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Oscillation results for the summatory functions of fake μ's

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Abstract. Martin, Mossinghoff, and Trudgian [19] recently introduced a family of arithmetic functions called "fake μ's," which are multiplicative functions for which there is a $\{-1, 0, 1\}$-valued sequence $(\varepsilon_j)_{j=1}^{\infty}$ such that $f(p^j) = \varepsilon_j$ for all primes p. They investigated comparative number-theoretic results for fake μ's and, in particular, proved oscillation results at scale $\sqrt{x}$ for the summatory functions of fake μ's with $\varepsilon_1 = -1$ and $\varepsilon_2 = 1$. In this article, we establish new oscillation results for the summatory functions of all nontrivial fake μ's at scales $x^{1/2\ell}$ where ℓ is a positive integer (the "critical index") depending on f; for $\ell = 1$ this recovers the oscillation results in [19]. Our work also recovers results on the indicator functions of powerfree and powerfull numbers; we generalize techniques applied to each of these examples to extend to all fake μ's.

1 Introduction

A major topic in comparative prime number theory is the behavior of summatory functions of various multiplicative functions. For example, let $\mu(n)$ be the Möbius function, and let $M(x) = \sum_{n\le x} \mu(n)$ be its summatory function. In 1897, Mertens [23] conjectured that $|M(x)| \le \sqrt{x}$ for all $x \ge 1$, an assertion that subsequently became known as the *Mertens conjecture*. Similarly, let $\lambda(n) = (-1)^{\Omega(n)}$ be the Liouville function, and let $L(x) = \sum_{n\le x} \lambda(n)$ be its summatory function. In 1919, Pólya [31] asked whether $L(x) \le 0$ holds for all x, a question that became known as the *Pólya problem* (often mistakenly called "Pólya's conjecture").

One of the motivations for studying these problems was the fact that a positive answer to either would imply the Riemann hypothesis (RH), as Pólya noted for $L(x)$ in [31]. Indeed, by 1942, it was "well known," as reported by Ingham [14], that both RH and the simplicity of all zeros of the Riemann zeta function $\zeta(s)$ would follow from either of $M(x)/\sqrt{x}$ or $L(x)/\sqrt{x}$ being bounded either above or below by an absolute constant. However, Ingham showed that any of these one-sided bounds would also imply that there were infinitely many linear relations with integer coefficients among the positive imaginary parts of the zeros of $\zeta(s)$, which cast doubt upon both the Mertens conjecture and a positive answer to the Pólya problem.

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The Pólya problem was first resolved in the negative by Haselgrove [12], who showed that $L(x)$ changes sign infinitely often. Similarly, the Mertens conjecture was first disproved by Odlyzko and te Riele [29]. Currently, the best known lower bound on $\limsup_{x\to\infty} L(x)/\sqrt{x}$ and the best known upper bound on $\liminf_{x\to\infty} L(x)/\sqrt{x}$ are due to Mossinghoff and Trudgian [28], and the corresponding best known bounds for $M(x)/\sqrt{x}$ are due to Hurst [13]. We still do not know how to disprove either $L(x) \ll \sqrt{x}$ or $M(x) \ll \sqrt{x}$ or even any of the corresponding one-sided bounds. The Mertens conjecture and the Pólya problem motivated substantial work in comparative prime number theory; we refer the reader to an annotated bibliography for comparative prime number theory [20] for further sources.

Recently, Martin, Mossinghoff, and Trudgian [19] developed comparative number-theoretic results for a family of arithmetic functions they called "fake μ's," which are multiplicative functions f such that for each positive integer j, there is a constant $\varepsilon_j \in \{-1, 0, 1\}$ such that $f(p^j) = \varepsilon_j$ holds for all primes p. We say f is *defined via the sequence* $(\varepsilon_j)_{j=1}^{\infty}$, and throughout this article, we always identify f with its defining sequence $(\varepsilon_j)_{j=1}^{\infty}$. Certainly fake μ's include the Möbius function (the real μ!) and the Liouville function. These authors focused on the "persistent bias" and "apparent bias," at the scale of $\sqrt{x}$, of the summatory function of a fake μ. In particular, they showed [19, Theorem 3] that if f is a fake μ with $\varepsilon_1 = -1$ and $\varepsilon_2 = 1$, then its summatory function $F_f(x)$ satisfies

$$F_f(x) - \mathfrak{a}\sqrt{x} = \Omega_{\pm}(\sqrt{x}), \tag{1.1}$$

where the apparent bias $\mathfrak{a}$ is twice the residue at $s = \frac{1}{2}$ of the meromorphic function defined by the Dirichlet series $\sum_{n=1}^{\infty} f(n)n^{-s}$. In other words, $F_f(x) - \mathfrak{a}\sqrt{x}$ is infinitely often larger than some positive constant times $\sqrt{x}$ and also infinitely often smaller than some negative constant times $\sqrt{x}$ (these constants can depend on f). There is a small gap in their proof, but it can be filled with a few additional observations (see Example 3.1).

These authors included the comment that "a function with no bias at scale $\sqrt{x}$ could well see one at a smaller scale." This remark motivates the study of the current article, namely, to establish new oscillation results for the summatory functions of a larger family of fake μ's, with oscillations at potentially smaller scales than $\sqrt{x}$. Indeed, we will unconditionally establish such an oscillation result for the summatory function of every nontrivial fake μ (see Theorems 1.7, 1.10, 1.14, and 3.10 below). We also establish upper bounds, both unconditional and assuming RH, on the error terms in the asymptotic formulas for all these summatory functions (see Theorems 1.16 and 1.18 below).

1.1 Existing examples of fake μ's

Before introducing our main results formally, we first describe a few subfamilies of fake μ's whose summatory functions have been studied extensively. (We refer the reader to the survey [32, Chapter VI] for more results related to these fake μ's.) These examples also motivate us to divide fake μ's into three categories (see Definition 1.8).

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Example 1.1 (Tanaka's Möbius functions) Given an integer $k \ge 2$, recall that an integer is called k-free if it is not divisible by the kth power of any prime (for $k = 2$ and $k = 3$, these numbers are commonly called *squarefree* and *cubefree*, respectively). Tanaka [35] defined the generalized Möbius function μ_k by declaring that $\mu_k(n) = (-1)^{\Omega(n)}$ if n is k-free and $\mu_k(n) = 0$ otherwise. Note that these functions interpolate between the Möbius and Liouville functions in the sense that $\mu_2 = \mu$ and $\lim_{k\to\infty} \mu_k = \lambda$ as a pointwise limit.

- We see that μ_k is the fake μ corresponding to the sequence (ε_j) defined by $\varepsilon_j = (-1)^j$ for $1 \le j \le k-1$ and $\varepsilon_j = 0$ for $j \ge k$.
- The corresponding Dirichlet series $\sum_{n=1}^{\infty} \mu_k(n) n^{-s}$ can be written down explicitly and admits a nice factorization in terms of the Riemann zeta function. For example, when $k \ge 3$ is odd, we have the identity $\sum_{n=1}^{\infty} \mu_k(n) n^{-s} = \zeta(2s)\zeta(ks)/\zeta(s)\zeta(2ks)$.
- Let $M_k(x) = \sum_{n \le x} \mu_k(n)$. Tanaka [35] showed that $M_k(x) - \tau_k\sqrt{x} = \Omega_\pm(\sqrt{x})$, where $\tau_2 = 0$, $\tau_k = \zeta(\frac{k}{2})/\zeta(\frac{1}{2})\zeta(k)$ if $k \ge 3$ is odd, $\tau_k = 1/\zeta(\frac{1}{2})\zeta(\frac{k}{2})$ if $k \ge 4$ is even, and $\tau_\infty = 1/\zeta(\frac{1}{2})$. These statements are special cases of the result (1.1) that was proved later.

Example 1.2 (Indicator functions of k-free numbers) Note that $\mu_k^2(n)$ is the indicator function of k-free numbers (generalizing the fact that $\mu^2(n)$ is the indicator function of squarefree numbers).

- We see that μ_k^2 is the fake μ corresponding to the sequence defined by $\varepsilon_1 = \cdots = \varepsilon_{k-1} = 1$ and $\varepsilon_j = 0$ for $j \ge k$.
- The corresponding Dirichlet series also factors nicely in terms of $\zeta(s)$: we have the identity $\sum_{n=1}^{\infty} \mu_k^2(n) n^{-s} = \zeta(s)/\zeta(ks)$.
- Let $Q_k(x)$ be the number of k-free numbers up to x, and let $R_k(x) = Q_k(x) - x/\zeta(k)$. Montgomery and Vaughan [24] showed under RH that $R_k(x) \ll_\varepsilon x^{1/(k+1)+\varepsilon}$. (This result has been slightly improved by various authors; for k sufficiently large, the best known bound is due to Graham and Pintz [11].) Meng [21] gave, under the additional assumption $\sum_{0<\gamma\le T} |\zeta'(\rho)|^2 \ll_\varepsilon T^{1+\varepsilon}$ for all $\varepsilon > 0$, a bound on an integral involving $R_k(x)$ which implies that $R_k(x) \ll x^{1/2k}$ on average.
- On the other hand, Evelyn and Linfoot [9] first proved that $R_k(x) = \Omega_\pm(x^{1/2k})$. The recent paper [27] by Mossinghoff, Oliveira e Silva, and Trudgian provides the best known explicit lower bounds on the oscillations of the error term (see also the survey [30] by Pappalardi).

Example 1.3 (Apostol's Möbius functions) Apostol [1] described a different generalization of the Möbius function. For each $k \ge 1$, let ν_k be the multiplicative function such that for each prime p, we set $\nu_k(p^j) = 1$ if $j < k$, $\nu_k(p^j) = -1$ if $j = k$, and $\nu_k(p^j) = 0$ if $j > k$. Note that $\nu_1 = \mu$.

- We see that ν_k is the fake μ corresponding to the sequence defined by $\varepsilon_1 = \cdots = \varepsilon_{k-1} = 1$, $\varepsilon_k = -1$, and $\varepsilon_j = 0$ for $j \ge k+1$.
- While the corresponding Dirichlet series does not admit (for $k \ge 2$) an exact factorization in terms of $\zeta(s)$, it does possess [3, Lemma 2.5] a useful partial factorization of the form $\zeta(s)/\zeta(ks)^2 \cdot A_k^*(s)$, where $A_k^*(s)$ is an Euler product that is absolutely convergent (and thus analytic) for $\Re(s) > 1/(k+1)$.

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- For $k \geq 2$, Apostol [1] showed that there is a constant ϕ_k such that $\sum_{n\leq x} \nu_k(x) = \phi_k x + O(x^{1/k}\log x)$; the factor multiplying $x^{1/k}$ in the error term has recently been improved by Banerjee et al. [3]. Under RH, Suryanarayana [34] improved the error term to $O\big(x^{4k/(4k^2+1)}\exp(C\frac{\log x}{\log\log x})\big)$ for some positive constant C.

Example 1.4 (Indicator functions of k-full numbers) Given an integer $k \geq 2$, recall that an integer is called *k-full* if every prime that divides it does so with multiplicity at least k (for $k = 2$ these numbers are commonly called *powerfull* or *squarefull* numbers).

- We see that the indicator function of k-full numbers is the fake μ corresponding to the sequence defined by $\varepsilon_1 = \cdots = \varepsilon_{k-1} = 0$ and $\varepsilon_j = 1$ for $j \geq k$.
- When $k = 2$, the corresponding Dirichlet series is exactly $\zeta(2s)\zeta(3s)/\zeta(6s)$. When $k \geq 3$, the corresponding Dirichlet series admits [2, Proposition 1] the useful partial factorization

$$\left(\prod_{j=k}^{2k-1}\zeta(js)\middle/\prod_{j=2k+2}^{4k+3}\zeta(js)^{a_j}\right)V(s),$$

  where a_j are particular integers and where $V(s)$ is an Euler product that is absolutely convergent (and thus analytic) for $\Re(s) > 1/(4k+4)$.
- Let $N_k(x)$ be the number of k-full numbers up to x; then $N_k(x)$ admits an asymptotic formula of the form $N_k(x) = \sum_{k\leq j\leq 2k-1}\mathfrak{a}_f(j)x^{1/j} + \Delta^{N_k}(x)$. Various upper bounds on $\Delta^{N_k}(x)$ can be found in [18, 22] and the references therein; in particular, under the Lindelöf hypothesis, Ivić [15] showed that $\Delta^{N_k}(x) \ll_\varepsilon x^{1/2k+\varepsilon}$.
- Bateman and Grosswald [4] showed that if ρ is any zero of the Riemann zeta function such that $\zeta(\frac{\rho}{2}) \neq 0$ and $\zeta(\frac{\rho}{3}) \neq 0$, then $\Delta^{N_2}(x) = \Omega_\pm(x^{\Re(\rho)/6})$. Thus, in particular, by taking $\rho = \frac{1}{2} \pm i\cdot 14.1347\ldots$ to be a zero of the zeta function closest to the real axis, their result implies that $\Delta^{N_2}(x) = \Omega_\pm(x^{1/12})$. Balasubramanian, Ramachandra, and Subbarao [2] showed that $\Delta^{N_2}(x) = \Omega(x^{1/10})$ and that $\Delta^{N_k}(x) = \Omega(x^{1/(2k+\sqrt{8k}+3)})$ for $k \geq 3$; more precisely, they showed that $\Delta^{N_k}(x) = \Omega(x^{1/2(k+r)})$, where r is the smallest positive integer such that $r(r-1) \geq 2k$.

One commonality of the above examples is the idea of factoring out powers of $\zeta(s)$, $\zeta(2s)$, and so on from a Dirichlet series, either resulting in a complete factorization or else leaving a remaining factor with nicer analytic properties (a larger half-plane of absolute convergence, for example). This theme is present in many guises in analytic number theory. For example, if $f(n)$ is a multiplicative function such that κ is the average value of $f(p)$ over primes p, the Selberg–Delange method (see, for example, [36, Chapter II.5]) finds an asymptotic formula for the summatory function of $f(n)$ by factoring $\zeta(s)^\kappa$ out of the corresponding Dirichlet series, so that the leftover factor is typically analytic in a neighborhood of $s = 1$.

If in fact $f(p) = \kappa$ exactly for all primes p, then the resulting factorization is $\sum_{n=1}^\infty f(n)n^{-s} = \zeta(s)^\kappa U_1(s)$, where $U_1(s)$ is a Dirichlet series whose coefficients are supported on squarefull numbers. If $f(p^2)$ is also independent of p, one can further write $\sum_{n=1}^\infty f(n)n^{-s} = \zeta(s)^\kappa\zeta(2s)^{\kappa'}U_2(s)$ for an appropriate constant κ' and a Dirichlet series $U_2(s)$ whose coefficients are supported on 3-full numbers, and so on. These

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partial zeta-factorizations are already beneficial for analytic methods, and sometimes one can consider analogous *total zeta-factorizations* $\sum_{n=1}^{\infty} f(n)n^{-s} = \prod_{\nu=1}^{\infty} \zeta(\nu s)^{a_\nu}$. For example, Moree [26, Sections 2 and 3] and other authors have used these zeta-factorizations as a means of calculating certain number-theoretic constants to high precision; in another vein, Dahlquist [7, Section 2] recognized the importance of finite zeta-factorizations in the study of the natural boundary of analytic continuation for Dirichlet series (see also the later chapters of [8]). Moreover, when the exponents a_ν are integers, then the resulting functions are meromorphic and we expect Perron's formula and contour integration to yield asymptotic formulas, and explicit formulas involving the zeros of $\zeta(s)$, for our summatory functions.

Returning now to the examination of fake μ's, it turns out that partial zeta-factorizations of this type are important not only for the proofs, but even for the statements, of our oscillation results. In Section 2, we will describe an algorithm for computing such zeta-factorizations that is designed specifically for the Dirichlet series of fake μ's. We will write the result of such a zeta-factorization in the form

$$\sum_{n=1}^{\infty} f(n)n^{-s} = \prod_{j=1}^{\ell} \zeta(js)^{a_j} \cdot U_\ell(s), \tag{1.2}$$

where $a_1, \ldots, a_\ell$ are integers and $U_\ell(s)$ is of the form

$$U_\ell(s) = \prod_p \left(1 + \sum_{j=\ell+1}^{\infty} \frac{\eta_j}{p^{js}}\right) \tag{1.3}$$

for certain constants η_j (so that the coefficients in the Dirichlet series for $U_\ell(s)$ are supported on $(\ell+1)$-full numbers).

We now proceed to define the terminology required to state our main results.

1.2 Main results

We quickly observe that not all fake μ's exhibit oscillations in their partial sums. For instance, if f is the indicator function of $n = 1$ (corresponding to the case $\varepsilon_j \equiv 0$), there is no oscillation result. Similarly, if f is the indicator function of kth powers for some $k \geq 1$ (that is, if $\varepsilon_j = 1$ when $k \mid j$ and $\varepsilon_j = 0$ otherwise), then there is no oscillation result either beyond the trivial $\sum_{n\leq x} f(n) - x^{1/k} = \lfloor x^{1/k} \rfloor - x^{1/k} = \Omega_-(1)$. For this reason, we call the fake μ's mentioned above *trivial fake μ's*. These observations lead to the following definition.

Definition 1.5 Let $\mathcal{F}$ be the set of arithmetic functions consisting of all fake μ's that are not trivial. In other words, $f \in \mathcal{F}$ precisely when $f(n)$ is a multiplicative function such that:

(a) there exists a $\{-1, 0, 1\}$-valued sequence $(\varepsilon_j)_{j=1}^{\infty}$ such that $f(p^j) = \varepsilon_j$ for every prime p;

(b) $f(n)$ is neither the indicator function of $\{1\}$, nor the indicator function of the set of kth powers for any $k \geq 1$.

For any $f \in \mathcal{F}$, define $F_f(x) = \sum_{n\leq x} f(n)$ to be the summatory function of f, and define $D_f(s) = \sum_{n=1}^{\infty} f(n)n^{-s}$ to be the Dirichlet series associated with f.

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Our goal is to deduce an oscillation result for $F_f(x)$ based on analytic properties of $D_f(s)$, stated with the help of the indices introduced in the following definition.

Definition 1.6 If $f \in \mathcal{F}$ is defined via the sequence (ε_j), we define the *initial index* of f to be the smallest number j such that $\varepsilon_j \neq 0$. We define the *critical index* of f to be the smallest number j for which a power of $\zeta(js)$ appears in the denominator of the zeta-factorization of $D_f(s)$. More precisely, if for $\sigma > 1$ we can write $D_f(s)$ in the form given in equations (1.2) and (1.3), then the critical index of f equals ℓ precisely when $a_1, a_2, \ldots, a_{\ell-1} \geq 0$ and $a_\ell < 0$.

