Bull. Aust. Math. Soc. **103** (2021), 506–508
doi:10.1017/S0004972720001082

EXPONENTIAL SUMS AND ADDITIVE COMBINATORICS

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(Received 10 August 2020; first published online 9 November 2020)

2020 *Mathematics subject classification*: primary 11B30; secondary 11L07, 11T23, 11T30.

Keywords and phrases: exponential sum, additive combinatorics, finite field, sparse polynomial.

In this thesis we provide new results in additive combinatorics which in turn lead us to new bounds of certain exponential sums. We also use known bounds on exponential and character sums to give new results in additive combinatorics. Specifically, we will see how bounds on some quantities from additive combinatorics appear naturally when trying to bound multilinear exponential sums. We then find applications to bounds of exponential sums of sparse polynomials. We also give new bounds for an analogue of the energy variant of the sum-product problem over arbitrary finite fields. The thesis consists of four key chapters. They study the topics of collinear triples, decomposition of subsets of finite fields, multilinear exponential sums and multinomial exponential sums.

Bounds on the number of collinear triples are of particular importance when finding bounds on certain types of exponential sums as well as being a tool for giving bounds on sum and product sets. We define the number of collinear triples, $T(\mathcal{A}, \mathcal{B})$, to be the number of solutions of

$$(a_1 - a_2)(b_1 - b_2) = (a_1 - a_3)(b_1 - b_3), \quad a_i \in \mathcal{A},\ b_i \in \mathcal{B},\ i = 1, 2, 3,$$

for $\mathcal{A}, \mathcal{B} \subset \mathbb{F}_p$. We adapt existing techniques to give new bounds on the number of collinear triples, which are stronger when $\mathcal{A} \neq \mathcal{B}$. These results have been published in [3]. Previous results on this asymmetric case have been given using the Cauchy–Schwarz inequality by first finding bounds on $T(\mathcal{A}, \mathcal{A})$. Additionally, we also provide stronger bounds when our sets $\mathcal{A}$ and $\mathcal{B}$ are subgroups. We also consider a more general form of $T(\mathcal{A}, \mathcal{B})$ than what has been considered previously. Instead we consider collinear triples over two parameters λ and μ, which leads us to new bounds on multiplicative energy of shifted subgroups.

Additive and multiplicative energy have seen much study recently, some of which we outline in Chapter 3 of this thesis. Of particular importance is their relationship to

Thesis submitted to the University of New South Wales in January 2020: degree approved on 26 March 2020; supervisor Igor Shparlinski.

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https://doi.org/10.1017/S0004972720001082 Published online by Cambridge University Press

sum and product sets. We define the additive energy

$$E^{+}(\mathcal{A},\mathcal{B}) = |\{(a_1,a_2,b_1,b_2) \in \mathcal{A}^2 \times \mathcal{B}^2 : a_1 + b_1 = a_2 + b_2\}|.$$

Similarly, we define the multiplicative energy

$$E^{\times}(\mathcal{A},\mathcal{B}) = |\{(a_1,a_2,b_1,b_2) \in \mathcal{A}^2 \times \mathcal{B}^2 : a_1 b_1 = a_2 b_2\}|.$$

Here, we are most interested in the cases $\mathcal{A} = \mathcal{B}$ and thus define $E^{+}(\mathcal{A},\mathcal{A}) = E^{+}(\mathcal{A})$ and $E^{\times}(\mathcal{A},\mathcal{B}) = E^{\times}(\mathcal{A})$. We also define the sum and product sets respectively as

$$\mathcal{A} + \mathcal{B} = \{a + b : a \in \mathcal{A}, b \in \mathcal{B}\},$$
$$\mathcal{A} \cdot \mathcal{B} = \{ab : a \in \mathcal{A}, b \in \mathcal{B}\}.$$

