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LITTLEWOOD-PALEY AND MULTIPLIER THEOREMS FOR VILENKIN-FOURIER SERIES

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ABSTRACT. Let $S_{2^j}f$ be the 2^j-th partial sum of the Vilenkin-Fourier series of $f \in L^1$, and set $S_{2^{-1}}f = 0$. For $f \in L^p$, $1 < p < \infty$, we show that the ratio

$$\|(\sum_{j=-1}^{\infty} |S_{2^{j+1}}f - S_{2^j}f|^2)^{\frac{1}{2}}\|_p / \|f\|_p$$

is contained between two bounds (independent of f). From this we obtain the Marcinkiewicz multiplier theorem for Vilenkin-Fourier series.

1. **Introduction.** Let $\{p_i\}_{i\geq 0}$ be a sequence of integers with $p_i \geq 2$, and $G = \prod_{i=0}^{\infty} Z_{p_i}$ be the direct product of cyclic groups of order p_i. For $x = \{x_k\} \in G$, let $\phi_k(x) = \exp(2\pi i x_k/p_k)$, $k = 0, 1, 2, \ldots$. The Vilenkin system $\{\chi_n\}$ is the set of all finite products of $\{\phi_k\}$, which is enumerated in the following manner. Let $m_0 = 1$, $m_k = \prod_{i=0}^{k-1} p_i$, $k = 1, 2, \ldots$. Express each nonnegative integer n as a finite sum $n = \sum_{k=0}^{\infty} \alpha_k m_k$, where $0 \leq \alpha_k < p_k$, and let $\chi_n = \prod_{k=0}^{\infty} \phi_k^{\alpha_k}$. The functions $\{\chi_n\}$ are the characters of G, and they form a complete orthonormal system on G. For the case $p_i = 2$, $i = 0, 1, 2, \ldots$, $\{\phi_k\}$ are the Rademacher functions and $\{\chi_n\}$ are the Walsh functions. In this paper there is no restriction on the orders $\{p_i\}$, and the constants C, c_p and C_p that appear below are independent of $\{p_i\}$.

We consider Fourier series with respect to $\{\chi_n\}$. Let μ be the Haar measure on G normalized by $\mu(G) = 1$. For $f \in L^1$, let $\hat{f}(j) = \int_G f(t)\overline{\chi_j}(t)\,d\mu(t)$, $j = 0, 1, 2, \ldots$, and $S_n f = \sum_{j=0}^{n-1} \hat{f}(j)\chi_j$, $n = 1, 2, \ldots$. We prove the Vilenkin-Fourier series analogue of the Littlewood-Paley theorem [7, II, p. 224].

THEOREM 1. *Let $1 < p < \infty$. There exist positive constants c_p and C_p such that for any $f \in L^p$,*

$$c_p\|f\|_p \leq \left\|\left(\sum_{j=-1}^{\infty}\left|S_{2^{j+1}}f - S_{2^j}f\right|^2\right)^{1/2}\right\|_p \leq C_p\|f\|_p, \tag{1.1}$$

where $S_{2^{-1}}f = 0$.

For $p_i = 2$, $i = 0, 1, 2, \ldots$, Theorem 1 is Paley's result for Walsh-Fourier series [3]. On the other hand, if $p_0 \longrightarrow \infty$, $S_n f$ resembles the n-th trigonometric partial sum. Thus, when restricted to one cyclic group, Theorem 1 can be viewed as a discrete version of the Littlewood-Paley theorem for trigonometric Fourier series.

As a consequence of Theorem 1, we obtain the Marcinkiewicz multiplier theorem for Vilenkin-Fourier series (see [7, II, p. 232]).

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THEOREM 2. *Let $1 < p < \infty$. There is a constant C_p such that if $\{\lambda(n)\}_{n\geq 0}$ is any sequence of numbers satisfying*

$$|\lambda(n)| \leq B, \quad n = 0, 1, 2, \ldots$$

and

$$\sum_{n=2^j}^{2^{j+1}-1} |\lambda(n+1) - \lambda(n)| \leq B, \quad j = 0, 1, 2, \ldots,$$

and if $f \in L^p$, then $\sum_{n=0}^{\infty} \lambda(n)\hat{f}(n)\chi_n$ is the Vilenkin-Fourier series of a function $T_\lambda f \in L^p$ and

$$\|T_\lambda f\|_p \leq C_p B \|f\|_p.$$

The proof of Theorem 2 is the same as that given for the trigonometric case (see [2, pp. 148–151]). Instead of using the vector-valued inequality for the partial sums of trigonometric Fourier series, we use the corresponding inequality for Vilenkin-Fourier series:

LEMMA 3. *Let $1 < p < \infty$. There exists a constant C_p such that for any sequence of functions $\{f_\ell\}$ in L^p and any sequence of positive integers $\{n_\ell\}$,*

$$\Big\|\Big(\sum_\ell |S_{n_\ell} f_\ell|^2\Big)^{1/2}\Big\|_p \leq C_p \Big\|\Big(\sum_\ell |f_\ell|^2\Big)^{1/2}\Big\|_p.$$

This lemma is proved in [6].