Given a zeta-factorization (1.2), we expect (when U_ℓ is nicely behaved) that the right-hand side will have real poles at $s = \frac{1}{j}$ whenever $a_j > 0$ (so that $\zeta(js)$ appears to some power in the numerator); we further expect that it will have complex poles with real parts equal to $\frac{1}{2j}$ whenever $a_j < 0$ (so that $\zeta(js)$ appears to some power in the denominator). When using Perron's formula and contour integration, the real poles are associated with the main term of the asymptotic formula, and the complex poles are associated with oscillatory terms. Consequently, we expect a main term for $F_f(x)$ of the form

$$G_f(x) = \sum_{j=1}^{2\ell} \operatorname{Res}\left(D_f(s)\frac{x^s}{s}, \frac{1}{j}\right), \tag{1.4}$$

where $\operatorname{Res}(g(s), s_0)$ denotes the residue of $g(s)$ at $s = s_0$. We will study oscillation results and upper bounds for the error term

$$E_f(x) = F_f(x) - G_f(x). \tag{1.5}$$

Note that ℓ here denotes the critical index of f, and that the sum defining G_f has been deliberately taken up to exactly 2ℓ for the following reason. As a function of x, the residue at $s = \frac{1}{j}$ in equation (1.4) will have order of magnitude $x^{1/j}$, while the residues at the complex poles with real part equal to $\frac{1}{2\ell}$ will oscillate with order of magnitude $x^{1/2\ell}$. Therefore, the rightmost 2ℓ potential real poles should be taken into account in the main term, but we expect that all subsequent real poles will give a negligible contribution compared to the oscillations of the error term.

We are now able to state the most general form of our main oscillation result, which holds for every nontrivial fake μ.

Theorem 1.7 *Let $f \in \mathcal{F}$. If ℓ is the critical index of f, then $E_f(x) = \Omega_\pm(x^{1/2\ell})$. In other words,*

$$F_f(x) = G_f(x) + E_f(x) = \sum_{j=1}^{2\ell} \operatorname{Res}\left(D_f(s)\frac{x^s}{s}, \frac{1}{j}\right) + \Omega_\pm(x^{1/2\ell}).$$

(In this theorem and throughout this article, all implicit constants in Ω and O-notation may depend upon f.)

Given more information about the specific $f \in \mathcal{F}$, of course, we should be able to be more specific about this main term and oscillation term. We would like to determine when the residues on the right-hand side equal 0 (as many of them will) and to write the nonzero residues more explicitly; we would like to increase the size of the

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oscillation term when possible (even if just by a logarithmic factor); and we would like to more explicitly determine what the critical index ℓ actually is. As it happens, we can already be quite a bit more specific simply by dividing the set of nontrivial fake μ's into three types.

Definition 1.8 Let $f \in \mathcal{F}$ be a nontrivial fake μ.

(a) We say that f is of *Möbius-type* if the initial index $k \geq 1$ of f has the property that $\varepsilon_k = -1$. The Möbius function μ and the Liouville function λ are certainly of Möbius-type, as are Tanaka's Möbius functions μ_k from Example 1.1. As we will see in Theorem 1.10, the critical index also equals k in this case.

(b) We say that f is of *powerfree-type* if $\varepsilon_1 = 1$ (so that the initial index of f is 1). When $k \geq 2$, the indicator functions μ_k^2 of k-free numbers (see Example 1.2) are certainly of powerfree-type, as are Apostol's Möbius functions ν_k from Example 1.3. In this case, it is important to consider the smallest positive number k with $\varepsilon_k \neq 1$, which is in some sense a "measure of powerfreeness" (since this yields the correct value of k when $f = \mu_k^2$, and also when $f = \nu_k$). As we will see in Theorem 1.14, the critical index equals this value of k in this case.

(c) We say that f is of *powerfull-type* if the initial index of f is $k \geq 2$ and $\varepsilon_k = 1$. When $k \geq 2$, the indicator functions of k-full numbers (see Example 1.4) are certainly of powerfull-type. The initial index k is in some sense a "measure of powerfullness" of f. Unlike the previous two cases, there is no simple formula for the critical index, although in Section 2, we give an algorithm for computing the critical index from the defining sequence (ε_j).

We will be able to be more concrete about the main terms for our summatory functions of fake μ's with the following notation.

Definition 1.9 For any $f \in \mathcal{F}$ and any positive integer j, define

$$\mathfrak{a}_f(j) = j \operatorname{Res}(D_f(s), \tfrac{1}{j}) \quad \text{and} \quad \mathfrak{b}_f(j) = j^2 \operatorname{Res}((s - \tfrac{1}{j}) D_f(s), \tfrac{1}{j}).$$

If $D_f(s)$ has at most a double pole at $s = \frac{1}{j}$, then the principal part of $D_f(s)$ is

$$\frac{\mathfrak{b}_f(j)}{j^2(s - 1/j)^2} + \frac{\mathfrak{a}_f(j)}{j(s - 1/j)},$$

so that subtracting this expression from $D_f(s)$ results in a function that is analytic at $s = \frac{1}{j}$. Note that either or both of $\mathfrak{a}_f(j)$ and $\mathfrak{b}_f(j)$ might equal 0.

We may now state the following refinement of Theorem 1.7 for Möbius-type fake μ's.

Theorem 1.10 *Let $f \in \mathcal{F}$ be of Möbius-type with initial index k. Then, the critical index of f also equals k. Moreover,*

$$G_f(x) = \sum_{\substack{k+1 \leq j \leq 2k \\ \varepsilon_j = 1}} \mathfrak{a}_f(j) x^{1/j} \quad \textit{and} \quad E_f(x) = \Omega_\pm(x^{1/2k}).$$

Remark 1.11 A particular case of this theorem is when $f \in \mathcal{F}$ has $\varepsilon_1 = -1$ and $\varepsilon_2 = 1$, in which case f is of Möbius-type with initial index $k = 1$. We thus see that Theorem 1.10

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generalizes [19, Theorem 3] as stated in equation (1.1), which itself includes the Möbius function μ, the Liouville function λ, and Tanaka's Möbius functions μ_k from Example 1.1 as special cases.

Remark 1.12 When we provide upper bounds for $E_f(x)$ in Theorem 1.18, we will see that assuming RH, this oscillation result $\Omega_\pm(x^{1/2k})$ for fake μ's of Möbius type is best possible up to factors of x^ε.

Remark 1.13 A tempting heuristic suggests that Theorem 1.7 and its refinements might always yield best-possible oscillation results: One can write $E_f(x) = F_f(x) - G_f(x)$ as a contour integral involving a meromorphic function whose rightmost singularities are the poles coming from the negative power of $\zeta(\ell s)$ in equation (1.2) (where ℓ is the critical index of f), since $G_f(x)$ is designed to cancel all the real poles of $D_f(s)$ with $\Re(s) > \frac{1}{2\ell}$. Assuming RH, these rightmost singularities are all on the line $\Re(s) = \frac{1}{2\ell}$, and contour integration might plausibly result in an explicit formula whose dominant terms have order of magnitude $x^{1/2\ell}$. However, estimating the contribution to $F_f(x)$ from the shifted contour is not straightforward, and indeed we know that this heuristic can fail in general – a counterexample is given by the indicator function of k-full numbers (see Example 2.12 and Remark 2.13 below for more details).

We continue by stating the following refinement of Theorem 1.7 for powerfree-type fake μ's.

Theorem 1.14 *Let $f \in \mathcal{F}$ be of powerfree-type, and let k be the smallest positive integer such that $\varepsilon_k \neq 1$. Then, the critical index of f equals k, and*

$$G_f(x) = \mathfrak{a}_f(1)x + \sum_{\substack{k+1 \le j \le 2k-1 \\ \varepsilon_j > \varepsilon_{j-1}}} \mathfrak{a}_f(j)x^{1/j} + \sum_{\substack{k+1 \le j \le 2k-1 \\ \varepsilon_j = 1,\, \varepsilon_{j-1} = -1}} \mathfrak{b}_f(j)x^{1/j}\left(\tfrac{1}{j}\log x - 1\right)$$

$$+ \begin{cases} 0, & \text{if } \varepsilon_{2k} - \varepsilon_{2k-1} + \varepsilon_k \le 0, \\ \mathfrak{a}_f(2k)x^{1/2k}, & \text{if } \varepsilon_{2k} - \varepsilon_{2k-1} + \varepsilon_k = 1, \\ \mathfrak{a}_f(2k)x^{1/2k} + \mathfrak{b}_f(2k)x^{1/2k}\left(\frac{1}{2k}\log x - 1\right), & \text{if } \varepsilon_{2k} - \varepsilon_{2k-1} + \varepsilon_k = 2. \end{cases}$$

Moreover, $E_f(x) = \Omega_\pm(x^{1/2k}(\log x)^{|\varepsilon_k|})$.

Remark 1.15 Since the indicator function of k-free numbers is a powerfree-type fake μ, we see that Theorem 1.14 recovers the result of Evelyn and Linfoot [9] mentioned in Example 1.2 on oscillations of the error term in the counting function for k-free numbers. Theorem 1.14 also provides the first oscillation result for Apostol's Möbius functions ν_k from Example 1.3.

For the third category of fake μ's, namely, those of powerfull-type, a more precise statement is much more complicated, in large part because even computing the critical index itself is not straightforward. In Theorem 3.10, we will give a detailed version of our oscillation result for powerfull-type fake μ's.

We complement the oscillation results described above with upper bounds on the error terms $E_f(x)$. Since we are partly motivated by trying to understand how strong those oscillation results are, we provide two such results: the first one is unconditional, and the second one assumes RH.

Theorem 1.16 *Let $f \in \mathcal{F}$. Unconditionally, we have the following upper bounds on $E_f(x)$:*

(a) *If f is of powerfull-type with initial index k, then $E_f(x) \ll_\varepsilon x^{1/(k+1)+\varepsilon}$ for each $\varepsilon > 0$.*
(b) *If f is of Möbius-type or powerfree-type with critical index k, then*

$$E_f(x) \ll x^{1/k} \exp\left(- c\frac{(\log x)^{3/5}}{(\log\log x)^{1/5}}\right), \tag{1.6}$$

where c is some absolute positive constant.

Remark 1.17 Note that if $f \in \mathcal{F}$ has initial index k, then the first nonzero contribution to the main term (1.4) for $F_f(x)$ has order of magnitude $x^{1/k}$. Therefore, these unconditional upper bounds for the error term are only of modest strength for powerfull-type fake μ's, and when f is of Möbius-type we do not even have power savings in x. This challenge is already reflected in the classical cases $f = \mu$ and $f = \lambda$, where the upper bound (1.6) with $k = 1$ is the best known estimate for the Mertens sum $M(x)$ and for the error term $\Delta^L(x) = L(x) - \sqrt{x}/\zeta(\frac{1}{2})$ in Pólya's problem.

Theorem 1.18 *Let $f \in \mathcal{F}$. Assuming RH, we have the following upper bounds on $E_f(x)$:*

(a) *If f is of Möbius-type or powerfull-type with initial index k, then there exists a positive constant C such that $E_f(x) \ll x^{1/2k} \exp(C\frac{\log x}{\log\log x})$.*
(b) *If f is of powerfree-type with critical index k, then $E_f(x) \ll_\varepsilon x^{1/(k+1)+\varepsilon}$ for each $\varepsilon > 0$.*

Remark 1.19

(a) When f is of Möbius-type, Theorem 1.18(a) implies (assuming RH) that the oscillations given in Theorem 1.10 are essentially best possible, as mentioned in Remark 1.12.
(b) When $f = \mu$, the best-known conditional upper bound on $M(x)$ is due to Soundararajan [33], where he showed that $M(x) \ll x^{1/2} \exp\big((\log x)^{1/2}(\log\log x)^{14}\big)$.
(c) When f is the indicator function of k-full numbers, Theorem 1.18(a) improves Ivić's result mentioned in Example 1.4, although his result requires only the Lindelöf hypothesis rather than the full RH.
(d) There exist examples (see Example 2.10 below) of powerfull-type fake μ's with initial index k where the error-term oscillations are as large as $E_f(x) = \Omega_\pm(x^{1/(2k+2)})$; so Theorem 1.18(a) is at least reasonably sharp for powerfull-type fake μ's.
(e) Theorem 1.18(b) extends the result of Montgomery and Vaughan's result concerning the indicator function of k-free numbers (see Example 1.2) to all powerfree-type fake μ's with critical index k. In particular, Theorem 1.18(b) applies when $f = \nu_k$ (see Example 1.3) and improves Suryanarayana's result [34] that $E_f(x) \ll x^{4k/(4k^2+1)+o(1)}$.

The examples in Section 1.1 are far from being a complete list of fake μ's already studied in the literature. We introduce one additional family of fake μ's to further illustrate Theorems 1.16 and 1.18.

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Example 1.20 Bege [5] introduced the following generalization of Apostol's Möbius functions: given integers $2 \le k < m$, the function $\mu_{k,m}$ is the fake μ defined via the sequence $(\varepsilon_j)_{j=1}^{\infty}$ with

$$\varepsilon_j = \begin{cases} 1, & \text{if } 1 \le j \le k-1, \\ -1, & \text{if } j = m, \\ 0, & \text{otherwise.} \end{cases}$$

Note that $\mu_{k,m}$ is of powerfree-type with critical index k. Theorem 1.16(b) recovers Bege's unconditional bound [5, Theorem 3.1], while Theorem 1.18(b) improves Bege's conditional bound $E_{\mu_{k,m}}(x) \ll x^{2/(2k+1)+o(1)}$ [5, Theorem 3.2] under RH to $E_{\mu_{k,m}}(x) \ll x^{1/(k+1)+o(1)}$.

Notation. We use several notational conventions that are standard in analytic number theory. In this article, p always denotes a prime, and $\sum_p$ and $\prod_p$ represent sums and products over all primes. For a complex number s, we write $s = \sigma + it$, so that $\sigma = \Re(s)$ and $t = \Im(s)$. In addition, $\rho = \beta + i\gamma$ denotes a general nontrivial zero of the Riemann zeta function $\zeta(s)$, so that $\beta = \Re(\rho)$ and $\gamma = \Im(\rho)$. We write $f(x) = \Omega(g(x))$ to mean $\limsup_{x\to\infty} |f(x)|/g(x) > 0$, and $f(x) = \Omega_{\pm}(g(x))$ to mean both $\limsup_{x\to\infty} f(x)/g(x) > 0$ and $\liminf_{x\to\infty} f(x)/g(x) < 0$.

Outline of the article. In Section 2, we provide an algorithm to compute the critical index of a function $f \in \mathcal{F}$. In Section 3, we prove an oscillation result for $E_f(x)$ based on its critical index and a few other parameters from the algorithm (Theorem 3.5 is the most general statement). In particular, in Section 3.4, we complete the proofs of Theorems 1.10 and 1.14, which apply to Möbius-type and powerfree-type fake μ's, respectively, as well as giving a general result (Theorem 3.10 below) for powerfull-type fake μ's. Together these three results imply Theorem 1.7. Finally, in Section 4, we study upper bounds on the error term $E_f(x)$ and prove Theorems 1.16 and 1.18.

2 Zeta-factorizations and the critical index

In this section, we provide some precise statements about the (partial) zeta-factorizations mentioned in the introduction, including closed formulas and bounds for the resulting exponents and coefficients. With these statements in place, we then describe an algorithm for computing zeta-factorizations with enough factors to determine the critical index of a fake μ (recall Definition 1.6). We also provide several zeta-factorization examples using this algorithm, which we compare to known results from the literature.

One viewpoint we wish to stress is that given a specific Euler product with known numerical coefficients, all analytic number theorists who produced a zeta-factorization of that Euler product would arrive at the same numerical answer using essentially the same procedure as one another. The difficulties lie not in the calculations themselves, but rather in finding an accessible notation we can use to record the results of zeta-factorizations in a general setting.

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2.1 Zeta-factorization of Dirichlet series

We start with the following lemma for "one-step" zeta-factorization for a family of Dirichlet series relevant to our discussions. Later, we will apply the lemma recursively to obtain "multi-step" zeta-factorizations.

Lemma 2.1 *Let t be a positive integer, and let $(\eta_j)_{j=t}^{\infty}$ be a sequence of integers. Assume that the Euler product*

$$A(s) = \prod_p \left(1 + \sum_{j=t}^{\infty} \frac{\eta_j}{p^{js}}\right)$$

converges absolutely for $\sigma > 1$. Then for $\sigma > 1$,

$$A(s) = \zeta(ts)^{\eta_t} \cdot \prod_p \left(1 + \sum_{j=t+1}^{\infty} \frac{\eta'_j}{p^{js}}\right),$$

where the first several values of η'_j are

$$\eta'_j = \begin{cases} \eta_j, & \text{if } t+1 \le j \le 2t-1, \\ \eta_{2t} - \frac{1}{2}(\eta_t^2 + \eta_t), & \text{if } j = 2t. \end{cases}$$

Moreover, $|\eta'_j| \le (2j|\eta_t|)^{2|\eta_t|} \max_{n \le j} |\eta_n|$ for all $j \ge t+1$.

Proof For convenience, we extend the sequence (η_j) to all integers indices by defining $\eta_0 = 1$ and $\eta_j = 0$ when $1 \le j \le t-1$ or $j \le -1$. Assume throughout that $\sigma > 1$. Then

$$\zeta(ts)^{-\eta_t} A(s) = \prod_p \left(\sum_{j\in\mathbb{Z}} \frac{\eta_j}{p^{js}}\right)\left(1 - \frac{1}{p^{ts}}\right)^{\eta_t}. \tag{2.1}$$

The lemma is trivial if $\eta_t = 0$; we consider two cases according to the sign of η_t.

If $\eta_t > 0$, then equation (2.1) becomes

$$\zeta(ts)^{-\eta_t} A(s) = \prod_p \left(\sum_{j\in\mathbb{Z}} \frac{\eta_j}{p^{js}}\right)\left(\sum_{m=0}^{\eta_t} \frac{(-1)^m \binom{\eta_t}{m}}{p^{mts}}\right) = \prod_p \left(\sum_{k\in Z} \frac{1}{p^{ks}} \sum_{m=0}^{\eta_t} (-1)^m \binom{\eta_t}{m} \eta_{k-tm}\right).$$

It follows that

$$A(s) = \zeta(ts)^{\eta_t} \cdot \prod_p \left(1 + \sum_{j\in\mathbb{Z}} \frac{\eta'_j}{p^{js}}\right) \quad \text{with} \quad \eta'_j = \sum_{m=0}^{\eta_t} (-1)^m \binom{\eta_t}{m} \eta_{j-tm}. \tag{2.2}$$

In particular,

$$\eta'_j = \begin{cases} \eta_j = 0, & \text{if } j \le -1 \text{ or } 1 \le j \le t-1, \\ \eta_0 = 1, & \text{if } j = 0, \\ \eta_t - \binom{\eta_t}{1}\eta_0 = 0, & \text{if } j = t, \\ \eta_j - \binom{\eta_t}{1}\eta_{j-t} = \eta_j, & \text{if } t+1 \le j \le 2t-1, \\ \eta_{2t} - \binom{\eta_t}{1}\eta_t + \binom{\eta_t}{2}\eta_0 = \eta_{2t} - \frac{1}{2}(\eta_t^2 + \eta_t), & \text{if } j = 2t. \end{cases}$$

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From equation (2.2), we also deduce that

$$|\eta_j'| \le \sum_{m=0}^{\eta_t} \binom{\eta_t}{m} |\eta_{j-tm}| \le \max_{n\le j} |\eta_n| \sum_{m=0}^{\eta_t} \binom{\eta_t}{m} \le 2^{\eta_t} \max_{n\le j} |\eta_n|,$$

which establishes the lemma in the $\eta_t > 0$ case.