Using the Cauchy–Schwarz inequality and squaring, one can see that

$$|\mathcal{A} + \mathcal{A}| E^{+}(\mathcal{A}) \geq |\mathcal{A}|^4.$$

It follows that strong upper bounds on additive energy correspond to strong lower bounds on the size of the sum set. Similarly, for product sets. We prove an extension of results of Roche and Newton, Shparlinski, and Winterhof who show

$$\max(E^{+}(\mathcal{B}), E^{+}(f(C))) \ll |\mathcal{A}|^{3-\delta} \tag{1}$$

over $\mathbb{F}_q$, where q is a prime power, f is a suitably chosen function and $\mathcal{A}$ is of sufficient size. Our bounds rely on bounds on certain character sums. Our extensions will show that we can replace E^{+} with $E^{\times}$ in either or both terms in (1), as long as we suitably change our restriction on our function f. These results have been published in [4].

Multilinear exponential sums are those of the form

$$T(\mathcal{X}_1,\ldots,\mathcal{X}_n) = \sum_{x_1 \in \mathcal{X}_1} \cdots \sum_{x_n \in \mathcal{X}_n} \mathbf{e}_p(a x_1 \ldots x_n)$$

for $\mathcal{X}_i \subseteq \mathbb{F}_p$ for each $i = 1,\ldots,n$ and any $a \in \mathbb{F}_p^*$. The first results in this direction are due to Vinogradov who provided a bound on bilinear exponential sums. The focus of Chapter 4 is to consider multilinear exponential sums of the form

$$T(\mathcal{X}_1,\ldots,\mathcal{X}_n) = \sum_{x_1 \in \mathcal{X}_1} \cdots \sum_{x_n \in \mathcal{X}_n} \omega_1(\mathbf{x}) \ldots \omega_n(\mathbf{x})\, \mathbf{e}_p(a x_1 \ldots x_n) \tag{2}$$

where $a \in \mathbb{F}_p^*$ and the ω_i are $(n-1)$-dimensional complex weights, that is, complex numbers of modulus $|\omega_i| \leq 1$ depending on all but the ith coordinate of $\mathbf{x}$. Our results extend previous results of Petridis and Shparlinski and use similar techniques; however, some improvements are made in certain regions on trilinear and quadrilinear exponential sums due to estimates on collinear triples from Chapter 2 (see [3]). We have also been able to extend these results to general multilinear sums beyond $n = 4$. This extension is certainly nontrivial, and is due to some recent results in additive combinatorics. Some of these results have been published in [1, 2, 5].

https://doi.org/10.1017/S0004972720001082 Published online by Cambridge University Press

We define a t-sparse polynomial

$$\Psi_t(X) = \sum_{i=1}^{t} a_i X^{k_i}$$

with pairwise distinct, nonzero, integer exponents $k_1, \ldots, k_t$ with corresponding coefficients $a_1, \ldots, a_t \in \mathbb{F}_p^*$. We consider the multinomial exponential sum

$$S_\chi(\Psi_t) = \sum_{x \in \mathbb{F}_p^*} \chi(x)\, \mathbf{e}_p(\Psi_t(x)). \tag{3}$$

The bounds on such sums that appear in Chapter 5 come as a result of bounds on weighted multilinear sums from the previous chapter. By extending the sum over t multiplicative subgroups of $\mathbb{F}_p^*$ we are able to express the multinomial sum as a weighted multilinear sum. It is worth mentioning that in this chapter we find stronger results on multilinear exponential sums than those in Chapter 4 when the arbitrary sets are, instead, multiplicative groups.

The methods used to derive the bounds provide interesting results as our bounds do not depend directly on the size of the powers of the polynomials, but rather they depend on the size of some greatest common divisors of the powers. This is in contrast to the well-known Weil bound, which gives

$$|S_\chi(\Psi_t)| \le \max\{k_1, \ldots, k_t\} p^{1/2}.$$

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https://doi.org/10.1017/S0004972720001082 Published online by Cambridge University Press