The proof of Theorem 1 will be given in two parts. In §2 we show that it can be obtained as a result of a multiplier lemma. This lemma, which is a special case of Theorem 2, will be proved in §3.

In what follows, C will denote an absolute constant which may vary from line to line.

2. **Proof of Theorem 1.** The proof will be presented in several steps. To simplify our notation, set $\Delta_j f = S_{2^{j+1}} f - S_{2^j} f$, $j = -1, 0, 1, \ldots$. We first observe that, to prove Theorem 1, it suffices to prove the right side of (1.1), *i.e.*, for each $p \in (1, \infty)$, there is a constant C_p such that

$$\Big\|\Big(\sum_{j=-1}^{\infty} |\Delta_j f|^2\Big)^{1/2}\Big\|_p \leq C_p \|f\|_p, \quad f \in L^p. \tag{2.1}$$

The left side of (1.1) will then follow by a duality argument. To see this, let f and g be Vilenkin polynomials, $1 < p < \infty$ and $1/p + 1/q = 1$. Using the orthonormality of $\{\chi_n\}$, Hölder's inequality and (2.1), we obtain

$$\begin{aligned} \Big|\int_G f\bar{g}\,d\mu\Big| &= \Big|\sum_{j=-1}^{\infty} \int_G (\Delta_j f)(\overline{\Delta_j g})\,d\mu\Big| \\ &\leq \Big\|\Big(\sum_{j=-1}^{\infty} |\Delta_j f|^2\Big)^{1/2}\Big\|_p \Big\|\Big(\sum_{j=-1}^{\infty} |\Delta_j g|^2\Big)^{1/2}\Big\|_q \qquad (2.2) \\ &\leq C_q \Big\|\Big(\sum_{j=-1}^{\infty} |\Delta_j f|^2\Big)^{1/2}\Big\|_p \|g\|_q. \end{aligned}$$

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Since Vilenkin polynomials are dense in L^p, (2.2) holds for all $f \in L^p$ and $g \in L^q$. Taking the supremum over all $g \in L^q$ with $\|g\|_q \leq 1$, we get

$$\|f\|_p \leq C_q \Big\| \Big(\sum_{j=-1}^{\infty} |\Delta_j f|^2 \Big)^{1/2} \Big\|_p, \quad f \in L^p.$$

Since $\|\Delta_{-1} f\|_p = \|\hat{f}(0)\|_p \leq \|f\|_p$, (2.1) will be obtained if we prove

$$\Big\| \Big(\sum_{j=0}^{\infty} |\Delta_j f|^2 \Big)^{1/2} \Big\|_p \leq C_p \|f\|_p, \quad f \in L^p. \tag{2.3}$$

To prove (2.3), we introduce a related operator. Let L_k, $k = 0, 1, 2, \ldots$, be the integer such that $2^{L_k} \leq p_k < 2^{L_k+1}$. Note that $L_k \geq 1$. For $f \in L^1$, define

$$Qf = \Big[\sum_{k=0}^{\infty} \Big(\sum_{\ell=0}^{L_k-1} |S_{2^{\ell+1}m_k} f - S_{2^\ell m_k} f|^2 + |S_{m_{k+1}} f - S_{2^{L_k} m_k} f|^2 \Big) \Big]^{1/2}.$$

We shall show that

$$\Big\| \Big(\sum_{j=0}^{\infty} |\Delta_j f|^2 \Big)^{1/2} \Big\|_p \leq C_p \|Qf\|_p. \tag{2.4}$$

Let $\{n_i\}_{i \geq 0}$ be the enumeration of the set of integers $\{2^\ell m_k : \ell = 0, 1, \ldots, L_k, k = 0, 1, 2, \ldots\}$ with $n_0 < n_1 < n_2 < \cdots$. Also, let $\{\nu_i\}_{i \geq 0}$ be the enumeration of $\{2^j : j = 0, 1, 2, \ldots\} \cup \{n_i : i = 0, 1, 2, \ldots\}$ with $\nu_0 < \nu_1 < \nu_2 < \cdots$, and set $\delta_i f = S_{\nu_{i+1}} f - S_{\nu_i} f$, $i = 0, 1, 2, \ldots$. We observe that in each interval $[2^j, 2^{j+1})$, $j = 0, 1, 2, \ldots$, there are at most two n_i. Hence each $\Delta_j f$ is the sum of at most three $\delta_i f$. Therefore,

$$\sum_{j=0}^{\infty} |\Delta_j f|^2 \leq C \sum_{j=0}^{\infty} |\delta_j f|^2. \tag{2.5}$$