On the other hand, if $\eta_t < 0$, then equation (2.1) becomes

$$\begin{aligned}\zeta(ts)^{-\eta_t} A(s) &= \prod_p \bigg(\sum_{j\in\mathbb{Z}} \frac{\eta_j}{p^{js}} \bigg) \bigg(\sum_{m=0}^{\infty} \frac{\binom{m-\eta_t-1}{-\eta_t-1}}{p^{mts}} \bigg) \\ &= \prod_p \bigg(\sum_{k\in Z} \frac{1}{p^{ks}} \sum_{m=0}^{\infty} \binom{m-\eta_t-1}{-\eta_t-1} \eta_{k-tm} \bigg).\end{aligned}$$

and it follows that

$$A(s) = \zeta(ts)^{\eta_t} \cdot \prod_p \bigg(1 + \sum_{j\in\mathbb{Z}} \frac{\eta_j'}{p^{js}} \bigg) \quad \text{with} \quad \eta_j' = \sum_{m=0}^{\infty} \binom{m-\eta_t-1}{-\eta_t-1} \eta_{j-tm} \tag{2.3}$$

holds for all $j \ge 0$. In particular, we have $\eta_0' = 1$, and

$$\eta_j' = \begin{cases} \eta_j = 0, & \text{if } j \le -1 \text{ or } 1 \le j \le t-1, \\ \eta_0 = 1, & \text{if } j = 0, \\ \eta_t + \binom{-\eta_t}{1}\eta_0 = 0, & \text{if } j = t, \\ \eta_j + \binom{-\eta_t}{1}\eta_{j-t} = \eta_j, & \text{if } t+1 \le j \le 2t-1, \\ \eta_{2t} + \binom{-\eta_t}{1}\eta_t + \binom{1-\eta_t}{2}\eta_0 = \eta_{2t} - \frac{1}{2}(\eta_t^2 + \eta_t), & \text{if } j = 2t. \end{cases}$$

From equation (2.3), we also deduce that

$$\begin{aligned}|\eta_j'| \le \sum_{m=0}^{\lfloor j/t \rfloor} \binom{m-\eta_t-1}{-\eta_t-1} |\eta_{j-tm}| &\le \max_{n\le j} |\eta_n| \sum_{m=0}^{\lfloor j/t \rfloor} \binom{m+|\eta_t|-1}{|\eta_t|-1} \\ &= \max_{n\le j} |\eta_n| \cdot \frac{1}{|\eta_t|} \bigg\lfloor \frac{j}{t} + 1 \bigg\rfloor \binom{\lfloor j/t \rfloor + |\eta_t|}{|\eta_t| - 1} \\ &\le \max_{n\le j} |\eta_n| \bigg(\frac{j}{t} + 1 \bigg) \bigg(\frac{j}{t} + |\eta_t| \bigg)^{|\eta_t|} \le \max_{n\le j} |\eta_n| (2j|\eta_t|)^{2|\eta_t|},\end{aligned}$$

which establishes the lemma in the $\eta_t < 0$ case. ■

A follow-up lemma puts into context the significance of the coefficient bound at the end of Lemma 2.1.

Lemma 2.2 *Let n be a nonnegative integer. Suppose that there are positive constants A, B such that the Euler product*

$$U_n(s) = \prod_p \bigg(1 + \sum_{j=n+1}^{\infty} \frac{\eta_j}{p^{js}} \bigg) \tag{2.4}$$

satisfies $|\eta_j| \le (Aj)^B$ for all j. Then, $U_n(s)$ converges absolutely to an analytic function for $\sigma > \frac{1}{n+1}$.

Proof To show absolute convergence, we must bound

$$\sum_p \sum_{j=n+1}^{\infty} \frac{|\eta_j|}{|p^{js}|} \le \sum_p \sum_{j=n+1}^{\infty} \frac{(Aj)^B}{p^{j\sigma}} = A^B \sum_p \frac{1}{p^{(n+1)\sigma}} \sum_{i=0}^{\infty} \frac{(i+n+1)^B}{p^{i\sigma}}$$
$$\le A^B \sum_p \frac{1}{p^{(n+1)\sigma}} \sum_{i=0}^{\infty} \frac{(n+1)^B (i+1)\cdots(i+B)}{p^{i\sigma}}$$
$$= A^B \sum_p \frac{(n+1)^B}{p^{(n+1)\sigma}} B!(1-p^{-\sigma})^{-B-1} \ll_{A,B,n} \sum_p \frac{1}{p^{(n+1)\sigma}}$$

which converges by the assumption $\sigma > \frac{1}{n+1}$. Moreover, this convergence is locally uniform in s, which implies that the infinite product is indeed analytic. ∎

2.2 An algorithm for computing the critical index of $f \in \mathcal{F}$

We begin by using Lemma 2.1 to quickly compute the desired factorization of Dirichlet series $D_f(s)$ for a Möbius-type fake μ.

Proposition 2.3 *Let $f \in \mathcal{F}$ be of Möbius-type with initial index k. Then, the critical index of f also equals k. Moreover, for $\sigma > 1$,*

$$D_f(s) = U_{2k}(s) \cdot \prod_{j=k}^{2k} \zeta(js)^{\varepsilon_j}, \quad \textit{where} \quad U_{2k}(s) = \prod_p \left(1 + \sum_{j=2k+1}^{\infty} \frac{\eta_j}{p^{js}}\right) \tag{2.5}$$

with $|\eta_j| \le (2j)^{2(k+1)}$.

Remark 2.4 The coefficient bound $|\eta_j| \le (2j)^{2(k+1)}$ implies, by Lemma 2.2, that $U_{2k}(s)$ is analytic for $\sigma > 1/(2k+1)$.

Proof of Proposition 2.3 We begin by setting $\theta_j^{(k)} = \varepsilon_j$ for all j and writing

$$U_k(s) = \prod_p \left(1 + \sum_{j=k}^{\infty} \frac{\theta_j^{(k)}}{p^{js}}\right) = D_f(s). \tag{2.6}$$

We claim that for each $t = k, k+1, \dots, 2k+1$, we can write

$$D_f(s) = U_t(s) \prod_{j=k}^{t-1} \zeta(js)^{\varepsilon_j} \quad \text{with} \quad U_t(s) = \prod_p \left(1 + \sum_{j=t}^{\infty} \frac{\theta_j^{(t)}}{p^{js}}\right),$$

where $\theta_j^{(t)} = \varepsilon_j$ for all $t \le j \le 2k$ and $|\theta_j^{(t)}| \le (2j)^{2(t-k)}$ for all $j \in \mathbb{N}$. The base case $t = k$ is exactly equation (2.6), whereas deriving the case $t+1$ from the case t is a direct application of Lemma 2.1. (In the first step going from $t = k$ to $t = k+1$, it is important to note that $\varepsilon_k = -1$ implies that $\varepsilon_{2k} - \frac{1}{2}(\varepsilon_k^2 + \varepsilon_k) = \varepsilon_{2k}$. The fact that $\varepsilon_k = -1$ also confirms that the critical index of f equals k.) At the end of this recursive process, the final case $t = 2k+1$ is the statement of the proposition, with $\eta_j = \theta_j^{(2k+1)}$. ∎

Next, we consider $f \in \mathcal{F}$ of powerfree-type and powerfull-type. In this case, before applying Lemma 2.1, we need to first determine the critical index of f. Algorithm 1

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below computes the critical index of f, as well as principal indices of f defined below. Based on the algorithm, we further establish Theorem 2.6 on the partial zeta-factorization of $D_f(s)$ into the desired form (1.2). The introduction of principal indices plays a crucial role in describing Algorithm 1 as well as in stating Theorem 2.6.

Definition 2.5 Suppose that $f \in \mathcal{F}$ is defined via the sequence (ε_j) and has critical index ℓ, so that we can write $D_f(s)$ in the form given in equations (1.2) and (1.3) with $a_1, a_2, \ldots, a_{\ell-1} \geq 0$ and $a_\ell < 0$. We define the *principal indices* of f to be those numbers $1 \leq j \leq \ell - 1$ for which $a_j > 0$ (indicating that a power of $\zeta(js)$ is truly present in the numerator of the zeta-factorization (1.2)).

In the algorithm below, for a positive integer j and a set of positive integers $\{c_1, c_2, \ldots, c_m\}$, we define the number of *representations of j from* $\{c_1, c_2, \ldots, c_m\}$, denoted by n_j in the algorithm, to be the number of nonnegative integer solutions $(\alpha_1, \alpha_2, \ldots, \alpha_m)$ to the equation $\sum_{i=1}^m \alpha_i c_i = j$.

Algorithm 1: Compute the critical index and principal indices of $f \in \mathcal{F}$.

```
c_1 ← initial index of f
m ← 1
j ← c_1 + 1
while true do
    n_j ← the number of representations of j from {c_1, c_2, ..., c_m}
    if n_j = 0 and ε_j = 1 then
        c_{m+1} ← j
        m ← m + 1
    if n_j > ε_j then
        M ← m
        ℓ ← j
        return ℓ, c_1, c_2, ..., c_M
    j ← j + 1
```

Theorem 2.6 *Let $f \in \mathcal{F}$ be of powerfree-type or powerfull-type. Algorithm 1 terminates in finitely many steps and computes the critical index of f (denoted by ℓ), as well as the principal indices $c_1 < c_2 < \cdots < c_M$ of f. Moreover, for $\sigma > 1$, we have the factorization*

$$D_f(s) = U_{2\ell}(s) \cdot \frac{\prod_{j=1}^M \zeta(c_j s)}{\zeta(\ell s)^{n_\ell - \varepsilon_\ell}} \cdot \prod_{j=\ell+1}^{2\ell} \zeta(js)^{a_j}, \tag{2.7}$$

with

$$a_j = \begin{cases} \sum_{I \subset \{1,\ldots,M\}} (-1)^{\#I} \varepsilon_{j - \sum_{i\in I} c_i}, & \text{if } \ell+1 \leq j \leq 2\ell-1, \\ -\frac{(\varepsilon_\ell - n_\ell)^2 + \varepsilon_\ell - n_\ell}{2} + \sum_{I \subset \{1,\ldots,M\}} (-1)^{\#I} \varepsilon_{2\ell - \sum_{i\in I} c_i}, & \text{if } j = 2\ell, \end{cases} \tag{2.8}$$

where $M < \ell$ and n_ℓ are defined in Algorithm 1, and we have set $\varepsilon_0 = 1$ and $\varepsilon_j = 0$ for $j < 0$. Furthermore,

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$$U_{2\ell}(s) = \prod_p \left(1 + \sum_{j=2\ell+1}^{\infty} \frac{\eta_j}{p^{js}}\right),$$

and there exist constants A and B, depending only on ℓ, *such that* $|\eta_j| \le (Aj)^B$ *for all* $j \ge 2\ell + 1$.

Remark 2.7 Again, the coefficient bound $|\eta_j| \le (Aj)^B$ implies that $U_{2\ell}(s)$ is analytic for $\sigma > 1/(2\ell + 1)$ by Lemma 2.2.

Proof of Theorem 2.6 We first show that Algorithm 1 terminates in finitely many steps; equivalently, we show that the critical index of the sequence (ε_j) is finite. If there is a positive integer j such that $\varepsilon_j = -1$, then the second **if** statement of the algorithm ensures that the critical index of (ε_j) is at most j. Thus, we can assume that $\varepsilon_j \in \{0, 1\}$ for all j. Let k be the initial index of f.

- If $\varepsilon_j = 0$ for all indices j that are not multiples of k, then let k' be the smallest multiple of k such that $\varepsilon_{k'} \ne 1$ (such a k' must exist by the exclusion of trivial fake μ's from the family $\mathcal{F}$). Then, the second **if** statement of the algorithm ensures that the critical index of (ε_j) is at most k'.
- Otherwise, let k' be the smallest integer with $\varepsilon_j = 1$ that is not a multiple of k. Then, the first **if** statement of the algorithm sets $c_2 = k'$. Since $\operatorname{lcm}(k, k')$ has at least two representations from $\{k, k'\} = \{c_1, c_2\}$, the second **if** statement of the algorithm ensures that the critical index of (ε_j) is at most $\operatorname{lcm}(k, k')$.

Next, we make the initial definitions $D_0(s) = D_f(s)$ and $\theta_j^{(0)} = \varepsilon_j$ for all j.

Main goal: We will show inductively, for each $1 \le m \le M$, that c_m is the smallest positive integer j such that $\theta_j^{(m-1)}$ is nonzero, and moreover that $\theta_{c_m}^{(m-1)} = 1$. These claims are trivial for $m = 1$, as c_1 is the initial index of f, and $\varepsilon_{c_1} = 1$ since f is not of Möbius-type.

For the inductive step, fix $1 \le m \le M - 1$, and assume both that c_m is the smallest positive integer j such that $\theta_j^{(m-1)}$ is nonzero and that $\theta_{c_m}^{(m-1)} = 1$. We write $\theta_0^{(m-1)} = 1$, and for convenience, we adopt the convention that $\theta_j^{(m-1)} = 0$ when $j \le -1$. Set $D_m(s) = D_{m-1}(s)/\zeta(c_m s)$, and let $\theta_j^{(m)}$ be defined by

$$D_m(s) = \prod_p \left(\sum_{j=0}^{\infty} \frac{\theta_j^{(m)}}{p^{js}}\right)$$

for $\sigma > 1$. By the definition of $D_m(s)$, we have $\theta_j^{(m)} = \theta_j^{(m-1)} - \theta_{j-c_m}^{(m-1)}$, and then by induction, it is easy to show that

$$\theta_j^{(m)} = \sum_{I \subset \{1,\dots,m\}} (-1)^{\#I} \theta^{(0)}_{j-\sum_{i\in I} c_i} = \sum_{I \subset \{1,\dots,m\}} (-1)^{\#I} \varepsilon_{j-\sum_{i\in I} c_i}. \tag{2.9}$$

We define three parameters r_m, s_m, t_m as follows. Let r_m be the smallest positive integer with $\varepsilon_{r_m} \ne 0$ such that r_m has no representations from $\{c_1, c_2, \dots, c_m\}$. Let s_m be the smallest positive integer with $\varepsilon_{s_m} \ne 1$ that has exactly one representation from $\{c_1, c_2, \dots, c_m\}$. Finally, let t_m be the smallest positive integer that has at least

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two representations from $\{c_1, c_2, \ldots, c_m\}$. If any of these numbers r_m, s_m, t_m does not exist, we regard it as $+\infty$.

Now set $c_{m+1} = \min\{r_m, s_m, t_m\}$; we claim that c_{m+1} is finite. This is easy to see for $m \geq 2$, since, in this case, c_1c_2 has at least two representations from $\{c_1, c_2, \ldots, c_m\}$ and thus $c_{m+1} \leq t_m \leq c_1c_2$. It remains to consider the case $m = 1$, in which every nonnegative integer automatically has at most one representation from $\{c_1\}$. But $r_1 = s_1 = +\infty$ would mean that $\varepsilon_j = 1$ if j is a multiple of c_1 and $\varepsilon_j = 0$ otherwise; however, this sequence results in a trivial fake μ which has been ruled out in the definition of the family $\mathcal{F}$.

Subgoal: Next, we consider the range $1 \leq j \leq c_{m+1}$ and determine the values $\theta_j^{(m)}$ in this range. We will show that $\theta_j^{(m)} = 0$ when $j < c_{m+1}$, and also that $\theta_{c_{m+1}}^{(m)} = \varepsilon_{c_{m+1}} - n_{c_{m+1}} \neq 0$. We will need to consider three different cases depending on which of r_m, s_m, t_m is smallest. Note that $c_{m+1} \leq r_m$ and $c_{m+1} \leq s_m$ and $c_{m+1} \leq t_m$ by definition, so we will not need to consider values of j above any of these parameters.

(a) Assume that j has no representations from $\{c_1, c_2, \ldots, c_m\}$. For each subset I of $\{1, \ldots, m\}$, it follows that $j - \sum_{i\in I} c_i$ has no representations from $\{c_1, c_2, \ldots, c_m\}$ either and thus $\varepsilon_{j-\sum_{i\in I} c_i} = 0$ by the definition of r_m. Equation (2.9) then implies that $\theta_j^{(m)} = \varepsilon_j$. By the definition of r_m, if $j < r_m$ then $\theta_j^{(m)} = \varepsilon_j = 0$, while if $j = r_m$ then $j = c_{m+1}$ and $\theta_{c_{m+1}}^{(m)} = \varepsilon_{c_{m+1}} \in \{-1, 1\}$.

(b) Next, assume that $j = \sum_{i=1}^m \alpha_i c_i$ has exactly one representation from $\{c_1, c_2, \ldots, c_m\}$. Let $X = \{1 \leq i \leq m : \alpha_i > 0\}$. Then for each subset I of X, it follows that $j - \sum_{i\in J} c_i$ also has exactly one representation from $\{c_1, c_2, \ldots, c_m\}$ and thus $\varepsilon_{j-\sum_{i\in J} c_i} = 1$ by the definition of s_m. On the other hand, if I is a subset of $\{1, \ldots, m\}$ such that $I \not\subset X$, then $j - \sum_{i\in I} c_i$ has no representations from $\{c_1, c_2, \ldots, c_m\}$, and thus $\varepsilon_{j-\sum_{i\in I} c_i} = 0$ by the definition of r_m. Equation (2.9) and the binomial theorem then imply that

$$\theta_j^{(m)} = \sum_{I\subset X} (-1)^{\#I} \varepsilon_{j-\sum_{i\in I} c_i} = \varepsilon_j + \sum_{b=1}^{\#X} (-1)^b \binom{\#X}{b} = \varepsilon_j - 1.$$

By the definition of s_m, if $j < s_m$ then $\theta_j^{(m)} = \varepsilon_j - 1 = 0$, while if $j = s_m$ then $j = c_{m+1}$ and $\theta_{c_{m+1}}^{(m)} = \varepsilon_{c_{m+1}} - 1 \in \{-2, -1\}$.

(c) Finally, assume that j has $n_j \geq 2$ representations from $\{c_1, c_2, \ldots, c_m\}$; by the definition of t_m, we must have $j = t_m = c_{m+1}$. Write these representations as $j = \sum_{i=1}^m \alpha_i^{(h)} c_i$ for $1 \leq h \leq n_j$. Let $X_h = \{1 \leq i \leq m : \alpha_i^{(h)} > 0\}$; we claim that X_h are pairwise disjoint. Indeed, if we had $\alpha_i^{(h_1)} > 0$ and $\alpha_i^{(h_2)} > 0$ for some $1 \leq i \leq m$ and $1 \leq h_1 < h_2 \leq n_j$, then $j - c_i$ would have at least two representations from $\{c_1, c_2, \ldots, c_m\}$, violating the definition of t_m.

