k, x) = l$ can now be used to define $\sigma_2 \in Z(Y_1; K)$ namely

$$\sigma_2(z, t) = F_{\sigma_2}(z) = \delta\sigma_1(z) \quad \text{etc.}$$

We were motivated to study the structure of these extensions by our attempts to understand why our construction of eigenfunctions from cocycles in [4] 'worked'. This is explained in proposition 1.3 which states that for a proper choice of α an eigenfunction will appear in the supremum of the flows ext (x, σ) and ext $(x, \sigma)\alpha$.

As an added bonus our analysis allowed us to construct a weakly mixing flow with a non-weakly mixing quasi-factor. (Recall that a quasi-factor of a flow Z is a minimal subset of the flow induced on 2^Z, the space of closed subsets of Z.) To see how this is done suppose that in the procedure described above we take X to be weakly mixing, $K = \mathbb{K}$ the circle group and choose σ such that $Y = \mathbb{K} \times X$ and Y is also weakly mixing (this is possible by [6]). We then show that necessarily $Y_1 = \mathbb{K} \times \mathbb{K} \times X$ (proposition 1.7) and that Y_1 is weakly mixing (corollary 1.11). Take $\mathcal{Y}$ to be the orbit closure of say

$$\{(1, -1, x_0), (1, 1, x_0)\},$$

in 2^{Y_1}, the flow of closed subsets of Y_1. It is easy to see that the minimal flow $\mathcal{Y}$ has -1 as an eigenvalue, in particular it is not weakly mixing. We then show that $\mathcal{Y} = \mathfrak{a}(A, Y)$ for a certain τ-closed subgroup A of G.

These subjects together with the analysis of flows of the form

$$(Y, y_0) \vee (Y, y_1),$$

where y_0 and y_1 project onto the same point in X, on which our results rest, are the content of § 1. In § 2 we consider the higher order cocycles σ_n $(n \in \mathbb{N})$ and generalize some of the results of § 1.

We now formalize the above definitions. Each flow (X, t) will be assumed provided with a base point x_0 such that $x_0u = x_0$, where u is a fixed idempotent in some minimal subset M of $\beta\mathbb{Z}$. This allows us to pass back and forth between minimal sets and $\mathbb{Z}$-subalgebras of $C(M)$. The algebra corresponding to X will be denoted by al(X) and the flow corresponding to the algebra $\mathcal{A}$ by $|\mathcal{A}|$. (See [1] for details.)

Let K be a compact abelian topological group. Then $Z(M, K)$ will denote the set of cocycles on M to K and

$$Z(X, K) = \{\sigma \in Z(M, K): \text{al}(\sigma) \subset \text{al}(X)\}.$$

There is a bijective correspondence

$$\sigma \leftrightarrow F_\sigma: Z(X, K) \leftrightarrow C(X, K),$$

the set of continuous functions from X to K, given by

$$F_\sigma(x) = \sigma(x, t).$$

On the other hand there is a bijective correspondence

$$\sigma \to \delta\sigma : Z(M, K) \to C_0(M, K) = \{f \in C(M, K) : f(u) = e\}$$

(e, the identity element of K). Consequently given $\sigma \in Z(M, K)$ there exists $\partial_\sigma \in Z(M, K)$ such that

$$\delta\partial_\sigma(m) = F_\sigma(u)^{-1} F_\sigma(m) \quad (m \in M).$$

(For a detailed discussion of the correspondence $\sigma \to \delta\sigma$ see [2]. In the latter $\delta\sigma$ is denoted by f_σ.)

We write $\mathbb{R}$ for the real numbers, $\mathbb{Z}$ for the integers, and $\mathbb{K}$ for the multiplicative circle group.

Section 1

The first result of this section codifies the relationship among the various operations on a cocycle described above. Since the proof follows directly from the definitions, it will be omitted.

(1.1) Proposition. *Let* $\sigma \in Z(M, K)$. *Then*:
(1) $(\delta\partial_\sigma)(m) = F_\sigma(u)^{-1} F_\sigma(m)$.
(2) $\sigma(m, t) = F_\sigma(u)(\delta\partial_\sigma)(m)$.
(3) $F_\sigma(u)^{-1}\delta\sigma(m, t) = \delta\partial_\sigma(m)\delta\sigma(m)$.
(4) $(\delta\partial_\sigma)(m)(\delta\partial_\sigma)(n)^{-1} = F_\sigma(m)F_\sigma(n)^{-1}$ $(m, n \in M)$.

Notice that (2) implies that if σ is a cocycle on X to K, then ∂_σ is a coboundary on X to K.

(1.2) Proposition. *Let* $\sigma \in Z(M, K)$, $m \in M$, *and* ω *be that element of* $Z(M, K)$ *with*

$$\delta\omega(x) = \delta\sigma(x)^{-1}\delta\sigma(m)^{-1}\delta\sigma(mx) \quad (x \in M).$$

Then $F_\omega(x) = \delta\partial_\sigma(mx)\delta\partial_\sigma(x)^{-1} \quad (x \in M)$.

Proof.

$$\begin{aligned} F_\omega(x) &= \omega(x, t) = \delta\omega(x)^{-1}\delta\omega(xt) \\ &= \delta\sigma(mx)^{-1}\delta\sigma(m)\delta\sigma(x)\delta\sigma(xt)^{-1}\delta\sigma(m)^{-1}\delta\sigma(mxt) \\ &= \delta\sigma(mx)^{-1}\delta\sigma(x)\delta\sigma(xt)^{-1}\delta\sigma(mxt) \\ &= \sigma(x, t)^{-1}\sigma(mx, t) = F_\sigma(x)^{-1}F_\sigma(mx) \\ &= \delta\partial_\sigma(x)^{-1}\delta\partial_\sigma(mx). \end{aligned}$$

□

(1.3) Proposition. *Let* $\sigma \in Z(\mathscr{A}, K)$, $\alpha \in \mathfrak{g}(\partial_\sigma)$ *and* $F_\omega(m) = \delta\partial_\sigma(\alpha)$ $(m \in M)$. *Then*
(1) $\mathrm{al}\,(\delta\omega) \subset \mathrm{ext}\,(\mathscr{A}, \sigma) \vee \mathrm{ext}\,(\mathscr{A}, \sigma)\alpha$.
(2) *If* $\mathscr{A}\alpha = \mathscr{A}$, *then*

$$\mathrm{ext}\,(\mathscr{A}, \sigma) \vee \mathrm{ext}\,(\mathscr{A}, \sigma)\alpha = \mathrm{ext}\,(\mathscr{A}, \sigma) \vee \mathrm{ext}\,(\mathscr{A}, \omega).$$

Proof. (1). Set $m = \alpha$ in (1.2). Then

$$F_\omega(x) = \delta\partial_\sigma(\alpha x)\delta\partial_\sigma(x)^{-1} = \delta\partial_\sigma(\alpha)\delta\partial_\sigma(x)\delta\partial_\sigma(x)^{-1} = \delta\partial_\sigma(\alpha)$$

since $\alpha \in \mathfrak{g}(\partial_\sigma)$ (see [2]).