On the other hand, in each interval $[n_i, n_{i+1})$, $i = 0, 1, 2, \ldots$, there is at most one integer of the form 2^j. Hence $S_{n_{i+1}} f - S_{n_i} f$ is the sum of at most two $\delta_j f$. Moreover, each of these $\delta_j f$ is a difference of two partial sums of the Vilenkin-Fourier series of the function $S_{n_{i+1}} f - S_{n_i} f$. Hence it follows from Minkowski's inequality and Lemma 3 that

$$\begin{aligned} \Big\| \Big(\sum_{j=0}^{\infty} |\delta_j f|^2 \Big)^{1/2} \Big\|_p &\leq C_p \Big\| \Big(\sum_{i=0}^{\infty} |S_{n_{i+1}} f - S_{n_i} f|^2 \Big)^{1/2} \Big\|_p \\ &= C_p \|Qf\|_p. \end{aligned} \tag{2.6}$$

Combining (2.5) and (2.6), we obtain (2.4). A similar argument shows that we also have

$$\|Qf\|_p \leq C_p \Big\| \Big(\sum_{j=0}^{\infty} |\Delta_j f|^2 \Big)^{1/2} \Big\|_p.$$

Therefore, proving (2.3) is equivalent to proving

$$\|Qf\|_p \leq C_p \|f\|_p, \quad f \in L^p. \tag{2.7}$$

We shall simplify (2.7) further. Let

$$Rf = \Big(\sum_{k=0}^{\infty}\sum_{\ell=1}^{L_k-2}|S_{2^{\ell+1}m_k}f - S_{2^{\ell}m_k}f|^2\Big)^{1/2}.$$

(If $L_k \leq 2$, we interprete the sum $\sum_{\ell=1}^{L_k-2}$ to be zero.) We have

$$Qf \leq \Big(\sum_{k=0}^{\infty}|S_{2m_k}f - S_{m_k}f|^2\Big)^{1/2} + Rf + \Big(\sum_{k=0}^{\infty}|S_{2^{L_k}m_k}f - S_{2^{L_k-1}m_k}f|^2\Big)^{1/2}$$
$$+ \Big(\sum_{k=0}^{\infty}|S_{m_{k+1}}f - S_{2^{L_k}m_k}f|^2\Big)^{1/2}.$$

Each of the terms $S_{2m_k}f - S_{m_k}f$, $S_{2^{L_k}m_k}f - S_{2^{L_k-1}m_k}f$ and $S_{m_{k+1}}f - S_{2^{L_k}m_k}f$ is the difference of two partial sums of the Vilenkin-Fourier series of the function $S_{m_{k+1}}f - S_{m_k}f$. It thus follows from Lemma 3 that

$$\|Qf\|_p \leq \|Rf\|_p + C_p\Big\|\Big(\sum_{k=0}^{\infty}|S_{m_{k+1}}f - S_{m_k}f|^2\Big)^{1/2}\Big\|_p.$$

Since $\{S_{m_k}f\}$ is a martingale (see, *e.g.*, [6]), Burkholder's result for martingales [1] gives

$$\Big\|\Big(\sum_{k=0}^{\infty}|S_{m_{k+1}}f - S_{m_k}f|^2\Big)^{1/2}\Big\|_p \leq C_p\|f\|_p.$$

Therefore (2.7) will be proved if we show

$$\|Rf\|_p \leq C_p\|f\|_p, \quad f \in L^p. \tag{2.8}$$

We shall prove (2.8) using a multiplier transformation. Let $k = 0, 1, 2, \ldots$. If $L_k > 2$, define, for $\ell = 1, 2, \ldots, L_k - 2$, the sequence $\{a_{2^\ell m_k}(n)\}_{n\geq 0}$ by

$$a_{2^\ell m_k}(n) = \begin{cases} 1 & \text{if } 2^\ell m_k \leq n < 2^{\ell+1}m_k \\ \frac{j}{2^{\ell-1}} & \text{if } (2^{\ell-1}+j)m_k \leq n < (2^{\ell-1}+j+1)m_k, j = 0, 1, \ldots, 2^{\ell-1}-1 \\ 1 - \frac{j+1}{2^{\ell-1}} & \text{if } (2^{\ell+1}+j)m_k \leq n < (2^{\ell+1}+j+1)m_k, j = 0, 1, \ldots, 2^{\ell-1}-1 \\ 0 & \text{otherwise,} \end{cases}$$

and set

$$A_{2^\ell m_k}f = \sum_{n=0}^{\infty} a_{2^\ell m_k}(n)\hat{f}(n)\chi_n.$$

Let $r_i(t)$, $i = 0, 1, 2, \ldots$, be the Rademacher functions defined on $[0, 1]$. For $t \in [0, 1]$, $N = 1, 2, \ldots$, let

$$T_t^N f = \sum_{k=0}^{N-1}\sum_{\ell=1}^{L_k-2} r_{2^\ell m_k}(t)A_{2^\ell m_k}f.$$