Let I be a nonempty subset of $\{1, \ldots, m\}$. Then, $j - \sum_{i\in I} c_i$ has at most one representation from $\{c_1, c_2, \ldots, c_m\}$ by the definition of t_m. If $I \subset X_h$ for some $1 \leq h \leq n_j$, then $j - \sum_{i\in I} c_i > 0$ does have a representation from $\{c_1, c_2, \ldots, c_m\}$, and thus $\varepsilon_{j-\sum_{i\in I} c_i} = 1$ by the definition of s_m and the assumption that $t_m < s_m$. On the other hand, if $I \not\subset X_h$ for every $1 \leq h \leq n_j$, then $j - \sum_{i\in I} c_i$ has no representations from $\{c_1, c_2, \ldots, c_m\}$ (for any such representation would induce an additional

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representation of j itself). Thus, it follows that $\varepsilon_{j-\sum_{i\in I} c_i} = 0$ by the definition of r_m and the assumption that $t_m < r_m$. Equation (2.9) and the binomial theorem then imply that

$$\theta_{t_m}^{(m)} = \varepsilon_{t_m} + \sum_{h=1}^{n_j} \sum_{\substack{I\subset X_h \\ I\neq\varnothing}} (-1)^{\#I} \varepsilon_{t_m - \sum_{i\in I} c_i} = \varepsilon_{t_m} + \sum_{h=1}^{n_j} \sum_{b=1}^{\#X_h} (-1)^b \binom{\#X_h}{b} = \varepsilon_{t_m} - n_j \leq -1.$$

We have therefore achieved our subgoal of showing that $\theta_j^{(m)} = 0$ for $j < c_{m+1}$ and that $\theta_{c_{m+1}}^{(m)} = \varepsilon_{c_{m+1}} - n_{c_{m+1}} \neq 0$. We now consider the sign of $\theta_{c_{m+1}}^{(m)}$:

- The only way that $\theta_{c_{m+1}}^{(m)} > 0$ is when $\varepsilon_{c_{m+1}} = 1$ and $n_{c_{m+1}} = 0$, and thus $\theta_{c_{m+1}}^{(m)} = 1$. Therefore, we have finished the proof for the induction step for $m+1$. At this point, the first **if** statement of the algorithm appends c_{m+1} to the list of principal indices, increases m by 1, and repeats the **while** loop.
- On the other hand, $\theta_{c_{m+1}}^{(m)} < 0$ means that $\varepsilon_{c_{m+1}} < n_{c_{m+1}}$. In this event, the second **if** statement sets $\ell = c_{m+1}$ and $m = M$ and terminates the algorithm.

These observations complete the verification of our main goal.

Note that for $\sigma > 1$,

$$D_M(s) = \frac{D_f(s)}{\prod_{j=1}^M \zeta(c_j s)} = \prod_p \left(\sum_{j=0}^\infty \frac{\theta_j^{(M)}}{p^{js}} \right). \tag{2.10}$$

We have shown that $\theta_j^{(M)} = 0$ for $1 \leq j < \ell$ and that $\theta_\ell^{(M)} = \varepsilon_\ell - n_\ell < 0$. Also, note that equation (2.9) implies that $|\theta_j^{(M)}| \leq 2^M$ for all j. In particular,

$$\left| \theta_{2\ell}^{(M)} - \frac{(\theta_\ell^{(M)})^2 + \theta_\ell^{(M)}}{2} \right| \leq 2^M + \frac{4^M + 2^M}{2} < 2^{2M+1}.$$

Now, we can apply Lemma 2.1 inductively $\ell + 1$ times to $D_M(s)$ to obtain

$$D_M(s) = \prod_{j=\ell}^{2\ell-1} \zeta(js)^{\theta_j^{(M)}} \cdot \zeta(2\ell s)^{\theta_{2\ell}^{(M)} - \frac{1}{2}((\theta_\ell^{(M)})^2 + \theta_\ell^{(M)})} U_{2\ell}(s), \tag{2.11}$$

where

$$U_{2\ell}(s) = \prod_p \left(1 + \sum_{j=2\ell+1}^\infty \frac{\eta_j}{p^{js}} \right)$$

with $|\eta_j| \leq 2^M (2j \cdot 2^{2M+1})^{4^{M+1}(\ell+1)}$. (Note that this bound can be made to depend upon ℓ alone since $M < \ell$.) Combining equations (2.9)–(2.11) establishes equation (2.7), which completes the proof of the theorem. ∎

2.3 Examples of applying Algorithm 1

It will be illuminating to give several examples of fake μ's where we see explicitly the zeta-factorization resulting from Algorithm 1. In some of these examples, we will take note of the oscillation results implied by Theorem 1.7, even though that theorem

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has not yet been proved; we assure the reader that these examples are merely for the purposes of illustration and will not be used when we prove Theorem 1.7 in Section 3.

First, we apply Theorem 2.6 to study the factorization of $D_f(s)$ for a powerfree-type fake μ.

Proposition 2.8 *Let $f \in \mathcal{F}$ be of powerfree-type, and let k be the smallest positive integer such that $\varepsilon_k \neq 1$. Then, the critical index of f equals k. Moreover, for $\sigma > 1$, we can write*

$$D_f(s) = \frac{\zeta(s)}{\zeta(ks)^{1+|\varepsilon_k|}} \left(\prod_{j=k+1}^{2k-1} \zeta(js)^{\varepsilon_j - \varepsilon_{j-1}} \right) \zeta(2ks)^{\varepsilon_{2k} - \varepsilon_{2k-1} - |\varepsilon_k|} U_{2k}(s),$$

where

$$U_{2k}(s) = \prod_p \left(1 + \sum_{j=2k+1}^{\infty} \frac{\eta_j}{p^{js}} \right)$$

has the property that there exist $A, B > 0$ such that $|\eta_j| \le (Aj)^B$ for all j.

Proof We follow the notation used in Algorithm 1. We have $c_1 = 1$ and thus $n_j \ge 1$ for all j since $j = 1 + 1 + \cdots + 1$. Since $\varepsilon_2 = \cdots = \varepsilon_{k-1} = 1$ and $\varepsilon_k < 1 = n_k$, we have $\ell = k$ and $M = 1$. Also, note that $\varepsilon_k \in \{-1, 0\}$ implies that $-\frac{1}{2}((\varepsilon_k - 1)^2 + \varepsilon_k - 1) = \varepsilon_k = -|\varepsilon_k|$. The conclusion thus follows immediately from Theorem 2.6. ■

For powerfull-type fake μ's, there is no general way to simplify the factorization of $D_f(s)$ obtained in Theorem 2.6. For one thing, the initial index places no restriction at all on the critical index, as the following example shows.

Example 2.9 Given integers $N > k \ge 2$, suppose that

$$\varepsilon_j = \begin{cases} 1, & \text{if } j < N \text{ and } k \mid j, \\ 0, & \text{if } j < N \text{ and } k \nmid j, \\ -1, & \text{if } j = N. \end{cases}$$

Then (regardless of the values of ε_j for $j > N$) Algorithm 1 terminates with $M = 1$ and $c_1 = k$ and $\ell = N$, so that the initial index is k and the critical index is N.

In particular, for powerfull-type fake μ's, it is possible for any index exceeding the initial index to be the critical index. (Proposition 2.8 gives the same conclusion for powerfree-type fake μ's. For Möbius-type fake μ's, the initial and critical indices always coincide by Proposition 2.3.)

We can, however, give several examples of powerfull-type fake μ's for which the critical index can be deduced.

Example 2.10 Given integers $k > h \ge 1$, suppose that

$$\begin{cases} \varepsilon_j = 0, & \text{if } 1 \le j \le k-1, \\ \varepsilon_j = 1, & \text{if } j = k, \\ \varepsilon_j \in \{0,1\}, & \text{if } k+1 \le j \le k+h-1, \\ \varepsilon_j = -1, & \text{if } j = k+h, \end{cases}$$

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giving a powerfull-type fake μ with initial index k. Algorithm 1 reveals (regardless of the values of ε_j for $j > k+h$) that the principal indices correspond to those $k \le j \le k+h-1$ with $\varepsilon_j = 1$, and that the critical index equals $k+h$; more precisely

$$D_f(s) = \prod_{\substack{k \le j \le k+h-1 \\ \varepsilon_j = 1}} \zeta(js) \cdot \frac{1}{\zeta((k+h)s)} U_{k+h}(s)$$

where the Dirichlet series coefficients of $U_{k+h}(s)$ are supported on $(k+h+1)$-free numbers.

Note that in this case, Theorems 1.7 and 1.18 imply that $E_f(x) = \Omega_\pm(x^{1/2(k+h)})$ and (under RH) $E_f(x) \ll_\varepsilon x^{1/2k+\varepsilon}$. When h is small, these oscillation and upper bound results for $E_f(x)$ are close to each other. In particular, when $h \le \sqrt{2k}$, the oscillation result we obtain is better than the oscillation results mentioned in Example 1.4 for the indicator function of the k-full numbers themselves.

Example 2.11 Given integers $k' > k \ge 2$ such that $k \nmid k'$, suppose that

$$\varepsilon_j = \begin{cases} 1, & \text{if } j = ak + bk' \text{ for some nonnegative integers } a \text{ and } b, \\ 0, & \text{otherwise.} \end{cases}$$

Then, the initial index is k, and Algorithm 1 yields the principal indices $\{k, k'\}$. Note that the only way to have $n_j > \varepsilon_j$ in the second **if** statement is for $n_j \ge 2$, since by construction, the number of representations of j from $\{k, k'\}$ is $n_j = 0$ when $\varepsilon_j = 0$. Therefore, the critical index of (ε_j) equals $\ell = \operatorname{lcm}[k, k']$, which is the smallest integer that can be written as a nonnegative integer combination of $\{k, k'\}$ in two different ways. In fact, since every integer of the form $ak + bk'$ can be written uniquely in this form with $0 \le a < \frac{\ell}{k}$, in this case, we have the exact zeta-factorization

$$\begin{aligned} D_f(s) &= \prod_p \Bigg(1 + \sum_{\substack{j \ge 1 \\ j = ak+bk' \text{ for some } a,b \ge 0}} \frac{1}{p^{js}}\Bigg) \\ &= \prod_p \Bigg(\sum_{a=0}^{\ell/k-1} \sum_{b=0}^{\infty} \frac{1}{p^{(ak+bk')s}}\Bigg) = \prod_p \Bigg(\sum_{a=0}^{\ell/k-1} \frac{1}{p^{aks}}\Bigg) \prod_p \Bigg(\sum_{b=0}^{\infty} \frac{1}{p^{bk's}}\Bigg) = \frac{\zeta(ks)}{\zeta(\ell s)} \zeta(k's). \end{aligned}$$

One special case of this family is when $k = 2$ and $k' = 3$, so that $\varepsilon_1 = 0$ while $\varepsilon_j = 1$ for all $j \ge 2$, which recovers the factorization $\zeta(2s)\zeta(3s)/\zeta(6s)$ for the Dirichlet series corresponding to the indicator function of squarefull numbers.

Our last example is an in-depth examination of the Dirichlet series corresponding to the indicator function of k-full numbers.

Example 2.12 Let $k \ge 3$, and let f be the indicator function of k-full numbers, corresponding to the sequence with $\varepsilon_j = 0$ when $j \le k-1$ and $\varepsilon_j = 1$ when $j \ge k$. Then, $D_f(s)$ has a zeta-factorization of the form

$$D_f(s) = \sum_{n\, k\text{-full}} n^{-s} = \prod_{j=k}^{2k-1} \zeta(js) \cdot \prod_{j=2k+2}^{4k+4} \zeta(js)^{a_j} \cdot U_{4k+4}(s), \tag{2.12}$$

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where the Dirichlet series coefficients of U_{4k+4} are supported on $(4k+5)$-full numbers (a nearly equivalent statement appears as [2, Proposition 1(b)], for instance). We apply Theorem 2.6 to compute the exponents a_j explicitly.

Following the notation used in Algorithm 1, observe that the principal indices are $(c_1,\dots,c_k) = (k, k+1,\dots,2k-1)$. Note that $2k = k+k$ and $2k+1 = k+(k+1)$ have unique representations from $(c_1,\dots,c_k)$, while $2k+2 = k+(k+2) = 2(k+1)$ has two representations from $(c_1,\dots,c_k)$. Therefore, the critical index equals $\ell = 2k+2$, and $a_{2k} = \varepsilon_{2k} - n_{2k} = 1-1 = 0$ and similarly $a_{2k+1} = 0$, while $a_{2k+2} = \varepsilon_{2k+2} - n_{2k+2} = 1-2 = -1$. Note that the fraction in the second case of equation (2.8) equals 0, and so the formula for $a_{2\ell} = a_{4k+4}$ is the same sum as the formula for $a_{2k+3},\dots,a_{4k+3}$.

For example, when $k=3$, the zeta-factorization from Theorem 2.6 turns out to be

$$\sum_{n\,3\text{-full}} n^{-s} = \frac{\zeta(3s)\zeta(4s)\zeta(5s)}{\zeta(8s)} \frac{\zeta(13s)\zeta(14s)}{\zeta(9s)\zeta(10s)} U_{16}(s),$$

while when $k=4$, the zeta-factorization from Theorem 2.6 turns out to be

$$\sum_{n\,4\text{-full}} n^{-s} = \frac{\zeta(4s)\zeta(5s)\zeta(6s)\zeta(7s)}{\zeta(10s)} \frac{\zeta(16s)\zeta(17s)^2\zeta(18s)^2\zeta(19s)^2\zeta(20s)}{\zeta(11s)\zeta(12s)^2\zeta(13s)\zeta(14s)} U_{20}(s).$$

In both examples, the first fraction reflects the information in the previous paragraph, while the second fraction results from applying the formula in equation (2.8).

Going forward, we assume $k \ge 5$. Recall that $\varepsilon_j = 1$ when $j \ge k$ or $j = 0$, and $\varepsilon_j = 0$ otherwise; recall also that $c_i = k+i-1$ for $1 \le i \le k$. To compute the exponents a_j in equation (2.12) for $2k+3 \le j \le 4k+4$, we apply equation (2.8), which we can write in the form

$$a_j = \sum_{t=0}^{k} (-1)^t \big(S_t(j) + T_t(j)\big), \tag{2.13}$$

where

$$\begin{aligned} S_t(j) &= \#\Big\{ I \subset \{1,\dots,k\},\ \#I = t \colon j - \sum_{i\in I}(k+i-1) \ge k \Big\} \\ &= \#\Big\{ I \subset \{1,\dots,k\},\ \#I = t \colon \sum_{i\in I} i \le j - t(k-1) - k \Big\}, \\ T_t(j) &= \#\Big\{ I \subset \{1,\dots,k\},\ \#I = t \colon j - \sum_{i\in I}(k+i-1) = 0 \Big\} \\ &= \#\Big\{ I \subset \{1,\dots,k\},\ \#I = t \colon \sum_{i\in I} i = j - t(k-1) \Big\}. \end{aligned}$$

Note that we can restrict the sum in equation (2.13) to $0 \le t \le 3$, since $j - t(k-1) \le (4k+4) - 4(k-1) = 8$ when $t \ge 4$ but the sum of any four distinct elements of $\{1,\dots,k\}$ is at least 10.

Therefore, computing $a_{2k+3},\dots,a_{4k+4}$ reduces to finding, for $0 \le t \le 3$, the number of t-element subsets of $\{1,\dots,k\}$ whose sum equals or is bounded by particular numbers. This is an elementary but tedious problem, and we record only the results here, valid for all $k \ge 5$ and all $2k+3 \le j \le 4k+4$:

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- $S_0(j) = 1$ and $T_0(j) = 0$;
- $S_1(j) = \min\{k, j-2k+1\}$ and $T_1(j) = 0$;
- $S_2(j) = \begin{cases} 0, & \text{if } j \le 3k, \\ \lfloor \frac{1}{4}(j-3k+1)^2 \rfloor, & \text{if } 3k+1 \le j \le 4k-1, \\ \lfloor \frac{1}{4}(j-3k+1)^2 \rfloor - \binom{j-4k+2}{2}, & \text{if } j \ge 4k; \end{cases}$
- $T_2(j) = \begin{cases} \lfloor \frac{1}{2}\big(k - |j-3k+1|\big) \rfloor, & \text{if } j \le 4k-2, \\ 0, & \text{if } j \ge 4k-1; \end{cases}$
- $S_3(j) = \max\{0, j-4k-2\}$;
- $T_3(j) = \begin{cases} 0, & \text{if } j \le 3k, \\ [\frac{1}{12}(j-3k)^2], & \text{if } 3k+1 \le j \le 4k, \\ [\frac{1}{12}(j-3k)^2] - \lfloor \frac{1}{4}(j-4k+1)^2 \rfloor, & \text{if } j \ge 4k+1. \end{cases}$

For $T_3(j)$, we have used $[x]$ to denote rounding x to the nearest integer, in contrast to the greatest-integer function $\lfloor x \rfloor$ that also appears.

These formulas, together with equation (2.13), allow for the full computation of the zeta-factorization (2.12) for the indicator function of the k-full numbers, for all $k \ge 5$. For example, when $2k+3 \le j \le 3k-1$ the formula (2.13) simplifies to $a_j = k - \lfloor \frac{j}{2} \rfloor$.

Remark 2.13 The previous example has an interesting consequence for oscillations of the error term $E_f(x)$ of the counting function $F_f(x)$ for k-full numbers; let us restrict to $k \ge 16$ for ease of exposition. In Example 2.12, we saw that the critical index of f equals $\ell = 2k+2$, and thus Theorem 1.7 implies that $E_f(x) = \Omega_\pm(x^{1/(4k+4)})$. But in addition, Remark 1.13 described a tempting heuristic suggesting that these oscillations might be essentially best possible. We are now in a position to show that this heuristic does not hold here.

We computed in Example 2.12 that $a_{2k} = a_{2k+1} = 0$ and $a_{2k+2} = -1$, and that $a_j = k - \lfloor \frac{j}{2} \rfloor < 0$ for $2k+3 \le j \le 3k-1$. In particular, in view of equation (2.12), the Dirichlet series $D_f(s)$ is analytic at $\frac{1}{j}$ for all $2k \le j \le 3k-1$. Therefore, the expression (1.4) for the main term $G_f(x)$ of the counting function $F_f(x)$ can be written as

$$G_f(x) = \sum_{j=k}^{2k-1} \operatorname{Res}\left(D_f(s)\frac{x^s}{s}, \frac{1}{j}\right) + \sum_{j=3k}^{4k+4} \operatorname{Res}\left(D_f(s)\frac{x^s}{s}, \frac{1}{j}\right)$$
$$= \sum_{j=k}^{2k-1} \operatorname{Res}\left(D_f(s)\frac{x^s}{s}, \frac{1}{j}\right) + O(x^{1/(3k-1)}).$$

On the other hand, we saw in Example 1.4 that

$$F_f(x) = \sum_{j=k}^{2k-1} \operatorname{Res}\left(D_f(s)\frac{x^s}{s}, \frac{1}{j}\right) + \Omega\big(x^{1/(2k+\sqrt{8k}+3)}\big).$$

Since $2k + \sqrt{8k} + 3 < 3k-1$ when $k \ge 16$, we conclude that $E_f(x) = F_f(x) - G_f(x) = \Omega(x^{1/(2k+\sqrt{8k}+3)})$, which is rather larger than the $x^{1/(4k+4)}$ suggested by the heuristic.

In summary, for a general $f \in \mathcal{F}$, the heuristic prediction $E_f(x) \ll_\varepsilon x^{1/2\ell+\varepsilon}$ can be far away from the truth, and it is not immediately clear what the true order of magnitude of $E_f(x)$ should be in general.

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3 Oscillation results for $E_f(x)$

In this section, we prove a general oscillation result for the error term $E_f(x)$ of the summatory function $F_f(x)$ of a fake μ, based on the analytic properties of its Dirichlet series $D_f(s)$. More precisely, we state Theorem 3.5 in Section 3.1 and prove it in Section 3.3. From that theorem, we then deduce Theorems 1.10 and 1.14, which apply to Möbius-type and powerfree-type fake μ's, respectively, as well as giving a general result (Theorem 3.10) for powerfull-type fake μ's. Together these three results imply Theorem 1.7.

