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NONCLASSICAL ORTHOGONAL POLYNOMIALS AS SOLUTIONS TO SECOND ORDER DIFFERENTIAL EQUATIONS

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ABSTRACT. One of the more popular problems today in the area of orthogonal polynomials is the classification of all orthogonal polynomial solutions to the second order differential equation:

$$a_2(x, n)y''(x)+a_1(x, n)y'(x)+a_0(x, n)y(x)=\lambda_n y(x).$$

In this paper, we show that the Laguerre type and Jacobi type polynomials satisfy such a second order equation.

1. **Introduction.** Recently, there has been increasing interest in classifying all second order differential equations of the form

$$a_2(x, n)y''(x)+a_1(x, n)y'(x)+a_0(x, n)y(x)=\lambda_n y(x) \tag{1}$$

which have orthogonal polynomials as solutions. In his master's thesis [6], Shore found a second order differential equation like (1) for which the Legendre type polynomials [3] are solutions. It is the purpose of this article to show that the Laguerre type and Jacobi type polynomials [3] also satisfy a second order differential equation of the form (1).

All three of these polynomial sets are known to satisfy fourth order equations [5] of the form:

$$L_4(y)=\lambda_n y \tag{2}$$

In order to find second order differential equations for which these polynomials are solutions, we proceed as follows: we first find a sixth order differential equation for which these polynomials are solutions by using a method developed by Shore and H. L. Krall [6]. We then differentiate (2) twice to obtain another sixth order equation. By carefully combining the sixth order equations, we obtain a fourth order differential equation, different from (2). Finally, we combine this fourth order equation with (2) and eliminate the third and fourth derivatives to obtain the desired second order differential equation. We will see these equations are different from the so called classical second order equations

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of Legendre, Laguerre and Jacobi in that the coefficients of y' and y'' are function of x and n.

Recently, W. Hahn has attacked this classification problem from another point of view and has obtained some very interesting results. He has shown, in fact, that the minimal order of a differential equation having orthogonal polynomial solutions is either two or four [2].

2. **Notation and preliminaries.** The most general sixth order formally self adjoint differential equation is given by:

$$\begin{aligned} a_6(x)y^{(vi)}(x)+3a_6'(x)y^{(v)}(x)+a_4(x)y^{(iv)}(x)+(2a_4'(x)-5a_6'''(x))y'''(x) \\ +a_2(x)y''(x)+(a_2'(x)-a_4'''(x)+3a_6^{(v)}(x))y'(x)=\mu y(x) \end{aligned} \tag{3}$$

Recall that Lagrange's identity guarantees that if $L(y)$ is an nth order linear differential equation, then there exists a bilinear concomitant $P(u, v)$ so that

$$vL(u)-uL^*(v)=\frac{dP(u,v)}{dx} \quad [1] \tag{4}$$

If we let $w_{ij}=v^{(i)}u^{(j)}-v^{(j)}u^{(i)}$ and let $L(y)$ denote the left side of (3), then

$$\begin{aligned} P(u,v)=a_6w_{05}+2a_6'w_{04}-a_6w_{14}+a_6w_{23}-a_6'w_{13}-2a_6''w_{03}+3a_6''w_{12}-3a_6'''w_{02} \\ +3a_6^{(iv)}w_{01}+a_4w_{03}-a_4w_{12}+a_4'w_{02}-a_4''w_{01}+a_2w_{01} \end{aligned}$$

3. **The Legendre type polynomials.** The Legendre type polynomial

$$y_n(x)=\sum_{k=0}^{[n/2]}\frac{(-1)^k(2n-2k)!\left(\alpha+\frac{(n-1)}{2}+2k\right)x^{n-2k}}{2^nk!\,(n-k)!\,(n-2k)!} \tag{5}$$

satisfies the fourth order formally self adjoint differential equation

$$\begin{aligned} ((x^2-1)^2y'')''+4((\alpha(x^2-1)-2)y')' \\ =[8\alpha n+(4\alpha+12)n(n-1)+8n(n-1)(n-2)+n(n-1)(n-2)(n-3)]y \end{aligned} \tag{6}$$