We shall show that (2.8) will follow if we have

$$\|T_t^N f\|_p \leq C_p\|f\|_p, \quad f \in L^p,\ N = 1, 2, \ldots,\ t \in [0, 1]. \tag{2.9}$$

To see this, we note that, from (2.9),

$$\int_G \int_0^1 \Big| \sum_{k=0}^{N-1} \sum_{\ell=1}^{L_k-2} r_{2^\ell m_k}(t) A_{2^\ell m_k} f(x) \Big|^p \, dt \, d\mu(x) \le C_p \|f\|_p^p.$$

By Khintchin's inequality [7, I, p. 213], there is a constant B_p (depending only on p) such that

$$\int_0^1 \Big| \sum_{k=0}^{N-1} \sum_{\ell=1}^{L_k-2} r_{2^\ell m_k}(t) A_{2^\ell m_k} f(x) \Big|^p \, dt \ge B_p \Big(\sum_{k=0}^{N-1} \sum_{\ell=1}^{L_k-2} |A_{2^\ell m_k} f(x)|^2 \Big)^{p/2}.$$

Therefore,

$$\Big\| \Big(\sum_{k=0}^{N-1} \sum_{\ell=1}^{L_k-2} |A_{2^\ell m_k} f|^2 \Big)^{1/2} \Big\|_p \le C_p \|f\|_p.$$

Now, for $k = 0, 1, 2, \ldots$, $\ell = 1, 2, \ldots, L_k - 2$,

$$S_{2^{\ell+1} m_k} f - S_{2^\ell m_k} f = S_{2^{\ell+1} m_k}(A_{2^\ell m_k} f) - S_{2^\ell m_k}(A_{2^\ell m_k} f).$$

Combining this with Lemma 3, we get

$$\begin{aligned} \Big\| \Big(\sum_{k=0}^{N-1} \sum_{\ell=1}^{L_k-2} |S_{2^{\ell+1} m_k} f - S_{2^\ell m_k} f|^2 \Big)^{1/2} \Big\|_p &\le C_p \Big\| \Big(\sum_{k=0}^{N-1} \sum_{\ell=1}^{L_k-2} |A_{2^\ell m_k} f|^2 \Big)^{1/2} \Big\|_p \\ &\le C_p \|f\|_p. \end{aligned}$$

Letting $N \longrightarrow \infty$, we obtain (2.8).

We shall prove (2.9) in a slightly more general form. Since $a_{2^\ell m_k}(n) = 0$ for $n \notin [m_k, m_{k+1})$,

$$T_t^N f = \sum_{n=0}^{m_N - 1} \Big[\sum_{k=0}^{\infty} \sum_{\ell=1}^{L_k-2} r_{2^\ell m_k}(t) a_{2^\ell m_k}(n) \Big] \hat{f}(n) \chi_n.$$

Let $\lambda_{k,\ell}(n) = r_{2^\ell m_k}(t) a_{2^\ell m_k}(n)$, $n = 0, 1, 2, \ldots$, $k = 0, 1, 2, \ldots$, $\ell = 1, 2, \ldots, L_k - 2$, $t \in [0, 1]$. We notice that each sequence $\{\lambda_{k,\ell}(n)\}_{n \ge 0}$ has the following properties:

(2.10) $\lambda_{k,\ell}(n) = \lambda_{k,\ell}(\alpha m_k)$ for all $n \in [\alpha m_k, (\alpha+1) m_k)$, $\alpha = 0, 1, 2, \ldots$;

(2.11) $\lambda_{k,\ell}(\alpha m_k) = 0$ for $\alpha \notin [2^{\ell-1} + 1, 2^{\ell+2} - 1]$;

(2.12) $|\lambda_{k,\ell}(\alpha m_k)| \le 1$, $\alpha = 0, 1, 2, \ldots$;

(2.13) $\Big| \lambda_{k,\ell}(\alpha m_k) - \lambda_{k,\ell}\big((\alpha - 1) m_k\big) \Big| \le \dfrac{1}{2^{\ell-1}}$, $\alpha = 1, 2, \ldots$.

Hence (2.9) will be proved if we have the following lemma.