3.1 A motivating example and a general strategy

As an illustration, we begin with the family of fake μ's discussed in [19, Theorem 3], both to fill a gap in the original proof by Martin, Mossinghoff, and Trudgian, and to motivate our approach for the remainder of this section.

Example 3.1 Let $f \in \mathcal{F}$ with $\varepsilon_1 = -1$ and $\varepsilon_2 = 1$. This family of Möbius-type fake μ's was studied in [19]: it was shown there that $D_f(s) = U_2(s)\zeta(2s)/\zeta(s)$, where

$$U_2(s) = \prod_p \widetilde{C}_p(s) \quad \text{with} \quad \widetilde{C}_p(s) = 1 + \sum_{j\ge 3} \frac{\varepsilon_{j-1}+\varepsilon_j}{p^{js}}.$$

(Our Proposition 2.3 contains this expression as a special case, other than the exact expression for $\widetilde{C}_p(s)$ which, in this case, is easy to work out from the given zeta-factorization.) Their result [19, Theorem 3], translated into our notation, is that $E_f(x) = \Omega_\pm(\sqrt{x})$. We point out two small mistakes in their proof, partially to show how those gaps can be filled, and partially because describing the proof will help us motivate the approach we take in this section for general fake μ's.

From an integral representation for $D_f(s) - c\zeta(2s)$ for real constants c, they use a classical method of Landau to argue that if $E_f(x)$ is eventually of one sign then $D_f(s)$ must be analytic for $\sigma > \frac{1}{2}$. Normally, this would imply RH, as is stated in the first line of [19, p. 3242]; but in this case, the fact that $U_2(s)\zeta(2s)/\zeta(s)$ is analytic for $\sigma > \frac{1}{2}$ implies the slightly weaker statement that $\zeta(s)$ cannot have any zeros off the critical line *except possibly* at zeros of $U_2(s)$. While the authors of [19] did not directly use the assertion of RH in their proof, this detail helps us see why it is important to compare the zeros of the various terms in a partial zeta-factorization (as we will do explicitly in Proposition 3.4 below).

The proof of [19, Theorem 3] is then carried out using the assertion that $U_2(\rho_1) \ne 0$, where $\rho_1 = \frac{1}{2} + i\gamma_1 \approx \frac{1}{2} + i \cdot 14.135$ is the lowest zero of $\zeta(s)$ in the upper half-plane. The authors claim that $U_2(\rho_1) \ne 0$ follows from the fact that $U_2(s)$ is absolutely convergent for $\sigma > \frac{1}{3}$, so that $1/U_2(s)$ is analytic there. However, this argument actually shows that $U_2(\rho_1) \ne 0$ *unless* one of the individual Euler factors $\widetilde{C}_p(\rho_1)$ vanishes. (Indeed, the vanishing of an individual Euler factor is a significant contributor to [19, Theorem 1(iii)].) It is impossible for any of these factors with $p \ge 3$ to vanish, since by the triangle inequality

$$\left|\widetilde{C}_p(\rho_1)\right| \ge 1 - \sum_{j\ge 3} \frac{2}{p^{j/2}} = 1 - \frac{2}{p^{3/2}(1-p^{-1/2})} \ge 1 - \frac{2}{3\sqrt{3}-3} > 0$$

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(this argument appears at the top of [19, p. 3238]). We now show that $\widetilde{C}_2(\rho_1) \neq 0$ to fill this small gap in their proof.

If we look at that Euler factor

$$\widetilde{C}_2(\rho_1) = 1 + \sum_{j \geq 3} \frac{\varepsilon_{j-1} + \varepsilon_j}{2^{j\rho_1}} = 1 + \frac{1 + \varepsilon_3}{2^{3\rho_1}} + \sum_{j=4}^{6} \frac{\varepsilon_{j-1} + \varepsilon_j}{2^{j\rho_1}} + \sum_{j=7}^{\infty} \frac{\varepsilon_{j-1} + \varepsilon_j}{2^{j\rho_1}},$$

then another triangle-inequality argument gives

$$\begin{aligned} |\widetilde{C}_2(\rho_1)| &\geq \left| 1 + \frac{1 + \varepsilon_3}{2^{3\rho_1}} + \sum_{j=4}^{6} \frac{\varepsilon_{j-1} + \varepsilon_j}{2^{j\rho_1}} \right| - \sum_{j=7}^{\infty} \frac{2}{2^{j/2}} \\ &= \left| 1 + \frac{1 + \varepsilon_3}{2^{3\rho_1}} + \sum_{j=4}^{6} \frac{\varepsilon_{j-1} + \varepsilon_j}{2^{j\rho_1}} \right| - \frac{1}{4\sqrt{2} - 4}. \end{aligned}$$

We can split into 81 cases based on the values $\varepsilon_3, \ldots, \varepsilon_6 \in \{-1, 0, 1\}$; in each case, we directly compute the expression on the right-hand side, and in all 81 cases, the quantity happens to exceed 0.055. These inequalities prove that the Euler factor $\widetilde{C}_2(s)$ cannot vanish at $s = \rho_1$, which finishes the justification that $U_2(\rho_1) \neq 0$.

In Example 3.1, the critical index was $\ell = 1$, meaning that a power of $\zeta(s)$ was present in the denominator of our zeta-factorization; the fact that we were considering a zero of $\zeta(s)$ on the critical line is the reason why the geometric series we studied had common ratios whose size was $p^{-1/2}$. For general critical indices ℓ, the corresponding series evaluated at a zero of $\zeta(\ell s)$ would have common ratios of size $p^{-1/2\ell}$ and would thus converge more slowly. In principle, we could try to extend the above triangle-inequality arguments, although at the very least we would have to consider more primes in addition to $p = 2$ and more terms in each series; but such an extension would work only for one fixed ℓ at a time in any case.

Instead, we exploit the fact that the specific zero ρ_1 is not important to us – we can use any convenient zero of $\zeta(s)$, indeed without even needing to know its exact identity. We therefore adopt a suitably specified version of this strategy in Proposition 3.4 below. To facilitate precise statements of our results, we introduce some notation that will be in force throughout Section 3. We precede that notation with a lemma that will be familiar to analytic number theorists.

Lemma 3.2 *Fix $\sigma_0 > 0$, and let $h(s)$ be a meromorphic function with a pole of order ξ at σ_0. Then for any real number $x > 1$, the residue of $h(s)x^s/s$ at $s = \sigma_0$ equals $P(\log x)x^{\sigma_0}$ for some polynomial $P(t)$ (depending on $h(s)$) whose degree is $\xi - 1$.*

Note that the second sum in Theorem 1.14 contains expressions of exactly this type coming from poles of order $\xi = 2$.

Proof Since $h(s)/s$ is also meromorphic with a pole of order ξ at σ_0, we can write

$$\frac{h(s)}{s} = \sum_{k=-\xi}^{\infty} c_k (s - \sigma_0)^k \quad \text{and} \quad x^s = x^{\sigma_0} e^{(s-\sigma_0)\log x} = x^{\sigma_0} \sum_{k=0}^{\infty} \frac{(s - \sigma_0)^k}{k!} (\log x)^k$$

for some constants c_k with $c_{-\xi} \neq 0$. When we multiply these series together, the coefficient of $(s - \sigma_0)^{-1}$ depends on the first ξ terms in each series; more precisely,

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the residue we seek equals

$$x^{\sigma_0} \sum_{j=0}^{\xi-1} \frac{(\log x)^j}{j!} c_{-1-j} = P(\log x) x^{\sigma_0} \quad \text{where} \quad P(t) = \sum_{j=0}^{\xi-1} \frac{c_{-1-j}}{j!} t^j.$$

∎

Notation 3.3 *Let $f \in \mathcal{F}$ be defined via the sequence (ε_j). Let ℓ be the critical index of f. By Proposition 2.3 and Theorem 2.6, there are integers $a_1, a_2, \dots, a_{2\ell}$ and a function $U_{2\ell}(s)$ such that the following conditions hold:*

- *We have $a_j \ge 0$ for $1 \le j \le \ell - 1$, while $a_\ell < 0$;*
- *For $\sigma > 1$, we can write*

$$D_f(s) = U_{2\ell}(s) \cdot \prod_{j=1}^{2\ell} \zeta(js)^{a_j} \quad \textit{where} \quad U_{2\ell}(s) = \prod_p \left(1 + \sum_{j=2\ell+1}^{\infty} \frac{\eta_j}{p^{js}}\right); \tag{3.1}$$

- *There are positive constants A, B, depending only on ℓ, such that $|\eta_j| \le (Aj)^B$ for all j; in particular, $U_{2\ell}(s)$ is analytic for $\sigma > \frac{1}{2\ell+1}$ by Lemma 2.2, and is nonzero provided each of its factors is nonzero.*
- *Given equation (3.1), in the region $\sigma > \frac{1}{2\ell+1}$, the only possible real poles of $D_f(s)$ are at $s = 1, \frac{1}{2}, \dots, \frac{1}{2\ell}$. Let ξ_j be the order of the pole of $D_f(s)$ at $s = \frac{1}{j}$, so that $\xi_j = 0$ if $a_j \le 0$ and $0 \le \xi_j \le a_j$ if $a_j \ge 1$.*
- *By Lemma 3.2, equation (1.4) becomes*

$$G_f(x) = \sum_{j=1}^{2\ell} \operatorname{Res}\left(D_f(s)\frac{x^s}{s}, \frac{1}{j}\right) = \sum_{\substack{1 \le j \le 2\ell \\ \xi_j \ge 1}} P_j(\log x) x^{1/j},$$

  where each $P_j(t)$ is a polynomial of degree $\xi_j - 1$.
- *As always, we have $F_f(x) = \sum_{n \le x} f(n)$ and $E_f(x) = F_f(x) - G_f(x)$.*

We are now in a position to state an important proposition, which asserts that a zero of $\zeta(s)$ exists at which other zeta-factorization factors do not vanish, as motivated by the above discussion. Establishing this proposition is the goal of Section 3.2.

Proposition 3.4 *In the situation described by Notation 3.3, there exists a zero ρ of $\zeta(s)$ with $\Re(\rho) \ge \frac{1}{2}$ such that*

$$U_{2\ell}\left(\frac{\rho}{\ell}\right) \cdot \prod_{\substack{1 \le j \le 2\ell \\ j \ne \ell}} \zeta\left(\frac{j\rho}{\ell}\right) \ne 0. \tag{3.2}$$

That proposition will allow us to prove the following general oscillation theorem in Section 3.3.

Theorem 3.5 *In the situation described by Notation 3.3,*

$$E_f(x) = \Omega_\pm\left(x^{1/2\ell} (\log x)^{|a_\ell| - 1}\right).$$

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3.2 Finding a zero where other factors do not vanish

This section is devoted to the proof of Proposition 3.4. We begin by recording some consequences of classical zero-counting functions for $\zeta(s)$ and the Landau–Gonek formula.

Lemma 3.6 *Given $\ell \in \mathbb{N}$ and $T \geq 3$, define*

$$Z_\ell(T) = \left\{\rho = \beta + i\gamma \colon \zeta(\rho) = 0,\ 0 < \gamma \leq T,\ \frac{\ell}{2\ell+1} < \beta < \frac{\ell+1}{2\ell+1}, \right.$$
$$\left. \prod_{\substack{1 \leq j \leq 2\ell \\ j \neq \ell}} \zeta\left(\frac{j\rho}{\ell}\right)\zeta\left(\frac{j(1-\rho)}{\ell}\right) \neq 0\right\}.$$

Then, $\#Z_\ell(T) = \frac{T}{2\pi}\log T + O_\ell(T)$.

Proof In the usual notation

$$N(T) = \#\{\rho = \beta + i\gamma \colon \zeta(\rho) = 0,\ 0 < \gamma \leq T\}$$
$$N(\sigma, T) = \#\{\rho = \beta + i\gamma \colon \zeta(\rho) = 0,\ 0 < \gamma \leq T,\ \beta \geq \sigma\},$$

we know that $N(T) = \frac{T}{2\pi}\log T + O(T)$ (see, for example, [25, Corollary 14.2]), while well-known zero-density estimates (first proved by Bohr and Landau [6]) imply that $N(\sigma, T) \ll_\sigma T$ for $\sigma > \frac{1}{2}$. We use this latter estimate to bound how many of the $N(T)$ zeros up to height T are not included in $Z_\ell(T)$. For the rest of this proof, all implicit constants may depend on ℓ.

The upper bound $\beta < \frac{\ell+1}{2\ell+1}$ excludes $N(\frac{\ell+1}{2\ell+1}, T) \ll T$ zeros; also, by the symmetries of $\zeta(s)$, the lower bound $\beta > \frac{\ell}{2\ell+1}$ excludes $N(1 - \frac{\ell}{2\ell+1}, T) \ll T$ zeros. Suppose that ρ is a zero that has not been excluded so far. If $\ell + 1 \leq j \leq 2\ell$, then $\Re(\frac{j}{\ell}\rho) \geq \frac{\ell+1}{\ell}\frac{\ell}{2\ell+1} = \frac{\ell+1}{2\ell+1} > \frac{1}{2}$, and so the number of these $\frac{j}{\ell}\rho$ that can be zeros of $\zeta(s)$ is at most $N(\frac{\ell+1}{2\ell+1}, T) \ll T$. Similarly, if $1 \leq j \leq \ell - 1$, then $\Re(\frac{j}{\ell}\rho) \leq \frac{\ell-1}{\ell}\frac{\ell+1}{2\ell+1} < \frac{1}{2}$, and so the number of these $\frac{j}{\ell}\rho$ that can be zeros of $\zeta(s)$ is at most $N(1 - \frac{\ell-1}{\ell}\frac{\ell+1}{2\ell+1}, T) \ll T$. Similar observations hold for the $\zeta(\frac{j(1-\rho)}{\ell})$ factors. We conclude that $\#Z_\ell(T) = N(T) + O_\ell(T) = \frac{T}{2\pi}\log T + O_\ell(T)$ as desired. ∎

Lemma 3.7 *If $x > 0$ with $x \neq 1$, then*

$$\sum_{0<\gamma\leq T} x^\rho \ll_x T.$$

Proof Landau's formula [17] tells us that if $x > 1$,

$$\sum_{0<\gamma\leq T} x^\rho = -\frac{T}{2\pi}\Lambda(x) + O(\log T),$$

where $\Lambda(x)$ is the von Mangoldt function when x is an integer and $\Lambda(x) = 0$ otherwise. When $0 < x < 1$, one can derive from Landau's formula (see, for example,

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[10, "Corollary"]) that

$$\sum_{0<\gamma\le T} x^\rho = -\frac{Tx}{2\pi}\Lambda\left(\frac{1}{x}\right) + O(\log T).$$

In both cases, the right-hand side is $\ll_x T$ as required. ∎

It will be helpful to give names to the Euler factors appearing in Notation 3.3.

Notation 3.8 *For each prime p, define*

$$C_p(s) = 1 + \sum_{j=1}^{\infty} \frac{\varepsilon_j}{p^{js}} \quad \textit{and} \quad \widetilde{C}_p(s) = 1 + \sum_{j=2\ell+1}^{\infty} \frac{\eta_j}{p^{js}},$$

so that $D_f(s) = \prod_p C_p(s)$ for $\sigma > 1$ and $U_{2\ell}(s) = \prod_p \widetilde{C}_p(s)$ for $\sigma > \frac{1}{2\ell+1}$. Note that the bounds $|\varepsilon_j| \le 1$ and $|\eta_j| \le (Aj)^B$ imply that the series defining $C_p(s)$ and $\widetilde{C}_p(s)$ both converge for $\sigma > 0$.

Lemma 3.9 *In the situation described by Notation 3.3, there exists a positive integer P, depending only on ℓ, such that for each zero ρ of ζ with $\Re(\rho) > \frac{\ell}{2\ell+1}$,*

$$U_{2\ell}\left(\frac{\rho}{\ell}\right) \ne 0 \quad \textit{if and only if} \quad \prod_{p\le P} C_p\left(\frac{\rho}{\ell}\right) \ne 0.$$

Proof We certainly have

$$\left|1 - \widetilde{C}_p\left(\frac{\rho}{\ell}\right)\right| \le \sum_{j=2\ell+1}^{\infty} \frac{(Aj)^B}{p^{j\Re\rho/\ell}} < \sum_{j=2\ell+1}^{\infty} \frac{(Aj)^B}{p^{j/(2\ell+1)}}.$$

The right-hand side is a positive series that is decreasing in p and tends to 0 termwise; by the monotone convergence theorem, the series itself tends to 0 as $p \to \infty$. In particular, there exists a positive integer P such that the series is less than 1 when $p \ge P$, and for these primes, we deduce that $\widetilde{C}_p(\rho/\ell) \ne 0$. Moreover, since A and B depend only on ℓ, the same is true of P.

As we observed in Notation 3.3, $U_{2\ell}(\rho/\ell) = 0$ only if $\widetilde{C}_p(\rho/\ell) = 0$ for some prime p, and such a prime must necessarily be less than P. On the other hand, observe that equation (3.1) implies that $\widetilde{C}_p(s) = C_p(s)\prod_{j=1}^{2\ell}(1-p^{-js})^{-a_j}$ for $\Re s > 1$, and so this identity remains true in the larger-half plane $\Re s > 0$ where all terms converge. In particular,

$$\widetilde{C}_p\left(\frac{\rho}{\ell}\right) = C_p\left(\frac{\rho}{\ell}\right)\prod_{j=1}^{2\ell}(1-p^{-j\rho/\ell})^{-a_j},$$

and thus $\widetilde{C}_p(\rho/\ell) = 0$ if and only if $C_p(\rho/\ell) = 0$, which completes the proof of the lemma. ∎

We now have all the ingredients to establish Proposition 3.4.