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Since $\operatorname{al}(\delta\sigma) \subset \operatorname{ext}(\mathscr{A}, \sigma)$, $\delta\sigma$ defines a continuous function on $|\operatorname{ext}(\mathscr{A}, \sigma)|$, the expression for $\delta\omega$ given in (1.2) shows that $\delta\omega$ defines a continuous function on

$$|\operatorname{ext}(\mathscr{A}, \sigma) \vee \operatorname{ext}(\mathscr{A}, \sigma)\alpha|.$$

(2). By (1), $\mathcal{N} = \operatorname{ext}(\mathscr{A}, \sigma) \vee \operatorname{ext}(\mathscr{A}, \omega) \subset \operatorname{ext}(\mathscr{A}, \sigma) \vee \operatorname{ext}(\mathscr{A}, \sigma)\alpha = \mathscr{S}$. Since $\mathscr{A}\alpha = \mathscr{A}$, $\mathscr{S}$ is a distal extension of $\mathscr{A}$ and so the proof may be completed by showing that

$$N = \mathfrak{g}(\mathcal{N}) \subset S = \mathfrak{g}(\mathscr{S}).$$

To this end let

$$\beta \in N = \mathfrak{g}(\operatorname{ext}(\mathscr{A}, \sigma)) \cap \mathfrak{g}(\operatorname{ext}(\mathscr{A}, \omega)).$$

Then

$$\delta\sigma(\beta) = e = \delta\omega(\beta)$$

whence by (1.2)

$$e = \delta\omega(\beta) = \delta\sigma(\beta)^{-1}\delta\sigma(\alpha)^{-1}\delta\sigma(\alpha\beta)$$
$$= \delta\sigma(\alpha)^{-1}\delta\sigma(\alpha\beta).$$

Hence

$$\delta\sigma(\alpha) = \delta\sigma(\alpha\beta) = \delta\alpha(\alpha\beta\alpha^{-1}\alpha) = \delta\sigma(\alpha\beta\alpha^{-1})\delta\sigma(\alpha)$$

(recall that $\mathscr{A}\alpha = \mathscr{A}$ implies that $\alpha A \alpha^{-1} = A$ whence $\alpha\beta\alpha^{-1} \in A \subset \mathfrak{g}(\sigma)$). This implies that

$$\alpha\beta\alpha^{-1} \in \ker \delta\sigma$$

and so

$$\beta \in \alpha^{-1} \ker(\delta\sigma)\alpha^{-1} \cap A = \mathfrak{g}(\operatorname{ext}(\mathscr{A}, \sigma)\alpha).$$

Consequently

$$\beta \in S = \mathfrak{g}(\operatorname{ext}(\mathscr{A}, \sigma)) \cap \mathfrak{g}(\operatorname{ext}(\mathscr{A}, \sigma)\alpha). \qquad \square$$

(1.4) COROLLARY. *Let*

$$\sigma \in Z(\mathscr{A}, K)$$

and

$$\mathscr{A}\alpha = \mathscr{A} \ (\alpha \in \mathfrak{g}(\partial_\sigma)).$$

Then

$$\bigvee\{\operatorname{ext}(\mathscr{A}, \sigma)\alpha \mid \alpha \in \mathfrak{g}(\partial_\sigma)\} \subset \operatorname{ext}(\mathscr{A}, \sigma) \bigvee \mathscr{E}.$$

(Here $\mathscr{E}$ is the set of all almost periodic functions on $\mathbb{Z}$.)

(1.5) *Remarks.* (1). Let F be a τ-closed subgroup of G and $\mathscr{A}$ a $\mathbb{Z}$-sub-algebra of $\mathfrak{a}(u)$. Then it is natural to define $\mathscr{A}$ to be *F-regular* if $\mathscr{A}\alpha = \mathscr{A}$ $(\alpha \in F)$ and $r_F(\mathscr{A})$, the *F-regularizer* of $\mathscr{A}$, as the supremum of $\{\mathscr{A}\alpha \mid \alpha \in F\}$. Then (1.4) states that

$$r_{\mathfrak{g}(\partial_\sigma)}(\operatorname{ext}(\mathscr{A}, \sigma)) \subset \operatorname{ext}(\mathscr{A}, \sigma) \vee \mathscr{E}$$

if $\mathscr{A}$ is $\mathfrak{g}(\partial_\sigma)$-regular.

(2). Let $S = \mathfrak{g}(\operatorname{ext}(\mathscr{A}, \sigma))$. Then (1.4) implies that

$$S \cap E \subset \cap\{\alpha S \alpha^{-1} \mid \alpha \in \mathfrak{g}(\partial_\sigma)\}$$

when $\mathscr{A}$ is $\mathfrak{g}(\partial_\sigma)$-regular. (Here $E = \mathfrak{g}(\mathscr{E})$.)

(1.6) PROPOSITION. *Let* $\sigma \in Z(\mathscr{A}, K)$ *and* $\omega \in Z(M, K)$ *with* $F_\omega = \delta\sigma$. *Then*

(1) $\partial_\omega = \sigma$.
(2) $\delta\omega(t^k) = \prod_{j=0}^{k-1} \delta\sigma(t^j)$.
(3) $\delta\omega(pt) = \delta\omega(p)\delta\sigma(p)$ $(p \in \beta\mathbb{Z})$.
(4) $\delta\omega(pt^k) = \delta\omega(p)\prod_{j=0}^{k-1}\delta\sigma(pt^j)$ $(p \in \beta\mathbb{Z})$.
(5) $\delta\omega(\beta t^k) = \delta\omega(\beta)\delta\sigma(\beta)^k\delta\omega(t^k)$ $(\beta \in A = \mathfrak{g}(\mathscr{A}))$.

Proof. (1)

$$(\delta\sigma)(m) = F_\omega(u)^{-1}F_\omega(m) = (\delta\partial_\omega)(u)^{-1}(\delta\partial_\omega)(m) = (\delta\partial_\omega)(m) \quad (m \in M).$$

Hence $\sigma = \partial_\omega$.

(2) $\delta\omega(t) = \omega(u, t) = F_\omega(u) = \delta\sigma(u) = e$ shows that

$$\delta\omega(t^k) = \prod_{j=0}^{k-1} \delta\sigma(t^j) \quad \text{for } k = 1.$$

Now assume that it holds for $1 \le k \le r$. Then

$$\begin{aligned}\delta\omega(t^{r+1}) &= \omega(u, t^{r+1}) = \omega(u, t^r)\omega(t^r, t)\\ &= \delta\omega(t^r)F_\omega(t^r) = \left(\prod_{j=0}^{r-1}\delta\omega(t^j)\right)\delta\omega(t^r)\\ &= \prod_{j=0}^{r}\delta\omega(t^j).\end{aligned}$$

(3) Let $p \in \beta\mathbb{Z}$ and $t^{k_i} \to p$. Then $t^{k_i+1} \to pt$ and

$$\begin{aligned}\delta\omega(pt) = \lim \delta\omega(t^{k_i+1}) &= \left(\lim_i \prod_{j=0}^{k_i-1}\delta\sigma(t^j)\right)\lim_i \delta\sigma(t^{k_i})\\ &= \lim_i \delta\omega(t^{k_i})\lim_i \delta\sigma(t^{k_i}) = \delta\omega(p)\delta\sigma(p).\end{aligned}$$

(4) This follows from (3) by induction on k.