LEMMA 4. *Suppose, for $k = 0, 1, 2, \ldots$ and $\ell = 1, 2, \ldots, L_k - 2$, $\{\lambda_{k,\ell}(n)\}_{n \ge 0}$ are sequences satisfying (2.10)–(2.13), and*

(2. 14) $$\lambda(n) = \sum_{k=0}^{\infty} \sum_{\ell=1}^{L_k-2} \lambda_{k,\ell}(n), \quad n = 0, 1, 2, \ldots .$$

Then, for $1 < p < \infty$, *there is a constant* C_p, *independent of* $\{\lambda_{k,\ell}(n)\}$, *such that*

$$T^N f = \sum_{n=0}^{m_N - 1} \lambda(n)\hat{f}(n)\chi_n$$

satisfies

$$\|T^N f\|_p \le C_p \|f\|_p \tag{2.15}$$

for every $f \in L^p$, $N = 1, 2, \ldots$.

The proof of this lemma will conclude the proof of Theorem 1.

3. **Proof of Lemma 4.** Because of (2.11), we notice that for each n, at most three terms on the right side of (2.14) can be nonzero. From this and (2.12), we get

$$|\lambda(n)| \le C, \quad n = 0, 1, 2, \ldots .$$

Thus it follows from Parseval's identity that

$$\|T^N f\|_2 \le C\|f\|_2, \quad f \in L^2,\ N = 1, 2, \ldots . \tag{3.1}$$

The lemma will be proved if we have the weak-type inequality

$$\mu\{|T^N f| > y\} \le C y^{-1} \|f\|_1, \quad f \in L^1,\ y > 0,\ N = 1, 2, \ldots . \tag{3.2}$$

The case $1 < p < 2$ of (2.15) will follow from (3.1), (3.2) and the Marcinkiewicz interpolation theorem [7, II, p. 112]. A duality argument will then give us the case $2 < p < \infty$ of (2.15).

We shall use the following notation. For $k = 0, 1, 2, \ldots$, let

$$\lambda_k(n) = \sum_{\ell=1}^{L_k - 2} \lambda_{k,\ell}(n), \quad n = 0, 1, 2, \ldots,$$

and

$$T_k f = \sum_{n=0}^{\infty} \lambda_k(n)\hat{f}(n)\chi_n.$$

Observe that $\lambda_k(n) = 0$ for $n \notin [m_k, m_{k+1})$. We have

$$T^N f = \sum_{k=0}^{N-1} T_k f. \tag{3.3}$$

We shall write $T_k f$ in an integral form. By (2.10),

$$\begin{aligned}\sum_{n=0}^{\infty} \lambda_k(n)\chi_n &= \sum_{\alpha=1}^{p_k - 1} \lambda_k(\alpha m_k) \sum_{n=\alpha m_k}^{(\alpha+1)m_k - 1} \chi_n \\ &= \sum_{\alpha=1}^{p_k - 1} \lambda_k(\alpha m_k)\phi_k^{\alpha} D_{m_k},\end{aligned}$$

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where $D_n = \sum_{j=0}^{n-1} \chi_j$, $n = 1, 2, \ldots$, denotes the n-th Dirichlet kernel. To describe D_{m_k}, let $\{G_k\}$ be a sequence of subgroups of G defined by

$$G_0 = G,\; G_k = \prod_{i=0}^{k-1} \{0\} \times \prod_{i=k}^{\infty} Z_{p_i}, \quad k = 1, 2, \ldots .$$

It is proved in [4] that $D_{m_k} = m_k \chi_{G_k}$. Note that $\mu(G_k) = m_k^{-1}$. Therefore

$$\begin{aligned} T_k f(x) &= \int_G f(t) \Big[\sum_{n=0}^{\infty} \lambda_k(n) \chi_n(x - t) \Big] \, d\mu(t) \\ &= \frac{1}{\mu(G_k)} \int_{x+G_k} f(t) M_k(x - t) \, d\mu(t), \end{aligned} \tag{3.4}$$

where

$$M_k(t) = \sum_{\alpha=1}^{p_k - 1} \lambda_k(\alpha m_k) \phi_k^{\alpha}(t).$$

We shall identify G with the unit interval $(0, 1)$ by associating with each $\{x_i\} \in G$, $0 \le x_i < p_i$, the point $\sum_{i=0}^{\infty} x_i m_{i+1}^{-1} \in (0, 1)$. If we disregard the countable set of p_i-rationals, this mapping is one-one, onto and measure-preserving. On the interval $(0, 1)$, cosets of G_k are intervals of the form $\big(j m_k^{-1}, (j+1) m_k^{-1}\big)$, $j = 0, 1, \ldots, m_k - 1$. An interval $I \subset (0, 1)$ is said to belong to $\mathcal{J}_k$, $k = 0, 1, 2, \ldots$, if I is a proper subset of a coset of G_k and is the union of cosets of G_{k+1}. For $I \in \mathcal{J}_k$, we define the set $3I$ as follows: Suppose $I \subset x + G_k$, $x \in G$. If $\mu(I) \ge \mu(G_k)/3$, let $3I = x + G_k$. If $\mu(I) < \mu(G_k)/3$, consider $x + G_k$ as a circle, and define $3I$ to be the interval in this circle which has the same center as I and has measure $\mu(3I) = 3\mu(I)$.