Proof of Proposition 3.4 For notational convenience, set $\varepsilon_0 = 1$. Note that uniformly for all primes p, all positive integers n, and all zeros ρ of $\zeta(s)$ with $\Re(\rho) \geq \frac{\ell}{2\ell+1}$,

$$\begin{aligned} C_p\left(\frac{\rho}{\ell}\right) &= \sum_{j=0}^{n} \frac{\varepsilon_j}{p^{j\rho/\ell}} + O\left(\sum_{j=n+1}^{\infty} \frac{1}{p^{j\Re\rho/\ell}}\right) \\ &= \sum_{j=0}^{n} \frac{\varepsilon_j}{p^{j\rho/\ell}} + O\left(\sum_{j=n+1}^{\infty} \frac{1}{p^{j/(2\ell+1)}}\right) \\ &= \sum_{j=0}^{n} \frac{\varepsilon_j}{p^{j\rho/\ell}} + O\left(\frac{1}{p^{(n+1)/(2\ell+1)}}\right). \end{aligned}$$

Let P be the positive integer (depending only on ℓ) from Lemma 3.9, and set

$$Q_n = \left\{ \prod_{p \leq P} p^{\alpha(p)} : 0 \leq \alpha(p) \leq n \right\}.$$

Note that $|Q_n| \leq (n+1)^P$. By expanding the product, we see that uniformly for all $n \in \mathbb{N}$ and for all zeros ρ of $\zeta(s)$ with $\frac{\ell}{2\ell+1} < \Re(\rho) < \frac{\ell+1}{2\ell+1}$,

$$\begin{aligned} &\prod_{p \leq P} C_p\left(\frac{\rho}{\ell}\right) C_p\left(\frac{1-\rho}{\ell}\right) \\ &\quad = \prod_{p \leq P} \left(\sum_{j=0}^{n} \frac{\varepsilon_j}{p^{j\rho/\ell}} + O\left(\frac{1}{p^{(n+1)/(2\ell+1)}}\right)\right)\left(\sum_{j=0}^{n} \frac{\varepsilon_j}{p^{j(1-\rho)/\ell}} + O\left(\frac{1}{p^{(n+1)/(2\ell+1)}}\right)\right) \\ &\quad = \left(\sum_{q \in Q_n} \frac{f(q)}{q^{\rho/\ell}} \right)\left(\sum_{q \in Q_n} \frac{f(q)}{q^{(1-\rho)/\ell}} \right) + O\left(\frac{(n+1)^{2P} - 1}{2^{(n+1)/(2\ell+1)}} \right). \end{aligned}$$

In particular, we can choose n sufficiently large in terms of ℓ so that

$$\left| \prod_{p \leq P} C_p\left(\frac{\rho}{\ell}\right) C_p\left(\frac{1-\rho}{\ell}\right) \right| \geq \left| \left(\sum_{q \in Q_n} \frac{f(q)}{q^{\rho/\ell}} \right)\left(\sum_{q \in Q_n} \frac{f(q)}{q^{(1-\rho)/\ell}} \right) \right| - \frac{1}{2}. \tag{3.3}$$

For each nontrivial zero ρ of ζ, we have

$$\begin{aligned} \left(\sum_{q \in Q_n} \frac{f(q)}{q^{\rho/\ell}} \right)\left(\sum_{q \in Q_n} \frac{f(q)}{q^{(1-\rho)/\ell}} \right) &= \sum_{q_1, q_2 \in Q_n} \frac{f(q_1) f(q_2)}{q_2^{1/\ell}} \left(\frac{q_2}{q_1} \right)^{\rho/\ell} \\ &= \sum_{q \in Q_n} \frac{f(q)^2}{q^{1/\ell}} + \sum_{\substack{q_1, q_2 \in Q_n \\ q_1 \neq q_2}} \frac{f(q_1) f(q_2)}{q_2^{1/\ell}} \left(\frac{q_2}{q_1} \right)^{\rho/\ell}. \end{aligned}$$

By Lemma 3.7, for each $q_1, q_2 \in Q_n$ with $q_1 \neq q_2$,

$$\sum_{0 < \gamma \leq T} \left(\frac{q_2}{q_1} \right)^{\rho/\ell} \ll T;$$

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while the implicit constant depends on q_1 and q_2, both numbers come from the finite set Q_n which depends only on ℓ. We conclude that

$$\sum_{0<\gamma\le T}\Bigg(\sum_{q\in Q_n}\frac{f(q)}{q^{\rho/\ell}}\Bigg)\Bigg(\sum_{q\in Q_n}\frac{f(q)}{q^{(1-\rho)/\ell}}\Bigg)=\Bigg(\sum_{q\in Q_n}\frac{f(q)^2}{q^{1/\ell}}\Bigg)N(T)+O_\ell(T)$$
$$\ge 1\cdot N(T)+O_\ell(T)=\frac{T}{2\pi}\log T+O_\ell(T),$$

since $f(1)=1$. Also, note that for each nontrivial zero ρ of $\zeta(s)$, the trivial bound $|f(q)|\le 1$ implies that

$$\Bigg|\Bigg(\sum_{q\in Q_n}\frac{f(q)}{q^{\rho/\ell}}\Bigg)\Bigg(\sum_{q\in Q_n}\frac{f(q)}{q^{(1-\rho)/\ell}}\Bigg)\Bigg|\le |Q_n|^2\le (n+1)^{2P}.$$

Since the set $Z_\ell(T)$ defined in Lemma 3.6 excludes $\ll_\ell T$ zeros up to height T, it follows that

$$\sum_{\rho\in Z_\ell(T)}\Bigg(\sum_{q\in Q_n}\frac{f(q)}{q^{\rho/\ell}}\Bigg)\Bigg(\sum_{q\in Q_n}\frac{f(q)}{q^{(1-\rho)/\ell}}\Bigg)$$
$$=\sum_{0<\gamma\le T}\Bigg(\sum_{q\in Q_n}\frac{f(q)}{q^{\rho/\ell}}\Bigg)\Bigg(\sum_{q\in Q_n}\frac{f(q)}{q^{(1-\rho)/\ell}}\Bigg)+O_\ell(T(n+1)^{2P})$$
$$\ge\frac{T}{2\pi}\log T+O_\ell(T).$$

Finally, equation (3.3) implies that

$$\sum_{\rho\in Z_\ell(T)}\Bigg(\prod_{p\le P}C_p\Big(\frac{\rho}{\ell}\Big)C_p\Big(\frac{1-\rho}{\ell}\Big)\Bigg)\ge\frac{T}{2\pi}\log T+O_\ell(T)-\frac{1}{2}\#Z_\ell(T)$$
$$\ge\frac{T}{4\pi}\log T+O_\ell(T).$$

In particular, when T is sufficiently large in terms of ℓ, there exists $\rho\in Z_\ell(T)$ such that

$$\prod_{p\le P}C_p\Big(\frac{\rho}{\ell}\Big)C_p\Big(\frac{1-\rho}{\ell}\Big)\ne 0,$$

which implies that

$$U_{2\ell}\Big(\frac{\rho}{\ell}\Big)U_{2\ell}\Big(\frac{1-\rho}{\ell}\Big)\ne 0$$

by Lemma 3.9.

Since $\rho\in Z_\ell(T)$ implies $1-\rho\in Z_\ell(T)$, we may assume that $\Re\rho\ge\frac{1}{2}$. Then $\zeta(\rho)=0$ and

$$\prod_{\substack{1\le j\le 2\ell\\ j\ne\ell}}\zeta\Big(\frac{j\rho}{\ell}\Big)\ne 0.$$

by the definition of $Z_\ell(T)$, and we have just shown that $U_{2\ell}(\frac{\rho}{\ell})\ne 0$, which confirms equation (3.2) and hence establishes the proposition. ∎

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3.3 The general oscillation result

We are now able to establish our most general oscillation result: in the situation described by Notation 3.3 (which is used throughout the proof), we need to show that $E_f(x) = \Omega_\pm\big(x^{1/2\ell}(\log x)^{|a_\ell|-1}\big)$.

Proof of Theorem 3.5 We show that $E_f(x) = \Omega_-\big(x^{1/2\ell}(\log x)^{|a_\ell|-1}\big)$, as the proof of the corresponding Ω_+ result is almost identical. For each $1 \le j \le 2\ell$ with $\xi_j \ge 1$, let the coefficients $b_{j,k}$ for $0 \le k \le \xi_j - 1$ be defined by $P_j(y) = \sum_{k=0}^{\xi_j-1} b_{j,k} y^k$. Define

$$\widetilde{D}(s) = D_f(s) - \sum_{\substack{1\le j\le 2\ell \\ \xi_j\ge 1}} \sum_{k=0}^{\xi_j-1} \frac{b_{j,k} s \cdot k!}{(s-\frac{1}{j})^{k+1}}.$$

Note that for each $k \ge 0$,

$$\frac{k!}{(s-\frac{1}{j})^{k+1}} = \int_1^\infty \frac{x^{1/j}(\log x)^k}{x^{s+1}}\,dx.$$

It follows that

$$\widetilde{D}(s) = s\int_1^\infty \frac{E_f(x)}{x^{s+1}},$$

and thus $\widetilde{D}(s)$ has no real pole with $s \ge \frac{1}{2\ell}$.

Let $r > 0$ be a constant to be chosen later, and define

$$H(s) = \widetilde{D}(s) + \frac{rs(|a_\ell|-1)!}{(s-\frac{1}{2\ell})^{-a_\ell}}.$$

Then for $\sigma > 1$,

$$H(s) = s\int_1^\infty \frac{E_f(x) + rx^{1/2\ell}(\log x)^{|a_\ell|-1}}{x^{s+1}}\,dx. \tag{3.4}$$

Suppose that $E_f(x) + rx^{1/2\ell}(\log x)^{|a_\ell|-1}$ is positive when x is sufficiently large. Note that $\widetilde{D}(s)$ has no real singularity with $s \ge \frac{1}{2\ell}$, and the smallest real singularity of $(s - \frac{1}{2\ell})^{a_\ell}$ is at $s = \frac{1}{2\ell}$. Thus, Landau's theorem (see, for example, [25, Lemma 15.1]) implies that $H(s)$ is analytic for $\sigma > \frac{1}{2\ell}$ and equation (3.4) holds for $\sigma > \frac{1}{2\ell}$. On the other hand, by Proposition 3.4, there is a nontrivial zero ρ of ζ such that $\Re(\rho) \ge \frac{1}{2}$ and

$$U_{2\ell}\left(\frac{\rho}{\ell}\right) \cdot \prod_{\substack{1\le j\le 2\ell \\ j\ne \ell}} \zeta\left(\frac{j\rho}{\ell}\right) \ne 0. \tag{3.5}$$

In particular, since $\zeta(\ell s)$ has a zero at ρ/ℓ, $D_f(s)$ does indeed have a pole at ρ/ℓ by equation (3.1), and thus so does $H(s)$. Since $H(s)$ is analytic for $\sigma > \frac{1}{2\ell}$, we deduce

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that $\Re(\rho) = \frac{1}{2}$. Set $\rho = \frac{1}{2} + i\gamma$ and $\gamma' = \gamma/\ell$. Equation (3.4) implies that for $\sigma > \frac{1}{2\ell}$,

$$|H(\sigma + i\gamma')| \le |\sigma + i\gamma'| \int_1^\infty \frac{|E_f(x) + rx^{1/2\ell}(\log x)^{|a_\ell|-1}|}{x^{\sigma+1}}\,dx$$
$$= \frac{|\sigma + i\gamma'|}{\sigma} H(\sigma) < 2\ell|\rho| H(\sigma).$$

Let m be the order of ρ as a zero of $\zeta(s)$, and set $m' = |a_\ell| m$. The above inequality implies that

$$\lim_{\sigma \to \frac{1}{2\ell}^+} \left(\sigma - \frac{1}{2\ell}\right)^{m'} |H(\sigma + i\gamma')| \le 2\ell|\rho| \lim_{\sigma \to \frac{1}{2\ell}^+} \left(\sigma - \frac{1}{2\ell}\right)^{m'} H(\sigma). \tag{3.6}$$

Since $\widetilde{D}(s)$ is analytic at $\sigma = \frac{1}{2\ell}$, the right-hand side of inequality (3.6) is equal to

$$2\ell|\rho| \lim_{\sigma \to \frac{1}{2\ell}^+} \left(\sigma - \frac{1}{2\ell}\right)^{m'} \left| \frac{r\sigma(|a_\ell| - 1)!}{(\sigma - \frac{1}{2\ell})^{-a_\ell}} \right| = \begin{cases} r|\rho|(|a_\ell| - 1)!, & \text{if } m = 1, \\ 0, & \text{if } m > 1. \end{cases}$$

Since ρ is a zero of $\zeta(s)$ of order m, the left-hand side of inequality (3.6) is equal to

$$\lim_{\sigma \to \frac{1}{2\ell}^+} \left(\sigma - \frac{1}{2\ell}\right)^{m'} \left| D_f(\sigma + i\gamma') - \sum_{\substack{1 \le j \le 2\ell \\ \xi_j \ge 1}} \sum_{k=0}^{\xi_j - 1} \frac{b_{j,k} s \cdot k!}{(\sigma + i\gamma' - \frac{1}{j})^{k+1}} + \frac{r(\sigma + i\gamma')(|a_\ell| - 1)!}{(\sigma + i\gamma' - \frac{1}{2\ell})^{-a_\ell}} \right|$$
$$= \lim_{\sigma \to \frac{1}{2\ell}^+} \left(\sigma - \frac{1}{2\ell}\right)^{m'} |D_f(\sigma + i\gamma')| = \left(\frac{m!}{\ell^m |\zeta^{(m)}(\rho)|}\right)^{-a_\ell} \left| U_{2\ell}\left(\frac{\rho}{\ell}\right) \right| \prod_{\substack{1 \le j \le 2\ell \\ j \ne \ell}} \left| \zeta\left(\frac{j\rho}{\ell}\right) \right|^{a_j}.$$

Now inequality (3.5) implies that $m = 1$ and thus

$$\frac{|U_{2\ell}(\frac{\rho}{\ell})|}{(\ell|\zeta'(\rho)|)^{-a_\ell}} \cdot \prod_{\substack{1 \le j \le 2\ell \\ j \ne \ell}} \left| \zeta\left(\frac{j\rho}{\ell}\right) \right|^{a_j} \le r|\rho|(|a_\ell| - 1)!. \tag{3.7}$$

Note that inequality (3.5) implies that the left-right side of inequality (3.7) is nonzero, and thus inequality (3.7) can only hold when r is sufficiently large. In other words, for smaller positive values of r, the difference $E_f(x) + rx^{1/2\ell}(\log x)^{|a_\ell|-1}$ cannot be always positive for sufficiently large x, which shows that $E_f(x) = \Omega_-\big(x^{1/2\ell}(\log x)^{|a_\ell|-1}\big)$ as required. ∎

3.4 Applications

It is now a simple matter to use Theorem 3.5 to derive Theorems 1.10 and 1.14, which apply to Möbius-type and powerfree-type fake μ's, respectively. We also use Theorem 3.5 to give a general result (Theorem 3.10 below) that is our strongest oscillation result for powerfull-type fake μ's. Together these three results imply Theorem 1.7.

Proof of Theorem 1.10 For each $1 \le j \le 2k$, Proposition 2.3 tells us that $D_f(s)$ has at most a simple pole at $s = \frac{1}{j}$; therefore, the residue of $D_f(s) \cdot \frac{x^s}{s}$ at $s = \frac{1}{j}$ is $\mathfrak{a}_f(j)x^{1/j}$,

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where $\mathfrak{a}_f(j)$ is the constant from Definition 1.9. The theorem then follows from Theorem 3.5 by applying the partial zeta-factorization of $D_f(s)$ in Proposition 2.3. ■

Combining Theorem 2.6 and Theorem 3.5 results immediately in the following theorem.

Theorem 3.10 *Let $f \in \mathcal{F}$ be of powerfree-type or powerfull-type. In the notation in Theorem 2.6,*

$$G_f(x) = \sum_{j=1}^{M} \operatorname{Res}\left(D_f(s)\cdot\frac{x^s}{s}, \frac{1}{c_j}\right) + \sum_{j=\ell+1}^{2\ell} \operatorname{Res}\left(D_f(s)\cdot\frac{x^s}{s}, \frac{1}{j}\right)$$
$$E_f(x) = \Omega_\pm\left(x^{1/2\ell}(\log x)^{n_\ell-\varepsilon_\ell-1}\right).$$

Proof of Theorem 1.14 Let $1 \le j \le 2k$. By Proposition 2.8, $D_f(s)$ has a pole at $s = \frac{1}{j}$ with order at most 2. By Definition 1.9, the principal part of $D_f(s)$ is

$$\frac{\mathfrak{b}_f(j)}{j^2(s-1/j)^2} + \frac{\mathfrak{a}_f(j)}{j(s-1/j)}.$$

Thus, the residue of $D_f(s)\cdot\frac{x^s}{s}$ at $s = \frac{1}{j}$ is given by

$$\lim_{s\to\frac{1}{j}} \frac{d}{ds} D_f(s)\cdot\frac{x^s}{s} = \lim_{s\to\frac{1}{j}} \frac{d}{ds}\left(\frac{\mathfrak{b}_f(j)}{j^2(s-1/j)^2} + \frac{\mathfrak{a}_f(j)}{j(s-1/j)}\right)\cdot\frac{x^s}{s}$$
$$= \mathfrak{a}_f(j)x^{1/j} + \mathfrak{b}_f(j)x^{1/j}\left(\frac{\log x}{j} - 1\right).$$

The theorem now follows from Proposition 2.8 and Theorem 3.10. ■

4 Upper bounds on $E_f(x)$

In this section, we prove our upper bounds on $E_f(x)$, both unconditional (Theorem 1.16, which is proved in Section 4.2) and assuming RH (Theorem 1.18, which is proved for powerfree-type f in Section 4.2 and for Möbius- and powerfull-type f in Section 4.3). Our main motivation for establishing these upper bounds is to provide some sort of calibration against which to gauge the strength of our oscillation results. As it happens, this exercise also allows us to gather techniques from the literature and generalize their scope to all fake μ's; in doing so, we have often recovered, and sometimes even improved, the best known upper bounds for error terms in special cases. We will combine two different methods to prove upper bounds on $E_f(x)$, namely, the convolution method (or Dirichlet hyperbola method) and the method of contour integration.

4.1 Upper bounds on $E_f(x)$ for a special family of fake μ's

In this section, we prove Theorems 1.16 and 1.18 for a special family of fake μ's of powerfree-type.

Definition 4.1 For each $k \ge 1$, let $h_k(n)$ be the multiplicative function appearing in the Dirichlet series $\zeta(ks)^{-2} = \sum_{n=1}^{\infty} h_k(n)n^{-s}$. We can check that $h_k(p^k) = -2$ (so that

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h_k is not quite a fake μ) and $h_k(p^{2k}) = 1$ and that $h_k(p^j) = 0$ for all other $j \ge 1$. Define $H_k(x) = \sum_{n\le x} h_k(n)$.

Moreover, for each $k \ge 1$, let g_k be the fake μ defined via the sequence $(\varepsilon_j)_{j=1}^{\infty}$ with $\varepsilon_j = 1$ for $1 \le j \le k-1$ and $\varepsilon_j = -1$ for $k \le j \le 2k-1$ and $\varepsilon_j = 0$ for $j \ge 2k$. We can check that g_k is the Dirichlet convolution of h_k and the constant function 1, which is the same as saying that the sequence $(1, \varepsilon_1, \varepsilon_2, \dots)$ is the convolution of $(1,1,1,\dots)$ and $(1, h_k(p^1), h_k(p^2), \dots)$; in particular, $D_{g_k}(s) = \sum_{n=1}^{\infty} g_k(n)n^{-s} = \zeta(s)\zeta(ks)^{-2}$. Note that $g_1 = \mu$.

The proof of our more general results for all fake μ's of powerfree-type will build on the upper bounds on $E_{g_k}(x)$ in Propositions 4.5 and 4.7 below. As preparation, we need the following lemmas. In this section, $\tau(n)$ denotes the number of positive divisors of n.