(5) If $\beta \in A$ then $\delta\sigma(\beta x) = \delta\sigma(\beta)\delta\sigma(x)$ $(x \in \beta\mathbb{Z})$. Hence

$$\delta\omega(\beta t^k) = \delta\omega(\beta)\delta\sigma(\beta)^k \prod_{j=0}^{k-1}\delta\sigma(t^j) = \delta\omega(\beta)\delta\sigma(\beta)^k\delta\omega(t^k)$$

(by (4) and (2)). □

(1.7) PROPOSITION. *Let* $\sigma \in Z(\mathscr{A}, \mathbb{K})$ *be such that* $\mathscr{S} = \text{ext}(\mathscr{A}, \sigma)$ *is weak-mixing and* $\delta\sigma(A) = \mathbb{K}$. *Then* $\delta\omega(S) = \mathbb{K}$ *where* $\omega \in Z(M, \mathbb{K})$ *with* $F_\omega = \delta\sigma$ *and* $S = \mathfrak{g}(\mathscr{S})$.

Proof. $\delta\omega(S)$ is a closed subgroup of $\mathbb{K}$ whence $\delta\omega(S)$ is finite or all of $\mathbb{K}$. If $\delta\omega(S)$ is finite then $\delta\omega^n(S) = e$ for some integer n. Since $\delta\sigma^n(A) = \mathbb{K}$ and $F_{\omega^n} = \delta\sigma^n$, it suffices to rule out the possibility that $\delta\omega(S) = e$.

Let $\mathscr{B} = \text{ext}(\mathscr{S}, \omega)$, then $\delta\omega(S) = e$ implies that $\mathscr{B} = \mathscr{A}$ whence $\mathscr{B}\alpha = \mathscr{A}\alpha = \mathscr{A} = \mathscr{B}$ $(\alpha \in A)$. This is impossible since $\mathscr{B} \vee \mathscr{B}\alpha$ contains the eigenfunction δp where $F_p(m) = \delta\sigma(\alpha)$ $(m \in M)$ by (1.3). (Recall that $\partial_\omega = \sigma$ and $A \subset \mathfrak{g}(\sigma)$.) □

(1.8) *Remarks.* (1) The assumption $\delta\sigma(A) = \mathbb{K}$ implies that $|\mathscr{S}| \simeq \mathbb{K} \times |\mathscr{A}|$, and the conclusion $\delta\omega(S) = \mathbb{K}$ implies that

$$|\text{ext}(\mathscr{S}, \omega)| \simeq \mathbb{K} \times \mathbb{K} \times |\mathscr{A}|.$$

We shall see later (corollary 1.11) that ext $(\mathcal{S}, \omega)$ is also weak-mixing.

(2) With the assumptions and notation of (1.7) let $\mathcal{B} = \text{ext}(\mathcal{S}, \omega)$. (Observe that $F_\omega = \delta\sigma$ implies that $\omega \in Z(\mathcal{S}, \mathbb{K})$.) Then $r_A(\mathcal{B}) = \mathcal{B} \vee \mathcal{E}$. To see this first observe that S is a normal subgroup of A. This implies that $\mathcal{S}\alpha = \mathcal{S}$ $(\alpha \in A)$. Moreover $\partial_\omega = \sigma$ implies that $A \subset g(\partial_\omega)$. Hence by (1.4) $r_A(\mathcal{B}) \subset \mathcal{B} \vee \mathcal{E}$.

Now let f be a character on $\mathbb{Z}$. Then $f = \delta p$ where $F_p(m) = k$ for some $k \in \mathbb{K}$ and all $m \in M$. There exists $\alpha \in A$ with

$$\delta\partial_\omega(\alpha) = \delta\sigma(\alpha) = k$$

whence $f = \delta p \in r_A(\mathcal{S})$ by (1.3.). Since the characters generate

$$\mathcal{E}, \mathcal{B} \vee \mathcal{E} \subset r_A(\mathcal{B}).$$

(3) Proposition 1.7 as well as theorem 1.10 below are true when $\mathbb{K}$ is replaced by a finite group of prime order.

(1.8) Lemma. *Let $\sigma \in Z^1(\mathcal{A}, \mathbb{K})$ with $\delta\sigma(A) = \mathbb{K}$, $\varepsilon > 0$ and V a neighbourhood of u. Then there exists $p \in V \cap \bar{A} \subset M$ such that*

$$|\delta\sigma(p) - 1| \leq \varepsilon \quad \textit{and} \quad (\delta\sigma(pu))^n \neq 1 \quad \textit{if } n \neq 0.$$

(Notice that $p \in \bar{A}$ implies that $pu \in A$.)

Proof. Since $\delta\sigma(u) = 1$ there exists a neighbourhood W of u with $\bar{W} \subset V$ and

$$|\delta\sigma(x) - 1| \leq \varepsilon \quad (x \in W).$$

By a now standard argument we may assume that

$$\text{int}_\tau\, \text{cls}_\tau\, (W \cap A) \neq \varnothing.$$

(See [3: 4.4].) Since $\delta\sigma : (A, \tau) \to \mathbb{K}$ is onto and open there exists

$$\alpha \in \text{cls}_\tau\, (W \cap A) \quad \text{with } \delta\sigma(\alpha)^n \neq 1 \text{ if } n \neq 0.$$

Let (α_n) be a net on $W \cap A$ with $\alpha_n \underset{\tau}{\to} \alpha$ and $\alpha_n \to p \in \beta\mathbb{Z}$. Then

$$p \in \bar{W} \cap \bar{A} \subset V \cap \bar{A}, \qquad |\delta\sigma(p) - 1| \leq \varepsilon \quad \text{and} \quad \alpha_n \underset{\tau}{\to} pu.$$

Hence $\delta\alpha(\alpha) = \delta\sigma(pu)$. The proof is completed. □

(1.9) Proposition. *Let $\sigma \in Z(\mathcal{A}, \mathbb{K})$ with $\delta\sigma(A) = \mathbb{K}$, $\omega \in Z(M, \mathbb{K})$ with $F_\omega = \delta\sigma$, and $\mathcal{S} = \text{ext}(\mathcal{A}, \sigma)$. Then $\mathcal{B} = \text{ext}(\mathcal{S}, \omega)$ is not an almost periodic extension of $\mathcal{A}$.*

Proof. It will be convenient to identify $|\mathcal{B}|$ with a subset of $\mathbb{K} \times \mathbb{K} \times |\mathcal{A}|$. When this is done

$$x|\mathcal{B} = (\delta\omega(x), \delta\sigma(x), x|\mathcal{A}) \quad (x \in M).$$

Let $\varepsilon > 0$. We shall find $p, g \in \beta\mathbb{Z}$ such that

$$p|\mathcal{A} = u|\mathcal{A}, \qquad |\delta\omega(p) - 1| \leq \varepsilon, \qquad |\delta\sigma(p) - 1| \leq \varepsilon \quad \text{and} \quad |\delta\omega(pg)\delta\omega(g)^{-1} - 1| \geq \tfrac{1}{2}.$$

Thus $p|\mathcal{B}$ is close to $u|\mathcal{B}$, $p|\mathcal{A} = u|\mathcal{A}$, but $pg|\mathcal{B}$ is not close to $ug|\mathcal{B}$. Consequently $\mathcal{B}$ is not an almost periodic extension of $\mathcal{A}$.