We are now ready to prove (3.2). Let $f \in L^1$ and $y > 0$. We can assume $\|f\|_1 \le y$. Otherwise, there is nothing to prove. Applying the Calderón-Zygmund decomposition lemma (see [5]), we obtain a sequence $\{I_j\}$ of disjoint intervals in $\bigcup_{k=0}^{\infty} \mathcal{J}_k$ such that

$$y < \frac{1}{\mu(I_j)} \int_{I_j} |f| \, d\mu \le 3y, \quad \text{for all } I_j \tag{3.5}$$

and

$$|f(x)| \le y \quad \text{for a.e. } x \notin \bigcup_j I_j \equiv \Omega.$$

Let $f = g + b$ where

$$g(x) = \begin{cases} f(x) & \text{if } x \notin \Omega \\ \frac{1}{\mu(I_j)} \int_{I_j} f \, d\mu & \text{if } x \in I_j, j = 1, 2, \ldots . \end{cases}$$

Then g and b have the following properties:

$$|g(x)| \le 3y \quad \text{a.e.}; \tag{3.6}$$

$$\|g\|_1 \le \|f\|_1; \tag{3.7}$$

$$b(x) = 0 \quad \text{if } x \notin \Omega; \tag{3.8}$$

$$\int_{I_j} b \, d\mu = 0 \quad \text{for all } I_j; \tag{3.9}$$

$$\int_{I_j} |b| \, d\mu \le 2 \int_{I_j} |f| \, d\mu \quad \text{for all } I_j. \tag{3.10}$$

Since

$$\mu\{|T^N f| > y\} \le \mu\{|T^N g| > y/2\} + \mu\{|T^N b| > y/2\},$$

(3.2) will be proved if we show that each term on the right is bounded by $Cy^{-1}\|f\|_1$.

For the first term, we use (3.1), (3.6) and (3.7) to get

$$\mu\{|T^N g| > y/2\} \le Cy^{-2}\|T^N g\|_2^2 \le Cy^{-2}\|g\|_2^2 \le Cy^{-1}\|f\|_1.$$

To estimate $T^N b$, let $\Omega^* = \bigcup_j (3I_j)$. Then

$$\mu(\Omega^*) \le 3\sum_j \mu(I_j) \le Cy^{-1}\|f\|_1,$$

by (3.5). From (3.3), we have

$$\mu\{x \notin \Omega^* : |T^N b| > y/2\} \le Cy^{-1}\int_{{}^c\Omega^*} |T^N b|\, d\mu$$
$$\le Cy^{-1}\sum_{k=0}^{N-1}\int_{{}^c\Omega^*} |T_k b|\, d\mu.$$

Hence (3.2) will be proved if we show

(3. 11) $$\sum_{k=0}^{\infty}\int_{{}^c\Omega^*} |T_k b|\, d\mu \le C\|f\|_1.$$

Let $x \notin \Omega^*$, $I = x + G_k$ and $I' = x + G_{k+1}$. From (3.4),

$$T_k b(x) = \frac{1}{\mu(I)}\int_I b(t)M_k(x-t)\, d\mu(t).$$

We shall split the integral over I' and $I \setminus I'$. Note that neither I nor I' is contained in Ω. For $t \in I'$, $M_k(x-t) = \sum_{\alpha=1}^{p_k-1}\lambda(\alpha m_k)$. Therefore

$$\int_{I'} b(t)M_k(x-t)\, d\mu(t) = \sum_{\alpha=1}^{p_k-1}\lambda(\alpha m_k)\int_{I'} b\, d\mu$$
$$= \sum_{\alpha=1}^{p_k-1}\lambda(\alpha m_k)\sum_{I_j\subset I'}\int_{I_j} b\, d\mu = 0,$$

by (3.8) and (3.9). As for the second integral, we have, by (3.8),

$$\int_{I\setminus I'} b(t)M_k(x-t)\, d\mu(t) = \sum_{I_j\subset I,\, I_j\not\subset I'}\int_{I_j} b(t)M_k(x-t)\, d\mu(t)$$
$$= \sum_{I_j\subset I,\, I_j\in\mathfrak{J}_k}\int_{I_j} b(t)M_k(x-t)\, d\mu(t)$$
$$+ \sum_{\substack{I_j\subset I,\, I_j\not\subset I' \\ I_j\notin\mathfrak{J}_k}}\int_{I_j} b(t)M_k(x-t)\, d\mu(t).$$