Lemma 4.2 *We have $|h_1(n)| \le \tau(n)$ for all $n \ge 1$.*

Proof Since both h_1 and τ are multiplicative, it suffices to observe that $|h_1(p^j)| \le j+1 = \tau(p^j)$ for all prime powers p^j. ∎

Lemma 4.3 *For all $k \ge 1$, $\sum_{n\le x} |h_k(n)| \ll x^{1/k} \log x$.*

Proof Note that $h_k(n) = h_1(m)$ if $n = m^k$ is a perfect kth power and otherwise $h_k(n) = 0$. It follows that $\sum_{n\le x} |h_k(n)| = \sum_{n\le x^{1/k}} |h_1(n)|$; thus it suffices to prove the lemma for $k = 1$. But since $|h_1(n)| \le \tau(n)$ for all $n \ge 1$, the estimate $\sum_{n\le x} |h_1(n)| \ll x \log x$ follows immediately from the classical evaluation of $\sum_{n\le x} \tau(n)$ (see, for example, [25, Theorem 2.3]). ∎

Lemma 4.4 *There exists an absolute constant $c > 0$ such that $H_k(x) \ll x^{1/k} \exp\big(-c\frac{(\log x)^{3/5}}{(\log\log x)^{1/5}}\big)$ for all $k \ge 1$.*

Proof We have $H_k(x) = H_1(x^{1/k})$ by the same reasoning as in the proof of Lemma 4.3, and so it suffices to prove the lemma for $k = 1$. We leverage a known result for the Mertens function $M(x) = \sum_{n\le x} \mu(n)$, which uses contour integration to show that

$$M(x) \ll x \exp\left(-c\frac{(\log x)^{3/5}}{(\log\log x)^{1/5}}\right) \tag{4.1}$$

for some absolute positive constant c (see, for example, [16, Theorem 12.7]). It is possible to slightly modify that proof to deal with $H_1(x)$; instead, however, we give an alternative derivation that uses the result (4.1) directly rather than modifying its proof.

Since h_1 is the Dirichlet convolution of μ with itself, the hyperbola method implies that

$$\begin{aligned} H_1(x) &= \sum_{n\le\sqrt{x}} \mu(n)M\left(\frac{x}{n}\right) + \sum_{m\le\sqrt{x}} \mu(m)M\left(\frac{x}{m}\right) - M(\sqrt{x})M(\sqrt{x}) \\ &\ll \sum_{n\le\sqrt{x}} \left|M\left(\frac{x}{n}\right)\right| + |M(\sqrt{x})|^2. \end{aligned} \tag{4.2}$$

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Inequalities (4.1) and (4.2) imply that there are absolute constants $0 < c'' < c' < c$ such that

$$H_1(x) \ll \sum_{n \le \sqrt{x}} \frac{x}{n} \exp\left(- c \frac{(\log(x/n))^{3/5}}{(\log\log(x/n))^{1/5}} \right) + x \exp\left(- 2c \frac{(\log \sqrt{x})^{3/5}}{(\log\log \sqrt{x})^{1/5}} \right)$$
$$\ll \sum_{n \le \sqrt{x}} \frac{x}{n} \exp\left(- c' \frac{(\log x)^{3/5}}{(\log\log x)^{1/5}} \right)$$
$$\ll x \log x \cdot \exp\left(- c' \frac{(\log x)^{3/5}}{(\log\log x)^{1/5}} \right) \ll x \exp\left(- c'' \frac{(\log x)^{3/5}}{(\log\log x)^{1/5}} \right),$$

as desired. ∎

We now have the tools we need to establish an unconditional upper bound for the error term associated with the function g_k from Definition 4.1.

Proposition 4.5 *There exists an absolute constant $c > 0$ such that*

$$E_{g_k}(x) \ll x^{1/k} \exp\left(- c \frac{(\log x)^{3/5}}{(\log\log x)^{1/5}} \right)$$

for all $k \ge 2$.

Proof Since $D_{g_k}(s) = \zeta(s)\zeta(ks)^{-2}$, we have $G_{g_k}(x) = x/\zeta(k)^2$. Since g_k is the Dirichlet convolution of h_k and 1, we can apply the hyperbola method, with a parameter $z \in (\sqrt{x}, x)$ to be determined and with $y = \frac{x}{z}$:

$$F_{g_k}(x) = \sum_{mn \le x} h_k(n) = \sum_{n \le z} h_k(n) \left\lfloor \frac{x}{n} \right\rfloor + \sum_{m \le y} H_k\left(\frac{x}{m}\right) - \lfloor y \rfloor H_k(z)$$
$$= x \sum_{n \le z} \frac{h_k(n)}{n} + O\left(\sum_{n \le z} |h_k(n)| + \sum_{m \le y} \left| H_k\left(\frac{x}{m}\right)\right| + y|H_k(z)| \right).$$

Since $\sum_{n=1}^{\infty} h_k(n)/n = 1/\zeta(k)^2$, we use Lemma 4.3 to conclude that

(4.3)
$$E_{g_k}(x) = F_{g_k}(x) - G_{g_k}(x) \ll x \left| \sum_{n > z} \frac{h_k(n)}{n} \right| + z^{1/k} \log z + \sum_{m \le y} \left| H_k\left(\frac{x}{m}\right)\right| + y|H_k(z)|.$$

By Lemma 4.4 and partial summation,

$$\frac{1}{\zeta(k)^2} - \sum_{n \le z} \frac{h_k(n)}{n} \ll \frac{|H_k(z)|}{z} + \int_z^{\infty} \frac{|H_k(t)|}{t^2}\, dt$$
$$\ll z^{1/k-1} \exp\left(- c \frac{(\log z)^{3/5}}{(\log\log z)^{1/5}} \right). \tag{4.4}$$

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Since $y \le \sqrt{x}$, we have $x/m \ge \sqrt{x}$ for each $m \le y$. Thus, by Lemma 4.4, there is a constant $c' \in (0, c)$ such that

$$\begin{aligned}\sum_{m\le y}\left(H_k\left(\frac{x}{m}\right) - H_k(z)\right) &\ll \sum_{m\le y}\left(\frac{x}{m}\right)^{1/k} \exp\left(-c\frac{(\log(x/m))^{3/5}}{(\log\log(x/m))^{1/5}}\right)\\ &\ll x^{1/k}\sum_{m\le y} m^{-1/k}\exp\left(-c'\frac{(\log x)^{3/5}}{(\log\log x)^{1/5}}\right)\\ &\ll \frac{x}{z^{1-1/k}}\exp\left(-c'\frac{(\log x)^{3/5}}{(\log\log x)^{1/5}}\right). \qquad (4.5)\end{aligned}$$

Combining inequalities (4.3)–(4.5), we conclude that

$$\begin{aligned}E_g(x) &\ll \frac{x}{z^{1-1/k}}\exp\left(-c'\frac{(\log x)^{3/5}}{(\log\log x)^{1/5}}\right) + \sum_{n\le z}|h_k(n)|\\ &\ll \frac{x}{z^{1-1/k}}\exp\left(-c'\frac{(\log x)^{3/5}}{(\log\log x)^{1/5}}\right) + z^{1/k}\log x. \qquad (4.6)\end{aligned}$$

If we set

$$z = x\exp\left(-c'\frac{(\log x)^{3/5}}{(\log\log x)^{1/5}}\right),$$

then we conclude that there exists $c'' \in (0, c')$ such that $E_g(x) \ll x^{1/k}\exp\left(-c''\frac{(\log x)^{3/5}}{(\log\log x)^{1/5}}\right)$. ∎

If we assume RH, then we can strengthen the above bound on $E_{g_k}(x)$. To do so, we first need a conditional estimate on the tail of the Dirichlet series for $1/\zeta(s)^2$, which we express in the following lemma using the function h_1 from Definition 4.1.

Lemma 4.6 *Assume RH. Let $\varepsilon > 0$ and $\sigma_0 \ge \frac{1}{2} + \varepsilon$. Uniformly for $\frac{1}{2} + \varepsilon \le \sigma \le \sigma_0$,*

$$\frac{1}{\zeta(s)^2} - \sum_{n\le x} h_1(n)n^{-s} \ll_{\varepsilon,\sigma_0} x^{1/2-\sigma+\varepsilon}.$$

Proof Recall that $\sum_{n=1}^{\infty} h_1(n)n^{-s} = \zeta(s)^{-2}$ for $\sigma > 1$. Assume that $\frac{1}{2} + \varepsilon \le \sigma \le \sigma_0$. By Lemma 4.2, we have $|h_1(n)n^{-s}| < \tau(n)n^{-1/2} \ll n^{-1/2+\varepsilon}$. Thus, a truncated Perron's formula [25, Corollary 5.3] implies that

$$\sum_{n\le x} h_1(n)n^{-s} = \frac{1}{2\pi i}\int_{2-iT}^{2+iT} \frac{1}{\zeta(s+w)^2}\frac{x^w}{w}\,dw + R(x), \qquad (4.7)$$

where

$$R(x) \ll \sum_{\substack{x/2<n<2x\\ n\ne x}} |h_1(n)n^{-s}|\min\left(1, \frac{x}{T|x-n|}\right) + \frac{4^2 + x^2}{T}\sum_{n=1}^{\infty}\frac{|h_1(n)n^{-s}|}{n^2} \ll \frac{x^2}{T} \qquad (4.8)$$

(we may assume that x is an integer). Let $\varepsilon' = \varepsilon/(\sigma_0 + \frac{5}{2})$. We shift the contour integral in equation (4.7) leftwards from $\Re(w) = 2$ to $\Re(w) = \frac{1}{2} - \sigma + \varepsilon'$, noting that the only

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pole of $x^w/w\zeta(s+w)^2$ inside the contour is the simple pole at $w=0$. Thus,

$$\frac{1}{2\pi i}\int_{2-iT}^{2+iT}\frac{1}{\zeta(s+w)^2}\frac{x^w}{w}\,dw$$
$$(4.9)\quad = \frac{1}{\zeta(s)^2}+\frac{1}{2\pi i}\left(\int_{\frac12-\sigma+\varepsilon'+iT}^{2+iT}+\int_{\frac12-\sigma+\varepsilon'-iT}^{\frac12-\sigma+\varepsilon'+iT}+\int_{2-iT}^{\frac12-\sigma+\varepsilon'-iT}\right)\frac{1}{\zeta(s+w)^2}\frac{x^w}{w}\,dw.$$

Since $\zeta(z)^{-1}\ll_{\varepsilon'}T^{\varepsilon'}$ holds uniformly for $\frac12+\varepsilon'\le\Re(z)\le 2$ and $|\Im(z)|\le T$ (see [25, Theorem 13.23]), it follows that

$$\int_{\frac12-\sigma+\varepsilon'-iT}^{\frac12-\sigma+\varepsilon'+iT}\frac{1}{\zeta(s+w)^2}\frac{x^w}{w}\,dw\ll_{\varepsilon'}x^{1/2-\sigma+\varepsilon}T^{\varepsilon'}$$
$$\int_{\frac12-\sigma+\varepsilon'\pm iT}^{2\pm iT}\frac{1}{\zeta(s+w)^2}\frac{x^w}{w}\,dw\ll_{\varepsilon'}\sigma T^{\varepsilon'}\cdot\frac{x^2}{T}\ll_{\varepsilon',\sigma_0}\frac{x^2}{T^{1-\varepsilon'}}.$$

Combining these estimates with equations (4.7)–(4.9), and setting $T=x^{3/2+\sigma}$, we conclude that

$$\frac{1}{\zeta(s)^2}-\sum_{n<x}h_1(n)n^{-s}\ll_{\varepsilon',\sigma_0}x^{1/2-\sigma+\varepsilon'}T^{\varepsilon'}+\frac{x^2}{T^{1-\varepsilon'}}+\frac{x^2}{T}$$
$$\ll_{\varepsilon',\sigma_0}x^{1/2-\sigma+\varepsilon'(\sigma+5/2)}\le x^{1/2-\sigma+\varepsilon}$$

as required. ∎

Proposition 4.7 *Let $k\ge 2$. Assuming RH, $E_{g_k}(x)\ll_\varepsilon x^{1/(k+1)+\varepsilon}$ for each $\varepsilon>0$.*

Proof We adapt the proof that Montgomery and Vaughan [24] used for the indicator function of k-free numbers. Recall that $h_k(n)=h_1(m)$ if $n=m^k$ is a perfect kth power and $h_k(n)=0$ otherwise. Let $y=x^{1/(k+1)}$, and define

$$A(s)=\frac{1}{\zeta(s)^2}-\sum_{n\le y}h_1(n)n^{-s}.$$

If we set $\widetilde{h}(n)=h_k(n)$ for $n>y^k$ and $\widetilde{h}(n)=0$ otherwise, we see that the Dirichlet series for $\widetilde{h}$ is precisely $A(ks)$. Define $\widetilde{g}$ to be the Dirichlet convolution of $\widetilde{h}$ and 1, so that the Dirichlet series for $\widetilde{g}$ is $\zeta(s)A(ks)$. Also note that $|\widetilde{g}(n)|\le\sum_{d|n}|\widetilde{h}(d)|\le\sum_{d|n}\tau(d)\ll_\varepsilon n^\varepsilon$ for each $\varepsilon>0$ by Lemma 4.2. Using the truncated Perron's formula [25, Corollary 5.3] with the choice $T=x$,

$$(4.10)\qquad \sum_{n\le x}\widetilde{g}(n)=\frac{1}{2\pi i}\int_{c-ix}^{c+ix}\zeta(s)A(ks)\frac{x^s}{s}\,ds+R(x),$$

where $c=1+1/(k+1)$ and

(4.11)

$$R(x)\ll\sum_{\substack{x/2<n<2x\\ n\ne x}}|\widetilde{g}(n)|\min\left(1,\frac{x}{x|x-n|}\right)+\frac{4^c+x^c}{x}\sum_{n=1}^{\infty}\frac{|\widetilde{g}(n)|}{n^c}\ll x^\varepsilon+x^{1/(k+1)}.$$

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We shift the contour leftwards from $\sigma = c$ to $\sigma = \frac{1}{2}$ and notice that the only pole of $\zeta(s)A(ks)$ inside the contour is the simple pole at $s = 1$. Thus,

$$\begin{aligned} &\frac{1}{2\pi i}\int_{c-ix}^{c+ix} \zeta(s)A(ks)\frac{x^s}{s} \\ (4.12)\qquad &= xA(k) + \frac{1}{2\pi i}\left(\int_{\frac{1}{2}-ix}^{\frac{1}{2}+ix} + \int_{\frac{1}{2}+ix}^{c+ix} + \int_{c-ix}^{\frac{1}{2}-ix}\right)\zeta(s)A(ks)\frac{x^s}{s}\,ds. \end{aligned}$$

By [25, Theorem 13.18], $\zeta(s) \ll_\varepsilon x^\varepsilon$ uniformly for $\sigma \ge \frac{1}{2}$ and $1 \le |t| \le x$. By Lemma 4.6, $A(s) \ll_\varepsilon y^{1/2-\sigma+\varepsilon}$ holds uniformly for $\frac{1}{2} + \varepsilon \le \sigma \le k+2$, and thus $A(ks) \ll_\varepsilon y^{1/2-k\sigma+\varepsilon}$ holds uniformly for $\frac{1}{2} + \varepsilon \le \sigma \le c$. Therefore

$$\begin{aligned} \int_{\frac{1}{2}-ix}^{\frac{1}{2}+ix} \zeta(s)A(ks)\frac{x^s}{s} &\ll_\varepsilon x^{1/2+\varepsilon} y^{(1-k)/2+\varepsilon}, \\ \int_{\frac{1}{2}\pm ix}^{c\pm ix} \zeta(s)A(ks)\frac{x^s}{s} &\ll_\varepsilon x^{c-1+\varepsilon} y^{(1-k)/2+\varepsilon}. \end{aligned}$$

Combining these estimates with equations (4.10)–(4.12), we conclude that

$$\sum_{n\le x} \widetilde{g}(n) = xA(k) + O_\varepsilon\big(x^{1/(k+1)} + x^{1/2+\varepsilon} y^{(1-k)/2+\varepsilon}\big). \tag{4.13}$$

On the other hand,

$$\begin{aligned} \sum_{mn\le x} \big(h_k(n) - \widetilde{h}(n)\big) &= \sum_{\substack{mn\le x\\ n\le y^k}} h_k(n) = \sum_{\substack{mn^k\le x\\ n\le y}} h_1(n) = \sum_{n\le y} h_1(n)\left\lfloor \frac{x}{n^k}\right\rfloor \\ &= x\sum_{n\le y} \frac{h_1(n)}{n^k} + O\bigg(\sum_{n\le y} |h_1(n)|\bigg) \\ &= x\left(\frac{1}{\zeta(k)^2} - A(k)\right) + O(y\log y) \end{aligned}$$

by Lemmas 4.6 and 4.3. From this estimate and equation (4.13), it follows that

$$\begin{aligned} F_{g_k}(x) = \sum_{n\le x} g_k(n) &= \sum_{mn\le x} h_k(n) \\ &= \sum_{mn\le x} \big(h_k(n) - \widetilde{h}(n)\big) + \sum_{mn\le x} \widetilde{h}(n) \\ &= \sum_{mn\le x} \big(h_k(n) - \widetilde{h}(n)\big) + \sum_{n\le x} \widetilde{g}(n) \\ &= \frac{x}{\zeta(k)^2} + O_\varepsilon\big(y\log y + x^{1/(k+1)} + x^{1/2+\varepsilon} y^{(1-k)/2+\varepsilon}\big). \end{aligned}$$

Since $y = x^{1/(k+1)}$, this error term is $O_\varepsilon\big(x^{1/(k+1)+\varepsilon}\big)$ and therefore

$$E_{g_k}(x) = F_{g_k}(x) - G_{g_k}(x) \ll_\varepsilon x^{1/(k+1)+\varepsilon}$$

as required. ∎

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4.2 The convolution method

In this section, we apply the convolution method to prove both parts of Theorem 1.16 as well as Theorem 1.18(b). The following key lemma is inspired by [4, Lemma 1].

Lemma 4.8 *Let $Y(s) = \sum_{n=0}^{\infty} y_n n^{-s}$ and $Z(s) = \sum_{n=0}^{\infty} z_n n^{-s}$ be two Dirichlet series; assume that $Y(s)$ converges absolutely for $\sigma > 1$ and that k is a positive integer such that $Z(s)$ converges absolutely for $\sigma > \frac{1}{k+1}$. Suppose that the partial sum $T(x) = \sum_{n\le x} y_n$ has the form*

$$T(x) = \sum_{j=1}^{k} \operatorname{Res}\left(Y(s)\cdot\frac{x^s}{s}, \frac{1}{j}\right) + R(x) = \sum_{j=1}^{k} r_j x^{1/j} + R(x). \tag{4.14}$$

Assume further that there is $f \in \mathcal{F}$ such that $D_f(s) = Y(s)Z(s)$ for $\sigma > 1$.

(a) *If $R(x) \ll 1$, then $E_f(x) \ll_\varepsilon x^{1/(k+1)+\varepsilon}$ for every $\varepsilon > 0$.*
(b) *If $\theta > \frac{1}{k+1}$ is a real number, and $\widetilde{R}(x)$ is an eventually increasing function that satisfies $R(x) \ll x^\theta \widetilde{R}(x)$, then $E_f(x) \ll x^\theta \widetilde{R}(x)$.*

Remark 4.9 In addition to the dependence on ε in part (a), the implied constants in the above estimates on $E_f(x)$ depend on $Y(s)$ and $Z(s)$, as well as on θ and on the implied constant in the assumed upper bound for $R(x)$ in part (b). In the applications below, these ancillary quantities will all be chosen in terms of the function $f \in \mathcal{F}$; consequently, the implied constants will simply depend on f in those cases.

Remark 4.10 In part (b), we see that the deduced upper bound for $E_f(x)$ is the same as the assumed upper bound for $R(x)$ as long as that bound is sufficiently "nice" with respect to θ. We will have this sort of nice upper bound in all applications of Lemma 4.8(b) below.