To this end let V be a neighbourhood of u such that

$$|\delta\omega(r) - 1| \leq \varepsilon \quad (r \in V).$$

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By (1.8) there exists $p\in V\cap\bar{A}$ with $|\delta\sigma(p)-1|\le\varepsilon$ and $\delta\sigma(pu)^n\ne 1$ if $n\ne 0$. Since $p\in\bar{A}$, $p|\mathscr{A}=u|\mathscr{A}$.

Now choose $\lambda\in\mathbb{K}$ with $|\lambda\delta\omega(pu)-1|>\frac{1}{2}$. There exists a sequence of integers k_i with $\delta\sigma(pu)^{k_i}\to\lambda$. Let $r\in\beta\mathbb{Z}$ be adherent to the sequence t^{k_i}. Then by (5) of (1.6)

$$\delta\omega(pur)=\delta\omega(pu)\lambda\delta\omega(r),$$

whence $|\delta\omega(pur)\delta\omega(r)^{-1}-1|>\frac{1}{2}$. Now set $ur=g$ and recall that $\delta\omega(r)=\delta\omega(ur)$. The proof is completed. □

The following result is valid for any abelian group T.

(1.10) THEOREM. *Let $\mathscr{S}$ be an almost periodic extension of $\mathscr{A}$ such that $S\lhd A$ and A/S is a Lie group, and let $\mathscr{B}$ be an almost periodic extension of $\mathscr{S}$ such that $B\lhd S$ and $S/B\cong\mathbb{K}$. Then either*

(i) *$\mathscr{B}$ is an almost periodic extension of $\mathscr{A}$ or*

(ii) *$B\in S^{\perp\perp}$.*

(Recall that

$$\mathscr{R}^{\perp}=\{C|C \text{ is a } \tau\text{-closed subgroup of } G \text{ with } CR=G\ (R\in\mathscr{R})\}$$

where $\mathscr{R}$ is a collection of τ-closed subgroups of G.)

Proof. Assume that (i) does not hold and let $C\in S^{\perp}$; i.e. C is a τ-closed subgroup of G with $CS=G$. Then

$$CB\supset CS'\supset G'=E.$$

Since G/E is abelian, CB is a normal subgroup of G.

Let $L=CB\cap S\supset E\cap S\supset E\cap A^{\#}=A^{\#}$. Hence $\mathscr{L}=\mathfrak{a}(L)\cap\mathscr{A}^{\#}$ is an almost periodic extension of $\mathscr{A}$. The exact sequences

$$1\to S/L\to A/L\to A/S\to 0 \quad\text{and}\quad S/B\to S/L\to 0$$

show that S/L is a circle or a point and that in either case A/L is a Lie group.

Thus $\mathscr{A}\lhd\mathscr{L}\lhd\mathscr{B}$ and L/B is a subgroup of the circle group S/B. Hence L/B is finite or $L/B=S/B$. If L/B were finite, $\mathscr{B}$ would be an almost periodic extension of $\mathscr{A}$ [**7**: 5.7], a possibility which has been ruled out. Therefore $L/B=S/B$ and so $L=S$. Consequently $S\subset CB$ and $G=CS\subset CB$. The proof is completed. □

(1.11) COROLLARY. *Let $\sigma\in Z(\mathscr{A},\mathbb{K})$ with $\delta\sigma(A)=\mathbb{K}$, $\mathscr{S}=\operatorname{ext}(\mathscr{A},\sigma)$ weak-mixing and $\omega\in Z(M,\mathbb{K})$ with $F_\omega=\delta\sigma$. Then $\mathscr{B}=\operatorname{ext}(\mathscr{S},\omega)$ is weak-mixing and $\delta\omega(S)=\mathbb{K}$.*

Proof. Recall that when T is abelian, a flow (X,T) is weak-mixing if and only if $\mathfrak{g}(X)E=G$ (see [**5**: 3.7 and 4]). By (1.9) and (1.10) $B\in S^{\perp\perp}$. Since $\mathscr{S}$ is weak-mixing, $SE=G$; i.e. $E\in S^{\perp}$. Hence $BE=G$ and $\mathscr{B}$ is weak-mixing. That $\delta\omega(S)=\mathbb{K}$ follows from (1.7). □

We shall now use the results obtained to produce a weak-mixing flow with a non-weak-mixing quasi-factor.

(1.12) *Notation.* The following notation will be used for the remainder of this section: $\sigma\in Z(\mathscr{A},\mathbb{K})$, $\mathscr{S}=\operatorname{ext}(\mathscr{A},\sigma)$, $\omega\in Z(M,\mathbb{K})$ with $F_\omega=\delta\sigma$, $\mathscr{B}=\operatorname{ext}(\mathscr{S},\omega)$, $\alpha\in A$ with $\delta\sigma(\alpha)\ne 1$, $\langle\alpha\rangle$ the τ-closed subgroup of G generated by α and $\rho\in Z(M,\mathbb{K})$ with $F_\rho(m)=\delta\sigma(\alpha)\quad(m\in M)$.

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(1.13) PROPOSITION. *Let* $\delta\omega(\beta)=1$ $(\beta\in\langle\alpha\rangle)$. *Then*

(1) $\mathfrak{g}(\mathfrak{a}(\langle\alpha\rangle,\mathcal{B}))=\langle\alpha\rangle(\ker\delta\rho\cap B)$ *and*

(2) $\mathrm{al}\,(\delta\rho)\subset\mathfrak{a}(\langle\alpha\rangle,\mathcal{B}))$.