Furthermore, these polynomials are orthogonal on $[-1, 1]$ with respect to the weight distribution $w(x)=\frac{1}{2}[\delta(x-1)+\delta(x+1)]+\alpha/2$ [4]. Shore found that (5) satisfied the second order equation:

$$\begin{aligned} (x^2-1)[(4\alpha^2+4\alpha+\lambda_n)x^2-(4\alpha^2-4\alpha+\lambda_n)]y''(x) \\ +2x[(4\alpha^2+4\alpha+\lambda_n)x^2-(4\alpha^2-12\alpha+\lambda_n)]y'(x) \\ -[(\mu_n+4\alpha+96)x^2-(\mu_n+4\alpha+96-4\lambda_n)]y(x)=0 \end{aligned}$$

where $\lambda_n=n(n+1)(n^2+n+4\alpha-2)$ and

$$\mu_n=n(n+1)(n^4+2n^3-97n^2-98n+192-372\alpha-12\alpha^2)$$

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4. The Laguerre type polynomials. The Laguerre type polynomial

$$y_n(x) = \sum_{k=0}^{n} \frac{(-1)^k}{(k+1)!} \binom{n}{k} [k(R+n+1)+R]x^k \tag{7}$$

satisfies the fourth order equation $L_4(y) = \lambda_n y$ where

$$L_4(y) = x^2 y^{(iv)}(x) - (2x^2 - 4x)y'''(x) + (x^2 - (2R+6)x)y''(x) + ((2R+2)x - 2R)y'(x)$$

and $\lambda_n = (2R+2)n + n(n-1)$. These polynomials are orthogonal on $[0, \infty)$ with respect to the weight distribution $w(x) = (1/R)\delta(x) + e^{-x}$. [4] Let $a_{2i} = b_{2i}e^{-x}$, $i = 1, 2, 3$ and note that $L_4(y)$ is formally self adjoint when multiplied by e^{-x}. Then (3) becomes $L(y) = e^{-x}L_6(y) = \mu_n y$ where

$$\begin{aligned} L_6(y) = {} & b_6 y^{(vi)} + [3b_6' - 3b_6]y^{(v)} + b_4 y^{(iv)} \\ & + [2b_4' - 2b_4 - 5b_6''' + 15b_6'' - 15b_6' + 5b_6]y''' \\ & + b_2 y'' + [b_2' - b_2 - b_4''' + 3b_4'' + b_4 + 3b_6^{(v)} - 15b_6^{(iv)} \\ & + 30b_6''' - 30b_6'' + 15b_6' - 3b_6]y' \end{aligned}$$

Assume (7) satisfies $L_6(y_n) = \mu_n y_n$. Necessarily then, the coefficient of $y^{(i)}$ must be a polynomial of degree $\leq i$, $i = 1, 2, 3, 4, 5, 6$. This gives us one set of conditions to determine b_{2i}, $i = 1, 2, 3$. Now

$$\begin{aligned} \int_0^\infty [vL_6(u) - uL_6(v)]w(x)\,dx &= \frac{1}{R}[vL_6(u) - uL_6(v)]|_{x=0} \\ &\quad + \int_0^\infty [vL(u) - uL(v)]\,dx \\ &= \frac{1}{R}[vL_6(u) - uL_6(v)]|_{x=0} + [P(u, v)]|_0^\infty \end{aligned} \tag{8}$$

If we choose $v = y_k$, $u = y_l$, $k \neq l$, the left side of (8) is $(\lambda_k - \lambda_l)\int_0^\infty y_k y_l w\,dx = 0$ because of our orthogonality condition. Thus, with this choice of u and v, the right side of (8) is also zero. Using our notation, this right side becomes an equation involving $w_{ij}(0)$. It is sufficient that all the coefficients of $w_{ij}(0)$ be zero. This gives us another set of conditions on the b_{2i}'s. Upon considerable computation, we find that (7) satisfies the sixth order equation:

$$\begin{aligned} & x^3 y^{(vi)} + [-3x^3 + 9x^2]y^{(v)} + [3x^3 + 15x]y^{(iv)}) \\ & \quad + [-x^3 - 27x^2 + 60x]y''' + [18x^2 - 3(R^2 + 15R + 39)x - 3R]y'' \\ & \quad\quad + [3(R^2 + 15R + 14)x - 3R^2 - 42R]y' = \mu_n y \end{aligned}$$

where $\mu_n = (3R^2 + 45R + 42)n + 18n(n-1) - n(n-1)(n-2)$. The second order

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differential equation that y_n then satisfies is

$$L_6(y)-xL_4''(y)-(1-x)L_4'(y)+\left(-2R-22+\frac{1}{x}\right)L_4(y)$$
$$=\mu_n y-\lambda_n xy''-\lambda_n(1-x)y'+\left(-2R-22+\frac{1}{x}\right)\lambda_n y$$

which is

$$[(R^2+R+\lambda_n)x^2-Rx]y''+[-(R^2+R+\lambda_n)x^2+(R^2+2R+\lambda_n)x-2R]y'$$
$$+[(2R\lambda_n+22\lambda_n-\mu_n)x-\lambda_n]y=0$$