For $I_j \subset I$ and $I_j \notin \mathcal{J}_k$, $M_k(x-t)$ is constant on I_j. Thus the last term vanishes by (3.9). Let $t^j = \{t^j_k\}_{k\geq 0}$ be any fixed point in I_j. Again, by (3.9),

$$\int_{I_j} b(t) M_k(x-t^j)\, d\mu(t) = 0$$

for any I_j. Therefore

$$T_k b(x) = \frac{1}{\mu(I)} \sum_{I_j \subset I, I_j \in \mathcal{J}_k} \int_{I_j} b(t)[M_k(x-t) - M_k(x-t^j)]\, d\mu(t).$$

If I is any coset of G_k,

$$\int_{I \cap {}^c\Omega^*} |T_k b(x)|\, d\mu(x) \leq \sum_{I_j \subset I, \quad I_j \in \mathcal{J}_k} \int_{I_j} |b(t)| \frac{1}{\mu(I)} \int_{I \cap {}^c(3I_j)} |M_k(x-t) - M_k(x-t^j)|\, d\mu(x)\, d\mu(t).$$

We shall show

$$\text{(3.12)} \qquad \frac{1}{\mu(I)} \int_{I \cap {}^c(3I_j)} |M_k(x-t) - M_k(x-t^j)|\, d\mu(x) \leq C$$

for any coset I of G_k, $I_j \subset I$, $I_j \in \mathcal{J}_k$ and $t, t^j \in I_j$. With (3.12) we get

$$\begin{aligned} \int_{I \cap {}^c\Omega^*} |T_k b|\, d\mu &\leq C \sum_{I_j \subset I, \quad I_j \in \mathcal{J}_k} \int_{I_j} |b|\, d\mu \\ &\leq C \sum_{I_j \subset I, \quad I_j \in \mathcal{J}_k} \int_{I_j} |f|\, d\mu, \end{aligned}$$

by (3.10). Summing over all cosets I of G_k and then over all k, we obtain

$$\sum_{k=0}^{\infty} \int_{{}^c\Omega^*} |T_k b|\, d\mu \leq C \sum_{k=0}^{\infty} \sum_{I_j \in \mathcal{J}_k} \int_{I_j} |f|\, d\mu \leq C \|f\|_1.$$

Thus (3.11) will be proved if we have (3.12).

Set

$$M_{k,\ell}(t) = \sum_{\alpha=1}^{p_k - 1} \lambda_{k,\ell}(\alpha m_k) \phi_k^\alpha(t), \qquad \ell = 1, \ldots, L_k - 2.$$

Then

$$M_k(t) = \sum_{\ell=1}^{L_k-2} M_{k,\ell}(t).$$

To prove (3.12) it suffices to establish the following inequality:

$$\text{(3.13)} \qquad \begin{aligned} &\frac{1}{\mu(I)} \int_{I \cap {}^c(3I_j)} |M_{k,\ell}(x-t) - M_{k,\ell}(x-t^j)|\, d\mu(x) \\ &\qquad \leq C \min\left\{ \left[2^{-\ell} \frac{\mu(I)}{\mu(I_j)}\right]^{1/2}, \left[2^{\ell} \frac{\mu(I_j)}{\mu(I)}\right]^{1/2} \right\}, \quad \ell = 1, \ldots, L_k - 2, \end{aligned}$$

for any coset I of G_k, $I_j \subset I$, $I_j \in \mathcal{J}_k$ and $t, t^j \in I_j$. Then (3.12) will follow if we sum over all ℓ, using the second estimate for $\ell \leq \log_2 \frac{\mu(I)}{\mu(I_j)}$ and the first for $\ell > \log_2 \frac{\mu(I)}{\mu(I_j)}$.