Proof. (1) Set $\mathcal{L}=\mathfrak{a}(\langle\alpha\rangle,\mathcal{B})$ and $L=\mathfrak{g}(\mathcal{L})$, and let $b\in\ker\delta\rho\cap B$. Then

$$1=\delta\rho(b)=\delta\omega(b)^{-1}\delta\omega(\alpha)^{-1}\delta\omega(\alpha b)=\delta\omega(\alpha)^{-1}\delta\omega(\alpha b)$$

(by (1.2)). Thus

$$\delta\omega(\alpha)=\delta\omega(\alpha b)=\delta\omega(\alpha b\alpha^{-1}\alpha)=\delta\omega(\alpha b\alpha^{-1})\delta\omega(\alpha)\quad\text{since } \alpha b\alpha^{-1}\in S.$$

Consequently $1=\delta\omega(\alpha b\alpha^{-1})$ and $\alpha b\alpha^{-1}\in B=\ker(\delta\omega|S)$. Thus

$$\alpha(\ker\delta\rho\cap B)\alpha^{-1}\subset\ker\delta\rho\cap B.$$

Let $H=\{a\in\langle\alpha\rangle|a(\ker\delta\rho\cap B)\subset(\ker\delta\rho\cap B)\langle\alpha\rangle\}$. Then H is a closed sub-semigroup of $\langle\alpha\rangle$. Hence H is a closed subgroup of $\langle\alpha\rangle$ [**1**: 2.11]. Since $\alpha\in H$, $H=\langle\alpha\rangle$. Consequently

$$\langle\alpha\rangle(\ker\delta\rho\cap B)\subset(\ker\delta\rho\cap B)\langle\alpha\rangle$$

and

$$(\ker\delta\rho\cap B)\langle\alpha\rangle=(\langle\alpha\rangle\ker\delta\rho\cap B)^{-1}\subset((\ker\delta\rho\cap B)\langle\alpha\rangle)^{-1}$$
$$=\langle\alpha\rangle(\ker\delta\rho\cap B).$$

Thus $\langle\alpha\rangle(\ker\delta\rho\cap B)$ is a τ-closed subgroup of G.

Now L is the largest τ-closed subgroup of G which contains $\langle\alpha\rangle$ and is contained in $\langle\alpha\rangle B$ [**5**: 3.1]. Hence $\langle\alpha\rangle(\ker\delta\rho\cap B)\subset L$.

Let $b\in L\cap B$. Then $\alpha b\in L=L^{-1}\subset(\langle\alpha\rangle B)^{-1}=B\langle\alpha\rangle$. Hence $\alpha b=r\beta$ for some $r\in B$ and $\beta\in\langle\alpha\rangle$. Then

$$\delta\omega(\alpha b)=\delta\omega(r\beta)=\delta\omega(r)\delta\omega(\beta)=\delta\omega(\beta)=1.$$

Thus $\delta\rho(b)=\delta\omega(b)^{-1}\delta\omega(\alpha)^{-1}\delta\omega(\alpha b)=1$ and so $b\in\ker\delta\rho\cap B$.

Let $l\in L$. Then $l=kb$ for some $k\in\langle\alpha\rangle$, $b\in B$. Then

$$b\in L\cap B\subset\ker\delta\rho\cap B$$

and so $l\in\langle\alpha\rangle(\ker\delta\rho\cap B)$.

(2) $\delta\rho(\alpha)=\delta\omega(\alpha)^{-1}\delta\omega(\alpha)^{-1}\delta\omega(\alpha^2)=1$ shows that $\langle\alpha\rangle\subset\ker\delta\rho$. Hence

$$L=\langle\alpha\rangle(\ker\delta\rho\cap B)\subset\ker\delta\rho=\mathfrak{g}(\mathrm{al}\,(\delta\rho)).$$

Now $\mathrm{al}\,(\delta\rho)\subset\mathcal{E}$, the algebra of almost periodic functions, implies that

$$\mathfrak{a}(\langle\alpha\rangle,\mathcal{B})\vee\mathrm{al}\,(\delta\rho)$$

is an almost periodic extension of $\mathfrak{a}(\langle\alpha\rangle,\mathcal{B})$. Since the groups of these flows are the same, the flows are equal. The proof is completed. □

(1.14) LEMMA. *Let* $\delta\omega(\alpha)=1=\delta\omega(\alpha^2)$. *Then* $\delta\omega(\alpha^n)=1$ *for all integers* n.

Proof. The formula

$$\delta\rho(x)=\delta\omega(x)^{-1}\delta\omega(\alpha)^{-1}\delta\omega(\alpha x)=\delta\omega(x)^{-1}\delta\omega(\alpha x)\quad(1.2)\qquad(*)$$

shows that $\delta\rho(\alpha)=1$. Hence $\delta\rho(\langle\alpha\rangle)=1$ since $\delta\rho$ is a continuous homomorphism of (G,τ) into $\mathbb{K}$. Lemma 1.14 now follows from (*) by induction. □

(1.15) LEMMA. *$\delta\omega(\langle\alpha\rangle)=1$ if and only if $H(\langle\alpha\rangle,\tau)\subset B$ and $\delta\omega(\alpha)=1=\delta\omega(\alpha^2)$.*

Proof. Let $\delta\omega(\langle\alpha\rangle)=1$. Then of course $\delta\omega(\alpha)=1=\delta\omega(\alpha^2)$. Moreover $\alpha\in A$ implies that $\langle\alpha\rangle\subset A$. Hence

$$H(\langle\alpha,\tau\rangle)\subset H(A,\tau)\subset A^{\#}\subset S$$

whence $H(\langle\alpha\rangle,\tau)\subset B=\ker(\delta\omega|S)$ since $\delta\omega(H(\langle\alpha\rangle,\tau))\subset\delta\omega(\langle\alpha\rangle)=1$.

Now let $\delta\omega(\alpha)=1=\delta\omega(\alpha^2)$ and $H(\langle\alpha\rangle,\tau)\subset B$. Let $\beta\in\langle\alpha\rangle$. Choose a net (α^{N_i}) in $\langle\alpha\rangle$ with $\alpha^{N_i}\to_\tau\beta$ and let $\alpha^{N_i}\to p\in M$. Then

$$\delta\omega(p)=\lim\delta\omega(\alpha^{N_i})=1$$

by (1.14).

Moreover $(\alpha^{N_i})\subset\langle\alpha\rangle\subset A$ implies that $p|\mathscr{A}=u|\mathscr{A}$ and $pu\in\langle\alpha\rangle$. Hence $p=pu$ on $\mathscr{B}$ ($\mathscr{B}$ is a distal extension of $\mathscr{A}$) and $\delta\omega(pu)=\delta\omega(p)=1$. Also $\alpha^{N_i}\to_\tau pu$ shows that

$$\beta(pu)^{-1}\in H(\langle\alpha\rangle,\tau)\subset B.$$

Consequently $\delta\omega(\beta)=\delta\omega(\beta(pu)^{-1}pu)=\delta\omega(\beta(pu)^{-1})\delta\omega(pu)=1$. □

(1.16) *A construction.* Let $\mathscr{A}$ be a weak-mixing metric flow and $\sigma\in Z(\mathscr{A},\mathbb{K})$ such that $\mathscr{S}=\text{ext}(\mathscr{A},\sigma)$ is weak-mixing and $\delta\sigma(A)=\mathbb{K}$. (Such exist, see [**6**].) Set $\mathscr{B}=\text{ext}(\mathscr{S},\omega)$ where $F_\omega=\delta\sigma$. Then by (1.11) $\mathscr{B}$ is weak-mixing and $\delta\omega(B)=\mathbb{K}$. Hence $|\mathscr{B}|$ may be identified with the flow $(\mathbb{K}\times\mathbb{K}\times|\mathscr{A}|,t)$ where

$$(k,l,x)t=(\delta\sigma(mt),\delta\omega(pt),xt),$$

$$(k,l\in\mathbb{K},x\in|\mathscr{A}|\quad\text{and}\quad m,p\in M\quad\text{with }\delta\sigma(m)=k,\ \delta\omega(p)=l).$$

Let $\lambda\in\mathbb{K}$. Then the flow $(\mathbb{K},R_\lambda)$ is equicontinuous and so is disjoint from $\mathscr{B}$. Hence there exists a sequence (N_i) such that

$$(1,1,x_0)t^{N_i}\to(1,\lambda,x_0)\quad\text{and}\quad\lambda^{N_i}\to1.$$

(Here $x_0=u|\mathscr{A}$.)

