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A REMARK ON INTEGRATION OF ALMOST-PERIODIC FUNCTIONS

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Introduction. A result proved by Favard for scalar-valued almost-periodic functions has an immediate extension to Banach space valued functions (see [2] and [3] for explicit details).

The result says that integrals of almost-periodic functions whose 'spectrum' is at positive distance from 0 are again almost-periodic. Our aim here is to indicate a more general formulation of this result using strongly almost-periodic one-parameter groups of operators in Banach spaces.

1. Let us remember here a few fundamental definitions. Consider a Banach space X and a function $f(t)$, $-\infty < t < +\infty \to X$ which is continuous and almost-periodic; this means that the set of translates $(f(t+h))$ where $h \in (-\infty, +\infty)$ is a relatively compact family with respect to the space $C[-\infty, +\infty; X]$. It is known that for such a function $\lim_{T\to\infty} (1/T) \int_0^T e^{-i\lambda t} f(t)\, dt$, where λ is real number, exists in X and is $\neq 0$ (the null element of X) on at most countable set $(\lambda_n)_1^\infty$ (see [1] and [3] for proof); one denotes by $\sigma(f(t)) = (\lambda_n)_1^\infty$ the 'spectrum' of $f(t)$. Then the result by Favard (extended to Banach space valued functions) which was stated in the introduction is as follows:

THEOREM 1. *If* $\exists \alpha > 0$ *such that* $|\lambda_n| > \alpha > 0$, $\forall\ n = 1, 2, \ldots$ *then* $F(t) = \int_0^t f(\eta) d\eta$ *is again an almost-periodic function.*

Let us consider now a strongly continuous one-parameter group of linear operators in X, $G(t) \in \mathscr{L}(X, X)$, $\forall t \in (-\infty, +\infty)$, $G(t+s) = G(t)G(s)$ $\forall t, s \in (-\infty, +\infty)$. We say that $G(t)$ is a strongly almost-periodic group (as in our paper [4]) when, $\forall x \in X$, $G(t)x$ is almost-periodic, $-\infty < t + \infty \to X$.

Then, according to our previous remark, $\forall x \in X$ the spectrum σ_x of $G(t)x$ is a well defined at most countable set. We may, at this stage, assume the following.

The union when x varies in X of the σ_x: $\bigcup_{x \in X} \sigma_x$ is a countable set $(\mu_n)_1^\infty \subset R^1$; we say in this case that the group $G(t)$ has property (P) and denote $(\mu_n)_1^\infty = \bigcup_{x\in X} \sigma_x = \sigma_G$. We can state now our result in form of

THEOREM 2. *Let* $f(t)$ *be an almost-periodic function,* $-\infty < t < +\infty$ *to* X *and* $G(t)$, $-\infty < t < +\infty \to \mathscr{L}(X, X)$ *be a strongly almost-periodic one parameter group of operators, having property* (P). *Then if* $(\lambda_n)_1^\infty = \sigma(f)$, *and* $(\mu_m)_1^\infty = \sigma_G$, *and if* $\exists \alpha > 0$ *such that* $|\lambda_n - \mu_m| > \alpha > 0$ $\forall n, m = 1, 2, \ldots$, *we have that* $F(t) = \int_0^t G(t-\eta) f(\eta)\, d\eta$ *is an almost-periodic function.*

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REMARK 1. When $G(t)=I$ (the identity operator), $\forall t \in (-\infty, +\infty)$ we obtain Theorem 1. So Theorem 2 is a more general form of Theorem 1. Its proof consists in fact in a reduction, after some steps, to Favard's Theorem 1.

REMARK 2. If $f(t)$ is regular, say $C^1(X)$, then $F(t) \in D(A)$ $\forall t$ and $F'(t)=AF(t)+f(t)$ holds; here A is the infinitesimal generator of $G(t)$ defined through $Ax=\lim_{\eta\to 0}(1/\eta)(G(\eta)x-x)$, and $D(A)$, the domain of A, is dense in X.

Proof of Theorem 2. We use the following result which was given by us previously (see [3] and [4]); see also [5] for a proof using Maak's definition of almost-periodicity.