Proof of Lemma 4.8 Let $1 \le j \le k$ and assume that $D_f(s)$ has a pole at $\frac{1}{j}$. Since $Z(s)$ converges absolutely for $\sigma > \frac{1}{k+1}$ and $D_f(s) = Y(s)Z(s)$, it follows that $\frac{1}{j}$ is a pole of $Y(s)$, which must be simple since the residues in equation (4.14) have no logarithmic terms. We deduce that

$$\operatorname{Res}\left(D_f(s)\cdot\frac{x^s}{s}, \frac{1}{j}\right) = Z\left(\frac{1}{j}\right)\operatorname{Res}\left(Y(s)\cdot\frac{x^s}{s}, \frac{1}{j}\right) = Z\left(\frac{1}{j}\right) r_j x^{1/j}. \tag{4.15}$$

By partial summation, it follows that for any $0 < \varepsilon < \frac{1}{j} - \frac{1}{k+1}$,

$$\begin{aligned} Z\left(\frac{1}{j}\right) - \sum_{n\le x} z_n n^{-1/j} &\ll \sum_{n>x} |z_n| n^{-1/j} \le x^{1/(k+1)-1/j+\varepsilon} \sum_{n>x} |z_n| n^{-(1/(k+1)+\varepsilon)} \\ &\ll_\varepsilon x^{1/(k+1)-1/j+\varepsilon}. \end{aligned} \tag{4.16}$$

Since $(f(n))$ is the Dirichlet convolution of (y_n) and (z_n),

$$F_f(x) = \sum_{n\le x} f(n) = \sum_{n\le x} z_n T\left(\frac{x}{n}\right) = \sum_{j=1}^{k}\left(\sum_{n\le x} z_n n^{-1/j}\right) r_j x^{1/j} + \sum_{n\le x} z_n R\left(\frac{x}{n}\right). \tag{4.17}$$

In light of the definition (1.4) of $G_f(x)$, equations (4.15)–(4.17) together imply that

$$E_f(x) = F_f(x) - G_f(x) \ll_\varepsilon x^{1/(k+1)+\varepsilon} + \sum_{n\le x} |z_n| \left| R\left(\frac{x}{n}\right)\right|.$$

This bound implies both parts of the lemma:

(a) If $R(x) \ll 1$, then $\sum_{n\le x} |z_n||R(\frac{x}{n})| \ll \sum_{n\le x}|z_n| \ll x^{1/(k+1)+\varepsilon}$ by partial summation, and thus $E_f(x) \ll_\varepsilon x^{1/(k+1)+\varepsilon}$ as required.

(b) Since $\theta > \frac{1}{k+1}$, it follows from the assumptions that

$$\sum_{n\le x} |z_n| \left| R\left(\frac{x}{n}\right)\right| \ll x^\theta \widetilde{R}(x) \sum_{n\le x} \frac{|z_n|}{n^\theta} \ll x^\theta \widetilde{R}(x).$$

Choosing $\varepsilon < \theta - 1/(k+1)$, we conclude that $E_f(x) \ll_\varepsilon x^{1/(k+1)+\varepsilon} + x^\theta \widetilde{R}(x) \ll x^\theta \widetilde{R}(x)$, as required. here

■

Next, we give the proof of Theorem 1.16 on unconditional upper bounds on $E_f(x)$, the first half of which is extremely short.

Proof of Theorem 1.16(a) Let (y_n) be the indicator function of kth powers, whose Dirichlet series is $Y(s) = \sum_{n=0}^\infty y_n n^{-s} = \zeta(ks)$ and whose summatory function is $\sum_{n\le x} y_n = x^{1/k} + O(1)$. By Theorem 2.6, we can write $D_f(s) = Y(s)Z(s)$, where $Z(s)$ converges absolutely for $\sigma > \frac{1}{k+1}$. Lemma 4.8(a) then implies that $E_f(x) \ll_\varepsilon x^{1/(k+1)+\varepsilon}$ for each $\varepsilon > 0$. ■

Proof of Theorem 1.16(b) We divide our discussion into two cases; in each case, we will eventually apply Lemma 4.8(b) with the choices

$$\theta \in \left(\frac{1}{k+1}, \frac{1}{k}\right) \quad \text{and} \quad \widetilde{R}(x) = x^{1/k-\theta} \exp\left(-c\frac{(\log x)^{3/5}}{(\log\log x)^{1/5}}\right). \tag{4.18}$$

(i) We first consider the case that f is of Möbius-type. Let $Y(s) = \zeta(ks)^{-1}$. By Proposition 2.3, we can write $D_f(s) = Y(s)Z(s)$ for $\sigma > 1$, where $Z(s)$ is a Dirichlet series that converges absolutely for $\sigma > \frac{1}{k+1}$. Let (y_n) be the sequence supported on perfect kth powers such that $y_{m^k} = \mu(m)$ for each nonnegative integer m, so that $Y(s) = \sum_{n=0}^\infty y_n n^{-s}$. Note that $T(x) = \sum_{n\le x} y_n = \sum_{m\le x^{1/k}} \mu(m) = M(x^{1/k})$ in terms of the Mertens function. From the best known error term on $M(x)$ (see, for example, [16, Theorem 12.7]), we see that $T(x) \ll R(x)$, where $R(x)$ is given by the right-hand side of the estimate (1.6). Then, the function $\widetilde{R}(x)$ given in equation (4.18) is eventually increasing and satisfies $T(x) \ll x^\theta \widetilde{R}(x)$, and so Lemma 4.8 implies the required upper bound on $E_f(x)$.

(ii) Next, we consider the case that f is of powerfree-type. By Proposition 2.8, k is the smallest integer such that $\varepsilon_k \ne 1$. If $\varepsilon_k = 0$, let (y_n) be the indicator function of the k-free numbers; if $\varepsilon_k = -1$, let $y_n = g_k(n)$, where g_k is defined in Definition 4.1. In both cases, the Dirichlet series of (y_n) is $Y(s) = \sum_{n=1}^\infty y_n n^{-s} = \zeta(s)/\zeta(ks)^{1+|\varepsilon_k|}$, and Proposition 2.8 implies that we can write $D_f(s) = Y(s)Z(s)$ for $\sigma > 1$, where $Z(s)$ converges absolutely for $\sigma > \frac{1}{k+1}$. Let $T(x) = \sum_{n\le x} y_n =$

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$x\,\mathrm{Res}(Y(s)x^s/s,1) + R(x)$. Then, the upper bound on $R(x)$ is exactly the right-hand side of the estimate (1.6): this is from a result of Walfisz [37, Section 5.6] when $\varepsilon_k = 0$, and follows from Proposition 4.5 when $\varepsilon_k = -1$. The theorem in this case follows from Lemma 4.8 by setting θ and $\widetilde{R}(x)$ as in equation (4.18).

∎

We also present the proof of Theorem 1.18(b) on upper bounds on $E_f(x)$ for f of powerfree-type under RH. The proof is very similar to the proof of Theorem 1.16(b) that just concluded.

Proof of Theorem 1.18(b) As in case (ii) in the proof of Theorem 1.16, we let (y_n) be the indicator function of k-free numbers when $\varepsilon_k = 0$, and let $y_n = g_k(n)$ when $\varepsilon_k = -1$. We can then write $D_f(s) = Y(s)Z(s)$ for $\sigma > 1$, where $Y(s)$ is the Dirichlet series of the sequence (y_n) and $Z(s)$ converges absolutely for $\sigma > \frac{1}{k+1}$. We then have $T(x) = \sum_{n\le x} y_n = x/\zeta(k)^{1+|\varepsilon_k|} + O_\varepsilon(x^{1/(k+1)+\varepsilon})$ for each $\varepsilon > 0$: this is a result by Montgomery and Vaughan [24] when $\varepsilon_k = 0$, and follows from Proposition 4.7 when $\varepsilon_k = -1$. In both cases, we have $x/\zeta(k)^{1+|\varepsilon_k|} = \mathrm{Res}(Y(s)x^s/s,1)$ and that $s = 1$ is the only real pole of $Y(s)$. Thus, by setting $\theta = \frac{1}{k+1} + \varepsilon$ and $\widetilde{R}(x) = 1$, Lemma 4.8 implies that $E_f(x) \ll_\varepsilon x^{1/(k+1)+\varepsilon}$. ∎

4.3 The Möbius-type and powerfull-type cases under RH

In this section, we prove Theorem 1.18(a) via the method of contour integration. We need the following upper bounds on $\zeta(s)$ and $1/\zeta(s)$.

Lemma 4.11 *Assume RH.*

(a) *For each $\delta > 0$, the bounds $\zeta(s) \ll_\delta 1$ and $\zeta(s)^{-1} \ll_\delta 1$ hold uniformly in the compact region $\{s\colon \frac12 \le \sigma \le 1-\delta,\ |t| \le 13\}$.*

(b) *There is a positive constant C such that both*

$$|\zeta(s)| \le \exp\left(C\frac{\log x}{\log\log x}\right) \quad \text{and} \quad |\zeta(s)^{-1}| \le \exp\left(C\frac{\log x}{\log\log x}\right)$$

hold for $\sigma \ge \frac12 + \frac{1}{\log\log x}$ and $1700 \le |t| \le x-4$.

Proof Part (a) is immediate: since all nontrivial zeros ρ of ζ have imaginary part at least 14, both $\zeta(s)$ and $\zeta(s)^{-1}$ are continuous in the compact region $\{s\colon \frac12 \le \sigma \le 1-\delta, |t| \le 13\}$, and the statement follows (indeed without even invoking RH).

As for part (b), let $x \ge \exp(\exp(10))$ be fixed. Consider the two functions

$$g(y) = \frac{\log y}{\log\log y} \quad \text{and} \quad h(y) = \frac{\log y}{\log\log y}\log\frac{e\log\log x}{\log\log y}.$$

Note that $\log g(y) = \log\log y - \log\log\log y$ and so

$$\frac{g'(y)}{g(y)} = \frac{1}{y\log y} - \frac{1}{y\log y\log\log y} = \frac{1}{y\log y}\left(1 - \frac{1}{\log\log y}\right) > 0$$

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when $y \ge e^e$. Thus, $g(y)$ is increasing when $y \ge 16$, and in particular, $g(y) \le g(x)$ when $16 \le y \le x$. Similarly,

$$\frac{h'(y)}{h(y)} = \frac{1}{y\log y} - \frac{1}{y\log y\log\log y} - \frac{1}{y\log y}\frac{1}{\log\log y}\cdot\log\frac{e\log\log x}{\log\log y}.$$

When $1700 \le y \le x$, we have

$$\frac{1}{\log\log y} + \frac{1}{\log\log y\cdot\log\frac{e\log\log x}{\log\log y}} \le \frac{2}{\log\log y} < 0.997 < 1;$$

so $h(y)$ is increasing when $1700 \le y \le x$, and in particular, $h(y) \le h(x) = g(x)$ when $1700 \le y \le x$.

Write $\tau = |t| + 4$ so that bounds involving $\log\tau$ are valid even when t is small. By [25, Theorem 13.18], there is an absolute positive constant C such that

$$|\zeta(s)| \le \exp\left(C\frac{\log\tau}{\log\log\tau}\right) = \exp(Cg(\tau))$$

for $\sigma \ge \frac{1}{2}$ and $|t| \ge 1$. Thus, when $\sigma \ge \frac{1}{2}$ and $1 \le |t| \le x-4$, we have $|\zeta(s)| \le \exp(Cg(\tau)) \le \exp(Cg(x))$.

To bound $|1/\zeta(s)|$, we consider two cases. When $\sigma \ge \frac{1}{2} + \frac{1}{\log\log\tau}$, we apply a similar argument as above by using the first part of [25, Theorem 13.23]. Next assume $\frac{1}{2} < \sigma \le \frac{1}{2} + \frac{1}{\log\log\tau}$ and $1700 \le |t| \le x-4$. In this case, by the second part of [25, Theorem 13.23], there is an absolute positive constant C such that

$$|1/\zeta(s)| \le \exp(Ch(\tau)) \le \exp(Ch(x)) = \exp(Cg(x)),$$

as required. ■

We conclude the article with a proof of Theorem 1.18(a).

Proof of Theorem 1.18(a) Let $f \in \mathcal{F}$ be fixed and let k be its initial index. By the discussion in Section 3, for $\sigma > 1$, we can write

$$D_f(s) = W(s)\cdot\prod_{j=k}^{2k-1}\zeta(js)^{a_j}, \tag{4.19}$$

where a_j are integers and $W(s)$ converges absolutely for $\sigma > \frac{1}{2k}$.

Let $x > \exp(\exp(10))$ be sufficiently large and set

$$\sigma_1 = \frac{1}{k}\left(\frac{1}{2} + \frac{1}{\log\log x}\right), \qquad T = x^{1-\sigma_1},$$

so that $k\sigma_1 = \frac{1}{2} + \frac{1}{\log\log x}$; note that $(2k-1)\sigma_1 < 1$ and $2k(T+4) < x$. Let C_0 be the positive constant from Lemma 4.11(b). Let $B = \sum_{j=k}^{2k-1}|a_j|$ and let $C_1 = BC_0$.

Since $|f(n)| \le 1$ for all $n \in \mathbb{N}$, we may assume without loss of generality that x is a positive integer in the following discussion. Since $D_f(s)$ is analytic for $\sigma > 1$, we set

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$\sigma_0 = 1 + \frac{1}{\log x}$. By the truncated Perron's formula [25, Corollary 5.3],

$$F_f(x) = \sum_{n \le x} f(n) = \frac{1}{2\pi i} \int_{\sigma_0 - iT}^{\sigma_0 + iT} D_f(s) \frac{x^s}{s} \, ds + R(x),$$

where

$$R(x) \ll \sum_{\substack{x/2 < n < 2x \\ n \ne x}} |f(n)| \min\left(1, \frac{x}{T|x-n|}\right) + \frac{4^{\sigma_0} + x^{\sigma_0}}{T} \sum_{n=1}^{\infty} \frac{|f(n)|}{n^{\sigma_0}}.$$

Therefore,

$$R(x) \ll \frac{x \log x}{T} + \frac{x\zeta(\sigma_0)}{T} \ll \frac{x \log x}{T}.$$

We shift the contour leftwards from $\sigma = \sigma_0$ to $\sigma = \sigma_1$. Define the integrals

$$I_1(x) = \frac{1}{2\pi i} \int_{\sigma_0 - iT}^{\sigma_0 + iT} D_f(s) \frac{x^s}{s} \, ds, \quad I_2(x) = \frac{1}{2\pi i} \int_{\sigma_0 + iT}^{\sigma_1 + iT} D_f(s) \frac{x^s}{s} \, ds,$$
$$I_3(x) = \frac{1}{2\pi i} \int_{\sigma_1 + iT}^{\sigma_1 - iT} D_f(s) \frac{x^s}{s} \, ds, \quad I_4(x) = \frac{1}{2\pi i} \int_{\sigma_1 - iT}^{\sigma_0 - iT} D_f(s) \frac{x^s}{s} \, ds.$$

Since $k\sigma_1 > \frac{1}{2}$ and $W(s)$ is analytic for $\sigma > \sigma_1$, the only possible poles of the contour integral $I_1 + I_2 + I_3 + I_4$ come from $s = \frac{1}{k}, \frac{1}{k+1}, \ldots, \frac{1}{2k-1}$. Thus, the residue theorem implies that

$$I_1(x) + I_2(x) + I_3(x) + I_4(x) = \sum_{j=k}^{2k-1} \operatorname{Res}\left(D_f(s)\frac{x^s}{s}, \frac{1}{j}\right).$$

We first bound $I_3(x)$. For $\sigma = \sigma_1$ and for each $k \le j \le 2k-1$, since $\frac{1}{2} + \frac{1}{\log\log x} \le j\sigma_1 \le (2k-1)\sigma < 1$ and $j|t| + 4 < 2k(T+4) < x$, it follows from Lemma 4.11 that $\zeta(js)^{a_j} \ll \exp(|a_j| C_0 \frac{\log x}{\log\log x})$. Thus, for $\sigma = \sigma_1$, we have

$$D_f(s) = W(s) \cdot \prod_{j=k}^{2k-1} \zeta(js)^{a_j} \ll \prod_{j=k}^{2k-1} \zeta(js)^{a_j} \ll \prod_{j=k}^{2k-1} \exp\left(|a_j| C_0 \frac{\log x}{\log\log x}\right)$$
$$= \exp\left(C_1 \frac{\log x}{\log\log x}\right).$$

It follows that

$$I_3(x) \ll \int_{-T}^{T} \exp\left(C_1 \frac{\log x}{\log\log x}\right) \frac{x^{\sigma_1}}{|\sigma_1 + t|} \, dt \ll x^{\sigma_1} \exp\left(C_2 \frac{\log x}{\log\log x}\right),$$

where $C_2 = C_1 + 1$.

Next, we estimate $I_2(x)$. Let $k \le j \le 2k-1$ be fixed. When $\sigma \ge \sigma_1$, we have $j\sigma \ge k\sigma_1 \ge \frac{1}{2} + \frac{1}{\log\log x}$. Thus, Lemma 4.11 implies that

$$\zeta\big(j(\sigma + iT)\big) \ll \exp\left(C_0 \frac{\log x}{\log\log x}\right) \quad \text{and} \quad \zeta\big(j(\sigma + iT)\big)^{-1} \ll \exp\left(C_0 \frac{\log x}{\log\log x}\right)$$

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uniformly for $\sigma \geq \sigma_1$, and thus that $D(\sigma + ix) \ll \exp(C_2 \frac{\log x}{\log\log x})$ holds uniformly for $\sigma_1 \leq \sigma \leq \sigma_0$. Therefore,

$$I_2(x) \ll \int_{\sigma_1}^{\sigma_0} \exp\left(C_2 \frac{\log x}{\log\log x}\right) \frac{x^\sigma}{T}\, d\sigma \ll \frac{x^{\sigma_0}}{x} \exp\left(C_2 \frac{\log x}{\log\log x}\right)$$
$$\ll \frac{x}{T} \exp\left(C_2 \frac{\log x}{\log\log x}\right).$$

A similar argument provides the same upper bound for $I_4(x)$.

We have shown that

$$F_f(x) - \sum_{j=k}^{2k-1} \operatorname{Res}\left(D_f(s)\frac{x^s}{s}, \frac{1}{j}\right) \ll R(x) + I_2(x) + I_3(x) + I_4(x)$$
$$\ll \frac{x \log x}{T} + \left(x^{\sigma_1} + \frac{x}{T}\right) \exp\left(C_2 \frac{\log x}{\log\log x}\right).$$

Since $T = x^{1-\sigma_1}$, it follows that

$$F_f(x) - \sum_{j=k}^{2k-1} \operatorname{Res}\left(D_f(s)\frac{x^s}{s}, \frac{1}{j}\right) \ll x^{\sigma_1} \exp\left(C_2 \frac{\log x}{\log\log x}\right) \ll x^{1/2k} \exp\left(C \frac{\log x}{\log\log x}\right),$$

where $C = C_2 + 1$.

Finally, recall from equation (1.4) that

$$G_f(x) = \sum_{j=k}^{2\ell} \operatorname{Res}\left(D_f(s)\frac{x^s}{s}, \frac{1}{j}\right) = \sum_{j=k}^{2k-1} \operatorname{Res}\left(D_f(s)\frac{x^s}{s}, \frac{1}{j}\right) + O(x^{1/2k}(\log x)^{O(1)}),$$

from which we conclude the desired upper bound

$$E_f(x) = F_f(x) - G_f(x) \ll x^{1/2k} \exp\left(C \frac{\log x}{\log\log x}\right).$$

■

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