Let $p\in\beta\mathbb{Z}$ be a limit point of the sequence (t^{N_i}) and set $\alpha=upu\in G$.

(1.16.1) *$\delta\omega(\alpha^k)=1$ for all integers k.*

Proof. Since $(1,1,x_0)t^{N_i}\to(1,\lambda,x_0)$, $(1,1,x_0)p=(1,\lambda,x_0)$. Also

$$(1,1,x_0)p=(1,1,x_0)up=(\delta\omega(up),\delta\sigma(up),x_0up).$$

Hence $\delta\omega(up)=1$, $\delta\sigma(up)=\lambda$ and $up=u$ on $\mathscr{A}$. Thus $\alpha=upu=u$ on $\mathscr{A}$; i.e. $\alpha\in A$. Since $\mathscr{S}$ and $\mathscr{B}$ are both distal extensions of $\mathscr{A}$ and $up=upu$ on $\mathscr{A}$, $\alpha=upu=up$ on $\mathscr{S}$ and $\mathscr{B}$. Consequently $\delta\omega(\alpha)=\delta\omega(up)=1$ and $\delta\sigma(\alpha)=\lambda$.

Moreover by (5) of (1.6)

$$\delta\omega(\alpha t^{N_i})=\delta\omega(\alpha)\delta\sigma(\alpha)^{N_i}\delta\omega(t^{N_i})=\lambda^{N_i}\delta\omega(t^{N_i})$$

from which it follows that

$$\delta\omega(\alpha p)=\delta\omega(p)=\delta\omega(\alpha)=1.$$

Since $\alpha pu=pu=p=\alpha p$ on $\mathscr{A}$, $\alpha^2=\alpha pu=\alpha p$ on $\mathscr{B}$. Consequently $\delta\omega(\alpha^2)=\delta\omega(\alpha p)=1$ and (1.16.1) follows from (1.14). □

(1.16.2) *Let $\lambda^k=1$. Then $H(\langle\alpha\rangle,\tau)\subset B$.*

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Proof. $\delta\sigma(\alpha^k)=\delta\sigma(\alpha)^k=\lambda^k=1$ implies that $\alpha^k\in S$. By (1.16.1), $\delta\omega(\alpha^k)=1$. Hence

$$\alpha^k\in B=\ker(\delta\omega|S).$$

Consequently $\langle\alpha\rangle\cap B$ has finite index in $\langle\alpha\rangle$ whence

$$H(\langle\alpha\rangle,\tau)\subset\langle\alpha\rangle\cap B\subset B. \qquad \square$$

(1.16.3) *Let* $\lambda^k=1$ *with* $\lambda\neq 1$, $k\neq 0$. *Then* $\mathfrak{a}(\langle\alpha\rangle,\mathcal{B})$ *is a non-weak-mixing quasi-factor of the weak-mixing flow,* $\mathcal{B}$.

Proof. This follows from (1.16.2), (1.15), and (1.13). □

Section 2

(2.1) *Definition.* Let $\sigma\in Z(\mathcal{A},K)$. Then the *sequence built on* $(\mathcal{A},\sigma)$ is the sequence $(\mathcal{A}_n,\sigma_n)$ $(n\geq 0)$ defined inductively as follows: $\mathcal{A}_0=\mathcal{A}$, $\sigma_0=\sigma$, $\mathcal{A}_{n+1}=\mathrm{ext}(\mathcal{A}_n,\sigma_n)$ and $\sigma_{n+1}\in Z(\mathcal{A}_n,K)$ with $F_{\sigma_{n+1}}=\delta\sigma_n$. Thus $\partial_{\sigma_{n+1}}=\sigma_n$.

In § 1 we were concerned with the first two or three terms of the sequence built on $(\mathcal{A},\sigma)$. In particular (1.3) dealt with $\mathcal{A}_1\vee\mathcal{A}_1\alpha$ for $\alpha\in\mathfrak{g}(\partial_\sigma)$. Here we shall consider $\mathcal{A}_n\vee\mathcal{A}_n\alpha$ for $\alpha\in A=\mathfrak{g}(\mathcal{A})$.

(2.2) *Notation.* For most of this section we shall be dealing with a fixed flow $\mathcal{A}$, $\sigma\in Z(\mathcal{A},K)$, and $\alpha\in A$. Let $(\mathcal{A}_n,\sigma_n)$ be the sequence built on $(\mathcal{A},\sigma)$ and

$$k_n=\delta\sigma_n(\alpha)\in K \quad (n\geq 0).$$

Then $(\mathcal{S}_i^n,\rho_i^n)_{i\geq 0}$ will denote the sequence built on $(\mathbb{R},\rho^n)$ $(n\geq 0)$ where

$$\rho^n\in Z(M,K) \text{ with } F_{\rho^n}(m)=k_n \quad (m\in M).$$

The various flows and cocycles depend of course on α but this dependence has been suppressed for 'notational convenience'.

(2.3) Proposition. *For all positive integers* n,

$$\prod_{i=0}^{n}\delta\rho_{n-i}^i(x)=\delta\sigma_{n+1}(x)^{-1}\delta\sigma_{n+1}(\alpha)^{-1}\delta\sigma_{n+1}(\alpha x) \quad (x\in M).$$

Proof. The case $n=0$ is just proposition 1.2. Now assume that

$$\prod_{i=0}^{n-1}\delta\rho_{n-1-i}^i(x)=\delta\sigma_n(x)^{-1}\delta\sigma_n(\alpha)^{-1}\delta\sigma_n(\alpha x) \quad (x\in M).$$