LEMMA 1. *If $G(t)$ is a strongly almost-periodic one parameter group of operators, and $f(t)$ is an almost-periodic function, then $G(t)f(t)$ is again an almost-periodic function.*

Then we have the obvious

LEMMA 2. *If $G(t)$ is a strongly almost-periodic one-parameter group of operators, then $\check{G}(t)=G(-t)$ is again a strongly almost-periodic one-parameter group of operators.*

Then we see that $F(t)=G(t)\int_0^t G(-\eta)f(\eta)\,d\eta=G(t)\int_0^t \check{G}(\eta)f(\eta)\,d\eta$. We shall prove that $\int_0^t \check{G}(\eta)f(n)\,d\eta$ is an almost-periodic function and then will apply Lemma 1.

From Lemma 2 and Lemma 1, $\check{G}(\eta)f(\eta)$ is an almost-periodic function. If $(\nu_n)_1^\infty=\sigma(\check{G}(\eta)f(\eta))$ it will be enough to show (by Theorem 1) that $\exists\alpha>0$ such that $|\nu_\eta|>\alpha$, $\forall n=1,2,\ldots$.

We have then, if for a set of real numbers $A=\{a\}$, $-A$ means the set $\{-a\}_{a\in A}$, the

LEMMA 3. *The relation $\sigma_G=-\sigma_{\check{G}}$ holds.*

For this it is enough to see that, $\forall x \in X$, $\sigma(G(t)x)=-\sigma(\check{G}(t)x)$. This follows immediately from the relation

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T e^{i\mu t}\check{G}(t)x\,dt = \lim_{T\to\infty}\frac{1}{T}\int_0^T e^{-i\mu u}G(u)x\,du$$

which is true for each real μ and $x \in X$, and is of easy verification.

Now we prove that, considering the positive α which enters in the hypothesis of Theorem 2, we have

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T e^{-i\eta s}\check{G}(\eta)f(\eta)\,d\eta = \theta \quad \text{for } |s|<\alpha$$

(this implies $\sigma(\check{G}(\eta)f(\eta)) \subset \{s;\ |s| \geq \alpha\}$).

Let us consider now Bochner–Fejer trigonometrical polynomials approximating uniformly (on $-\infty<s<+\infty$) $f(\eta)$ (see [3] for a proof).

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If $P_m(\eta)$ is such a polynomial, it is of the form:

$$P_m(\eta) = \sum_{k=1}^{N(m)} a_k^m \, e^{i\lambda_k\eta}$$

where $a_k^m \in X$ and λ_k belong to $\sigma(f)$. Then we have the

LEMMA 4. *The relation* $\lim_{T\to\infty} (1/T) \int_0^T e^{-i\eta s}\check{G}(\eta)P_m(\eta)\, d\eta = \theta$ *holds for* $|s| < \alpha$ *and* $\forall m = 1, 2, \ldots$

The expression equals:

$$\sum_{k=1}^{N(m)} \lim_{T\to\infty} \frac{1}{T}\int_0^T e^{-i\eta s}(\check{G}(\eta)a_k^m)\, e^{i\lambda_k\eta}\, d\eta$$
$$= \sum_{k=1}^{N(m)} \lim_{T\to\infty} \frac{1}{T}\int_0^T e^{-i\eta(s-\lambda_k)}\check{G}(\eta)a_k^m\, d\eta,$$

We know that for each $k = 1, 2, \ldots, N(m)$ this limit is θ if $s - \lambda_k \neq -\mu_j$, $\forall j = 1, 2, \ldots$ (as $(-\mu_j)_1^\infty = \sigma_{\check{G}}$). But this holds indeed when $|s| < \alpha$, because, obviously $|s - \lambda_k + \mu_j| \geq |\mu_j - \lambda_k| - |s| > \alpha - \alpha = 0$.
So the lemma is proved.

Then using uniform convergence (on $-\infty, +\infty$) of the sequence $P_m(\eta)$ to $f(\eta)$ and the uniform bound $\|\check{G}(\eta)\| \leq M$ (which follows from the strong almost-periodicity and uniform boundness theorem) we obtain the desired result. So, our theorem is proved, using the remarks in the beginning.

REFERENCES

1. S. Bochner, *Abstrakte Fast-periodische Funktionen*, Acta Math., 1933.

2. B. M. Levitan, *Almost-periodic functions*, (Russian), Moscow, 1953.

3. S. Zaidman, *Solutions presque-périodiques des équations hyperboliques*, Ann. Sci. École Norm. Sup., Paris, 1962.

4. ———, *Sur la perturbation presque-périodique des groupes et semi-groupes de transformations d'un espace de Banach*, Rend. Mat. e Appl., Rome, 1957.

5. H. Gunzler and S. Zaidman, *Almost-periodic solutions of a certain abstract differential equation*, J. Math. Mech., 1969.

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