5. **The Jacobi type polynomials.** The Jacobi type polynomials

(9) $$y_n(x)=\sum_{k=0}^{n}\frac{(-1)^{n-k}\binom{n}{k}(1+\alpha)_{n+k}(k[n+\alpha][n+1]+[k+1]M)x^k}{(k+1)!\,(1+\alpha)_n}$$

satisfies the fourth order equation $L_4(y_n)=\lambda_n y_n$ where

$$L_4(y)=(x^2-x)^2y^{(iv)}(x)+2x(x-1)([\alpha+4]x-2)y'''$$
$$+x([\alpha^2+9\alpha+14+2M]x-[6\alpha+12+2M])y''$$
$$+([\alpha+2][2\alpha+2+2M]x-2M)y'$$

and

$$\lambda_n=(\alpha+2)(2\alpha+2+2M)n+(\alpha^2+9\alpha+14+2M)n(n-1)$$
$$+2(\alpha+4)n(n-1)(n-2)+n(n-1)(n-2)(n-3)$$

These polynomials are orthogonal on [0, 1] with respect to the weight distribution $w(x)=(1/M)\delta(x)+(1-x)^\alpha$, $\alpha>-1$ [4]. Let $a_{2i}=b_{2i}$ $(1-x)^\alpha$, $i=1,2,3$ and note that $L_4(y)$ is formally self adjoint when multiplied by $(1-x)^\alpha$. (2) then becomes $L(y)=(1-x)^\alpha L_6(y)=\mu_n y$. Assume y_n satisfies $L_6(y_n)=\mu_n y_n$. In spite of some extremely tedious calculations, we find that $y_n(x)$ satisfies the sixth order equation;

$$L_6(y)=(x^2-x)^3y^{(vi)}+3x^2(1-x)^2(6x+\alpha x-3)y^{(v)}$$
$$+[(3\alpha^2+15\alpha-12)x^4+(-3\alpha^2-15\alpha+27)x^3-15x]y^{(iv)}$$
$$+[(\alpha^3-21\alpha^2-274\alpha-696)x^3+(27\alpha^2+345\alpha+1062)x^2$$
$$-(60\alpha+360)x]y'''$$
$$+[-(18\alpha^3+270\alpha^2+1188\alpha+1440+3M^2+219M+45M\alpha)x^2$$
$$+(117\alpha^2+855\alpha+1242+3M^2+21M+45M\alpha)x+3M]y''$$
$$+[-(42\alpha^3+342\alpha^2+732\alpha+432+6M^2+438M$$
$$+309M\alpha+3M^2\alpha+45M\alpha^2)x+3M^2+216M+42M\alpha]y'$$
$$=\mu_n y$$

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where

$$\begin{aligned}\mu_n = &-n(42\alpha^3+342\alpha^2+732\alpha+432+6M^2+438M\\&+309M\alpha+3M^2\alpha+45M\alpha^2)\\&-n(n-1)(18\alpha^3+270\alpha^2+1188\alpha+1440+3M^2+219M+45M\alpha)\\&+n(n-1)(n-2)(\alpha^3-21\alpha^2-274\alpha-696)\\&+n(n-1)(n-2)(n-3)(3\alpha^2+15\alpha-12)\\&+(18+3\alpha)n(n-1)(n-2)(n-3)(n-4)\\&+n(n-1)(n-2)(n-3)(n-4)(n-5).\end{aligned}$$

We find that y_n satisfies the second order equation

$$\begin{aligned}L_6(y)-(x^2-x)L_4''(y)-[(2+\alpha)x-1]L_4'(y)+\left[(22\alpha+2M+110)-\frac{1}{x}\right]L_4(y)\\=\mu_n y-(x^2-x)\lambda_n y''-[(2+\alpha)x-1]\lambda_n y'+\left[(22\alpha+2M+110)-\frac{1}{x}\right]\lambda_n y\end{aligned}$$

which, when simplified, is:

$$\begin{aligned}&[(M^2+M\alpha+M+\lambda_n)x^3-(M^2+M\alpha+2M+\lambda_n)x^2+Mx]y''\\&\quad+[(2M^2+3M\alpha+2M+M^2\alpha+M\alpha^2+2\lambda_n+\alpha\lambda_n)x^2\\&\quad-(M^2+2M\alpha+4M+\lambda_n)x+2M]y'\\&\quad\quad+\{[-(22\alpha+2M+110)\lambda_n-\mu_n]x+\lambda_n\}y=0\end{aligned}$$

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