Then $\mathcal{A}_{n+1}=\mathrm{ext}(\mathcal{A}_n,\sigma_n)$ and $\partial_{\sigma_{n+1}}=\sigma_n$. By (1.2) if $\gamma\in Z(M,K)$ with

$$\delta\gamma(x)=\delta\sigma_{n+1}(x)^{-1}\delta\sigma_{n+1}(\alpha)^{-1}\delta\sigma_{n+1}(\alpha x)$$

then $F_\gamma(x)=\delta\sigma_n(\alpha x)\delta\sigma_n(x)^{-1}$ $(x\in M)$. Thus

$$F_\gamma(x)=\delta\sigma_n(\alpha)\prod_{i=0}^{n-1}\delta\rho_{n-1-i}^i(x)$$

$$=F_{\rho_0^n}(x)\prod_{i=0}^{n-1}F_{\rho_{n-i}^i}(x)=\prod_{i=0}^{n}F_{\rho_{n-i}^i}(x).$$

Consequently $\gamma=\prod_{i=0}^n\rho_{n-i}^i$ and $\delta\gamma=\prod_{i=0}^n\delta\rho_{n-i}^i$ (recall that K is abelian). The proof is completed. □

(2.4) *Remarks.* We should now like to use the cocycles $\gamma_n = \prod_{i=0}^{n} \rho_{n-i}^{i}$ to build a sequence of flows $(\mathcal{R}_n)$. To this end observe that

$$\delta\gamma_n = \prod_{i=0}^{n} \delta\rho_{n-i}^{i} = \prod_{i=0}^{n} F_{\rho_{n+i-i}^{i}} \quad \text{and} \quad F_{\gamma_{n+1}} = \prod_{i=0}^{n+1} F_{\rho_{n+1-i}^{i}}$$

whence

$$F_{\gamma_{n+1}} = (\delta\gamma_n) F_{\rho_0^{n+1}}.$$

Now set $\mathcal{R}_0 = \mathbb{R}$. Since γ_0 is the constant cocycle ρ_0^0, $\mathcal{R}_1 = \text{ext}(\mathcal{R}_0, \gamma_0)$ is an almost periodic extension of $\mathcal{R}_0$.

Assume that γ_n is a cocycle on $\mathcal{R}_n$ to K and set $\mathcal{R}_{n+1} = \text{ext}(\mathcal{R}_n, \gamma_n)$. Then $\delta\gamma_n$ defines a continuous function on $|\mathcal{R}_{n+1}|$ to K. Since $F_{\rho_0^{n+1}}$ is constant, $F_{\gamma_{n+1}} = (\delta\gamma_n)F_{\rho_0^{n+1}}$ defines a continuous function on $|\mathcal{R}_{n+1}|$ to K. Hence γ_{n+1} is a cocycle on $\mathcal{R}_{n+1}$ to K and the sequence $(\mathcal{R}_n)$ is well defined, where $\mathcal{R}_{k+1} = \text{ext}(\mathcal{R}_k, \gamma_k)$.

(2.5) PROPOSITION. *For all integers $n \geq 1$, $\mathcal{A}_n \vee \mathcal{A}_n\alpha = \mathcal{A}_n \vee \mathcal{R}_{n-1}$.*

Proof. When $n = 1$, $\mathcal{R}_0 = \mathbb{R}$ and $\mathcal{A}_1 \vee \mathcal{R}_0 = \mathcal{A}_1$. Moreover $\mathcal{A}_1 \vee \mathcal{A}_1\alpha = \mathcal{A}_1$ since $\alpha \in A = A_0$ and $A_1 \lhd A_0$.

The case $n = 2$ is just (1.3).

Assume that $\mathcal{A}_n \vee \mathcal{A}_n\alpha = \mathcal{A}_n \vee \mathcal{R}_{n-1}$. Since $\mathcal{A}_{n+1}$ and $\mathcal{R}_n$ are distal extensions of $\mathcal{A}_n$ and $\mathcal{R}_{n-1}$ respectively $\mathcal{A}_{n+1} \vee \mathcal{A}_{n+1}\alpha$ and $\mathcal{A}_{n+1} \vee \mathcal{R}_n$ are both distal extensions of the flow

$$\mathcal{A}_n \vee \mathcal{A}_n\alpha = \mathcal{A}_n \vee \mathcal{R}_{n-1}.$$

It thus suffices to show that their groups $A_{n+1} \cap \alpha^{-1}A_{n+1}\alpha$ and $A_{n+1} \cap R_n$ are equal.

Let $\beta \in A_{n+1} \cap \alpha^{-1}A_{n+1}\alpha \subset A_n \cap \alpha^{-1}A_n\alpha = A_n \cap R_{n-1}$. Then

$$\delta\gamma_{n-1}(\beta) = \prod_{i=0}^{n-1} \delta\rho_{n-1-i}^{i}(\beta) = \delta\sigma_n(\beta)^{-1}\delta\sigma_n(\alpha)^{-1}\delta\sigma_n(\alpha\beta)$$

and $\beta \in A_{n+1} \cap \alpha^{-1}A_{n+1}\alpha$ implies that $\delta\sigma_n(\beta) = e$ and

$$\delta\sigma_n(\alpha\beta) = \delta\sigma_n(\alpha\beta\alpha^{-1}\alpha) = \delta\sigma_n(\alpha\beta\alpha^{-1})\delta\sigma_n(\alpha).$$

Thus $\delta\gamma_{n-1}(\beta) = e$; which together with the fact that $\beta \in R_{n-1}$ implies that $\beta \in R_n$. Consequently

$$A_{n+1} \cap \alpha^{-1}A_{n+1}\alpha \subset A_{n+1} \cap R_n.$$

Now let $\beta \in A_{n+1} \cap R_n$. Then $e = \delta\gamma_{n-1}(\beta) = \delta\sigma_n(\beta)^{-1}\delta\sigma_n(\alpha)^{-1}\delta\sigma_n(\alpha\beta)$ whence $\delta\sigma_n(\alpha\beta) = \delta\sigma_n(\alpha)\delta\sigma_n(\beta)$. On the other hand

$$\delta\sigma_n(\alpha\beta) = \delta\sigma_n(\alpha\beta\alpha^{-1}\alpha) = \delta\sigma_n(\alpha\beta\alpha^{-1})\delta\sigma_n(\alpha).$$

(Recall $\beta \in A_{n+1} \cap R_n \subset A_n \cap R_{n-1} = A_n \cap \alpha^{-1}A_n\alpha$.) Hence $\delta\sigma_n(\alpha\beta\alpha^{-1}) = e$ and so $\beta \in \alpha^{-1}A_{n+1}\alpha$. The proof is completed. □

(2.6) *Remarks.* We should now like to consider the A-regularizer, $r_A(\mathcal{A}_n)$ of $\mathcal{A}_n$. To this end we introduce the following notation. Let $\kappa \in K$. Then $(\mathcal{S}_n^{\kappa} | n = 0, \ldots)$ will denote the sequence built on $(\mathbb{R}, \eta)$ where η is the cocycle on M to K with $F_\eta(m) = \kappa \quad (m \in M)$. We shall also denote by $\mathcal{R}_n(\alpha)$ the flow previously denoted by $\mathcal{R}_n$ in order to indicate its dependence on $\alpha \in A$.