We shall now prove the first estimate in (3.13). Note that

$$\frac{1}{\mu(I)}\int_{I\cap^c(3I_j)}|M_{k,\ell}(x-t)|\,d\mu(x)\le\Big(\frac{1}{\mu(I)}\int_I|M_{k,\ell}(x-t)|^2|\phi_k(x-t)-1|^2\,d\mu(x)\Big)^{1/2}$$
$$\times\Big(\frac{1}{\mu(I)}\int_{I\cap^c(3I_j)}|\phi_k(x-t)-1|^{-2}\,d\mu(x)\Big)^{1/2},$$

by Hölder's inequality. A direct computation shows

$$\frac{1}{\mu(I)}\int_{I\cap^c(3I_j)}|\phi_k(x-t)-1|^{-2}\,d\mu(x)\le C\frac{\mu(I)}{\mu(I_j)}. \tag{3.14}$$

From (2.11) we have

$$M_{k,\ell}(x)[\phi_k(x)-1]=\sum_{\alpha=2^{\ell-1}}^{2^{\ell+2}}\Big[\lambda_{k,\ell}\big((\alpha-1)m_k\big)-\lambda_{k,\ell}(\alpha m_k)\Big]\phi_k^{\alpha}(x). \tag{3.15}$$

By Parseval's identity and (2.13) we get

$$\frac{1}{\mu(I)}\int_I|M_{k,\ell}(x-t)|^2|\phi_k(x-t)-1|^2\,d\mu(x)=\sum_{\alpha=2^{\ell-1}}^{2^{\ell+2}}\Big|\lambda_{k,\ell}\big((\alpha-1)m_k\big)-\lambda_{k,\ell}(\alpha m_k)\Big|^2$$
$$\le C2^{-\ell}.$$

Therefore

$$\frac{1}{\mu(I)}\int_{I\cap^c(3I_j)}|M_{k,\ell}(x-t)|\,d\mu(x)\le C\Big[2^{-\ell}\frac{\mu(I)}{\mu(I_j)}\Big]^{1/2}.$$

The same inequality holds if we replace t by t^j. From these we obtain the first estimate in (3.13).

To obtain the second estimate in (3.13), we use the inequality

$$\frac{1}{\mu(I)}\int_{I\cap^c(3I_j)}|M_{k,\ell}(x-t)-M_{k,\ell}(x-t^j)|\,d\mu(x)$$
$$\le\Big(\frac{1}{\mu(I)}\int_I|M_{k,\ell}(x-t)-M_{k,\ell}(x-t^j)|^2|\phi_k(x-t)-1|^2\,d\mu(x)\Big)^{1/2}$$
$$\times\Big(\int_{I\cap^c(3I_j)}|\phi_k(x-t)-1|^{-2}\,d\mu(x)\Big)^{1/2}.$$

Let $s=t^j-t$. We observe that

$$[M_{k,\ell}(x)-M_{k,\ell}(x-s)][\phi_k(x)-1]$$
$$=M_{k,\ell}(x)[\phi_k(x)-1]-M_{k,\ell}(x-s)[\phi_k(x-s)-1]$$
$$-M_{k,\ell}(x-s)[\phi_k(x)-\phi_k(x-s)]$$
$$=\sum_{\alpha=2^{\ell-1}}^{2^{\ell+2}}\Big[\lambda_{k,\ell}\big((\alpha-1)m_k\big)-\lambda_{k,\ell}(\alpha m_k)\Big][1-\phi_k^{-\alpha}(s)]\phi_k^{\alpha}(x)$$
$$-\sum_{\alpha=2^{\ell-1}}^{2^{\ell+2}}\lambda_{k,\ell}\big((\alpha-1)m_k\big)\phi_k^{1-\alpha}(s)[1-\phi_k^{-1}(s)]\phi_k^{\alpha}(x),$$

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by (3.15) and (2.11). Using Parseval's identity, (2.13), (2.12) and the fact that $t, t^j \in I_j$, we obtain

$$\begin{aligned}\frac{1}{\mu(I)}\int_I |M_{k,\ell}(x-t)-M_{k,\ell}(x-t^j)|^2|\phi_k(x-t)-1|^2\,d\mu(x)\\ \leq C\sum_{\alpha=2^{\ell-1}}^{2^{\ell+2}}\Big|\lambda_{k,\ell}\big((\alpha-1)m_k\big)-\lambda_{k,\ell}(\alpha m_k)\Big|^2|1-\phi_k^{-\alpha}(t^j-t)|^2\\ +C\sum_{\alpha=2^{\ell-1}}^{2^{\ell+2}}\Big|\lambda_{k,\ell}\big((\alpha-1)m_k\big)\Big|^2|1-\phi_k^{-1}(t^j-t)|^2\\ \leq C2^\ell\left[\frac{\mu(I_j)}{\mu(I)}\right]^2.\end{aligned}$$

Combining this with (3.14) we get

$$\frac{1}{\mu(I)}\int_{I\cap^c(3I_j)}|M_{k,\ell}(x-t)-M_{k,\ell}(x-t^j)|\,d\mu(x)\leq C\left[2^\ell\frac{\mu(I_j)}{\mu(I)}\right]^{1/2}.$$

This proves (3.13) and hence concludes the proof of Lemma 4. The proof of Theorem 1 is now complete. ∎

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