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Let $\mathscr{E}_n = \bigvee_{\kappa \in K} \mathscr{S}_n^{\kappa}$. Then we shall show that $\delta\sigma_n(A_n) = K$ for all n implies that $r_A(\mathscr{A}_n) = \mathscr{A}_n \vee \mathscr{E}_{n-1}$ for all n.

(2.7) LEMMA. *Let $\mathscr{L}$, $\mathscr{K}$, and $\mathscr{N}$ be minimal flows such that $\mathscr{K}$ is a distal extension of $\mathscr{L}$, $\mathscr{L} \subset \mathscr{N}$, and $N \subset K$. Then $\mathscr{K} \subset \mathscr{N}$.*

Proof. By [1: 13.11], $\mathfrak{g}(\mathscr{L}^* \cap \mathscr{N}) = L^*N$ and $\mathfrak{g}(\mathscr{L}^* \cap \mathscr{K}) = L^*K \supset L^*N$. Consequently

$$\mathscr{K} = \mathscr{L}^* \cap \mathscr{K} \subset \mathscr{L}^* \cap \mathscr{N} \subset \mathscr{N}. \qquad \square$$

(2.8) LEMMA. *For all integers $n \geq 1$ and all $\alpha \in A$, $\mathscr{R}_n(\alpha) \subset \mathscr{E}_n$.*

Proof. Since $\mathscr{R}_1(\alpha) = \mathscr{S}_1^{F_{\gamma_0}(\alpha)}$, $\mathscr{R}_1(\alpha) \subset \mathscr{E}_1$. Assume $\mathscr{R}_n(\alpha) \subset \mathscr{E}_n$. Then

$$\mathscr{R}_{n+1}(\alpha) = \operatorname{ext}(\mathscr{R}_n(\alpha), \gamma_n) \quad \text{with } F_{\gamma_n} = \prod_{i=0}^{n} F_{\rho_{n-i}^i}.$$

Since

$$\mathscr{S}_0^{\kappa_{n+1}} \vee \cdots \vee \mathscr{S}_{n+1}^{\kappa_0} \subset \mathscr{E}_{n+1} \quad (\kappa_i = F_{\rho_0^i}(\alpha)), \qquad \delta\gamma_n(\beta) = e \quad (\beta \in E_{n+1}).$$

Lemma 2.8 now follows from (2.7). $\square$

(2.9) LEMMA. *Let $\delta\sigma_i(A_i) = K$, $\kappa_i \in K$ $(1 \leq i \leq n)$. Then there exists $\alpha \in A$ with*

$$\delta\sigma_i(\alpha) = \kappa_i \quad (1 \leq i \leq n).$$

Proof. The hypothesis implies that $\mathscr{A}_{n+2}$ may be identified with a flow whose underlying phase space is $K^n \times |\mathscr{A}|$ and such that

$$(e, \ldots, e, x_0)p = (\delta\sigma_n(p), \ldots, \delta\sigma(p), x_0 p) \quad (p \in M).$$

Hence there exists $p \in M$ with

$$\delta\sigma_i(p) = \kappa_i \quad (1 \leq i \leq n) \quad \text{and} \quad x_0 p = x_0;$$

i.e. $p = u$ on $\mathscr{A}$. Then

$$\alpha = upu \in A \quad \text{and} \quad \delta\sigma_i(\alpha) = \kappa_i \quad (1 \leq i \leq n). \qquad \square$$

(2.10) LEMMA. *Let $\delta\sigma_i(A_i) = K$ for all i. Then, for all n,*

$$\mathscr{E}_n \subset \vee \{\mathscr{R}_n(\alpha) | \alpha \in A\}.$$

Proof. Since $\mathscr{S}_1^{\delta\sigma_0(\alpha)} = \mathscr{R}_1(\alpha)$ and $\delta\sigma_0(A) = K$,

$$\mathscr{E}_1 \subset \vee \{\mathscr{R}_1(\alpha) | \alpha \in A\}.$$

Now assume that

$$\mathscr{E}_n \subset \vee \{\mathscr{R}_n(\alpha) | \alpha \in A\}$$

and let $\kappa \in K$. Choose $\alpha \in A$ with $\delta\sigma_0(\alpha) = \kappa$ and $\delta\sigma_i(\alpha) = e$ $(1 \leq i \leq n)$.

In this case the cocycles ρ^k are trivial for $1 \leq k \leq n$ and ρ^0 is just the cocycle with $F_{\rho^0} \equiv \kappa$. Consequently $\gamma_n = \rho_n^0$. Since

$$\mathscr{R}_{n+1}(\alpha) = \operatorname{ext}(\mathscr{R}_n(\alpha), \gamma_n) \quad \text{and} \quad \mathscr{S}_{n+1}^{\kappa} = \operatorname{ext}(\mathscr{S}_n^{\kappa}, \rho_n^0),$$

$$\delta\rho_n^0(\beta) = e \quad (\beta \in R_{n+1}(\alpha) = \mathfrak{g}(\mathscr{R}_{n+1}(\alpha))).$$

Hence $\mathfrak{g}(\vee \{\mathscr{R}_{n+1}(\beta) | \beta \in A\}) \subset \mathfrak{g}(\mathscr{S}_{n+1}^{\kappa})$ and so

$$\mathscr{S}_{n+1}^{\kappa} \subset \vee \{\mathscr{R}_{n+1}(\beta) | \beta \in A\}$$

by (2.7). Since κ was arbitrary, $\mathscr{E}_{n+1} \subset \vee \{\mathscr{R}_{n+1}(\beta) | \beta \in A\}$. $\square$

(2.11) PROPOSITION. *Let* $\delta\sigma_n(A_n)=K$ *for all* n. *Then* $r_A(\mathcal{A}_n)=\mathcal{A}_n \vee \mathcal{E}_{n-1}$ *for all* n.

Proof. Let $\alpha\in A$. Then $\mathcal{A}_n \vee \mathcal{A}_n\alpha=\mathcal{A}_n \vee \mathcal{R}_{n-1}(\alpha)\subset\mathcal{A}_n \vee \mathcal{E}_{n-1}$ by (2.5) and (2.8). Hence

$$r_A(\mathcal{A}_n)=\vee\{\mathcal{A}_n\alpha \mid \alpha\in A\}\subset\mathcal{A}_n \vee \mathcal{E}_{n-1}.$$

On the other hand

$$\mathcal{R}_{n-1}(\alpha)\subset\mathcal{A}_n \vee \mathcal{A}_n\alpha\subset r_A(\mathcal{A}_n) \quad (\alpha\in A)$$

implies that $\mathcal{A}_n \vee \mathcal{E}_{n-1}\subset r_A(\mathcal{A}_n)$ by (2.10). □

(2.12) *Remark.* Corollary 1.11 shows that the condition that $\delta\sigma_n(A_n)=K$ for all n is satisfied in the case when $\mathcal{A}$ is weak-mixing, $K=\mathbb{K}$, and $\delta\sigma(A)=\mathbb{K}$.

R.E.'s research was supported by NSF Grant no. 7800403 A02.

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